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IN APPRECIATION OF HIS EXCEPTIONAL SERVICES
TO THE MATHEMATICAL ASSOCIATION OF AMERICA
AND TO THIS MONTHLY
THIS VOLUME IS DEDICATED TO THE LATE

HERBERT ELLSWORTH SLAUGHT

WHO AT THE TIME OF HIS DEATH WAS
HONORARY PRESIDENT OF THE ASSOCIATION
AND HAD SERVED CONTINUOUSLY AS AN
EDITOR OF THE MONTHLY
FOR THIRTY YEARS

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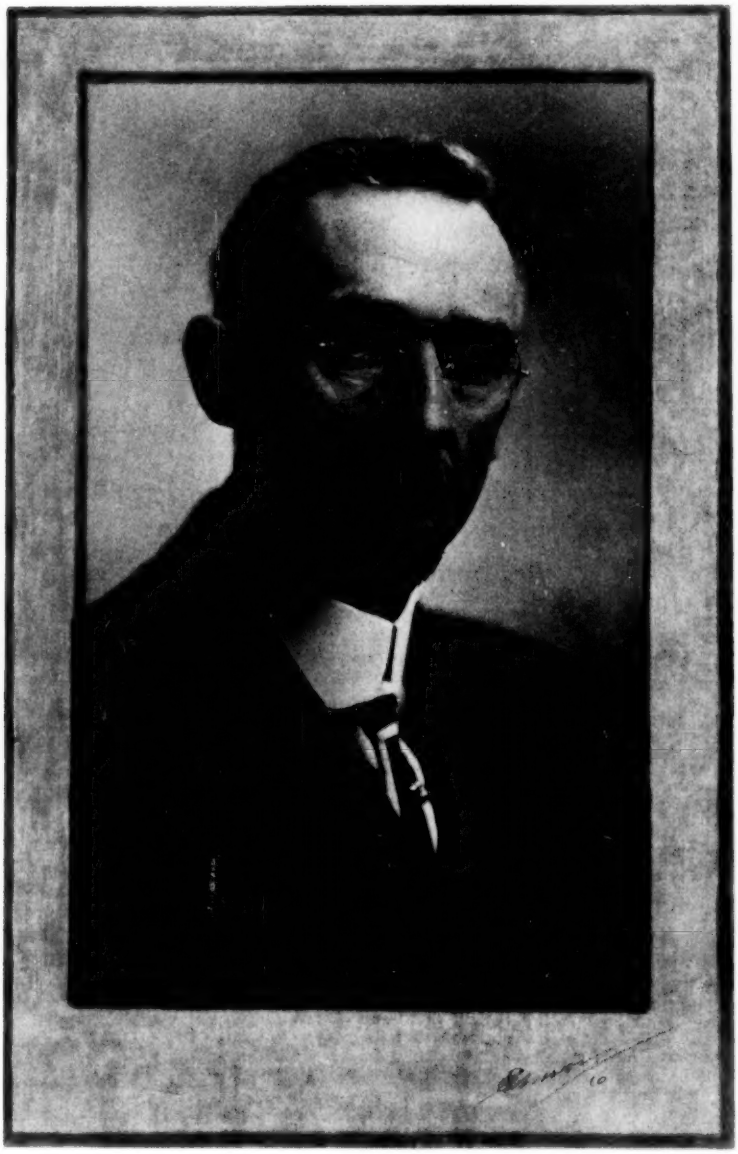

HERBERT ELLSWORTH SLAUGHT

HERBERT ELLSWORTH SLAUGHT—EDITOR AND ORGANIZER*

W. D. CAIRNS, Oberlin College

A great and unique leader in American mathematics closed his career on May 21, 1937. It is fitting that a record of his achievements should here be made so far as it concerns his relation to the AMERICAN MATHEMATICAL MONTHLY and the Mathematical Association of America.

The connection of Mr. Slaught with the MONTHLY began in January 1907, when Assistant Professor L. E. Dickson of the University of Chicago, who had been associated with Professor B. F. Finkel in editing the MONTHLY since October 1902, resigned because he found his time too fully occupied with research and with editorial duties for the *Transactions of the American Mathematical Society*. During the years 1907 and 1908 Mr. Slaught was in charge of all articles for publication. He recognized at once that there should be a wider support of the MONTHLY, and he soon gained the support of the University of Illinois; and for the years 1909 to 1912 inclusive Professors Slaught and G. A. Miller shared jointly in the editorial responsibility for articles, with Professor Finkel continuing in charge of the problem department and of the business management. A steady advance in the character of the MONTHLY is evident in the volumes for these years, strengthened as it was by articles by these two editors, for example, *The teaching of mathematics in the colleges* by Mr. Slaught in 1908 and *Mathematics beyond the calculus* by Mr. Miller in 1910.

In all these years Mr. Slaught worked persistently in his endeavor to broaden the basis of the journal. Forceful presentations that showed the breadth and the vividness of his vision won the support of more individuals and institutions, so that at the end of 1912 the editors were able to announce a new "Board of Editors," with Mr. Slaught as Managing Editor, representing eleven universities and colleges which, as institutions or privately, furnished a subsidy of fifty dollars each for three years for the journal. At this time the MONTHLY passed legally from the possession of its founder, Professor Benjamin F. Finkel, to that of the Board of Editors, a necessary preliminary, in Mr. Slaught's judgment, to placing the MONTHLY on a proper foundation. Support by one more university was gained in 1914 and by two more in 1915. With the issue for January 1913 the present form of the journal was adopted coincidentally with a change to the New Era Printing Company. Its scholarly character was further enhanced by such articles as the series on *The history of the logarithmic and exponential concepts* by Professor Cajori in the volume for 1913.

A perusal of the MONTHLY during this interval will verify the statement that Mr. Slaught, probably more than any other person, had the conviction that more must be done for the average teacher of mathematics and more must be done within the field of collegiate mathematics. As far back as 1912 the question was raised as to the possibility of having the American Mathematical Society extend its scope so as to include the publication of the MONTHLY. The proposi-

* Presented at the meeting of the Association at Pennsylvania State College, September 7, 1937.

tion took a more definite shape, under Mr. Slaught's impetus, at the April 1914 meeting of the Chicago Section of the Society. A committee of the Section reported at its meeting in December 1914 and these resolutions were unanimously adopted:

(1) The committee feels that there exists a large number of persons interested in mathematics but not efficiently reached by the present publications of the Society.

(2) Many of these persons might be stimulated to mathematical interest by a periodical covering the field of the AMERICAN MATHEMATICAL MONTHLY.

(3) Members of the Society are strongly interested in the questions connected with collegiate teaching and would value such a journal on that account.

(4) The financial situation of the AMERICAN MATHEMATICAL MONTHLY, while at present satisfactory, is nevertheless somewhat precarious and tentative.

(5) It appears therefore desirable for the progress of mathematical science in this country that the Society should assist and control a journal of this type.

These resolutions were transmitted to the Council of the Society with the request that it consider the feasibility of conducting, under the auspices of the Society, a journal for the field then covered by the AMERICAN MATHEMATICAL MONTHLY, and of allotting a portion of its activities to that field. It may be said here that the MONTHLY was then in good financial condition and not seeking rescue measures for itself. The movement was rather one of public concern for the higher welfare of mathematics in America.

The Council appointed a committee which reported at a large meeting of the Council in New York in April 1915. The gist of the report, adopted by an overwhelming majority, was embodied in the following resolution:

> It is deemed unwise for the American Mathematical Society to enter into the activities of the special field now covered by the AMERICAN MATHEMATICAL MONTHLY; but the Council desires to express its realization of the importance of the work in this field and its value to mathematical science, and to say that should an organization be formed to deal specifically with this work the Society would entertain toward such an organization only feelings of hearty good will and encouragement.

While some members of the Council committee and some others in the Society felt at the time that the Society might well broaden its scope of activity along the lines suggested and thus maintain its sphere of influence throughout the entire mathematical field, yet the decision of the Council was so emphatic as to leave no room for doubt concerning the policy of the Society both as to its own favorable attitude toward the field of activity in question and as to the desirability of having this field provided for by an organization formed to deal specifically with this work. It thus being made clear that the organization of a national body to deal with the collegiate field would meet with the hearty ap-

proval of the Society, events began to move rapidly. Mr. Slaught and the loyal group of mathematicians whom he had inspired gave wide publicity to the growing plans, both through the MONTHLY and through supplementary letters. About four hundred fifty persons signed the call for the organization meeting held in Columbus, Ohio, December 30, 1915. These represented every state in the Union, the District of Columbia, and Canada, including normal, college, and university teachers, consulting engineers, actuaries, and others interested in mathematics for its own sake. Over one hundred delegates attended the meeting, being about equally divided between college and university teachers.

Mr. Slaught preferred to suggest others than himself as officers, becoming formally the "Manager" under the new organization and joining arduously with the president and the secretary-treasurer, together with many helpers, in bringing about a wide publicity of the new plans and a notable accretion in the list of members of the new Association. Under such a stimulus the Association numbered 1045 individual members and 52 institutional members in the first three months and increased to 1382 individual members and 94 institutional members in the first six years. Not until the year 1919 could Mr. Slaught be prevailed upon to accept election to the presidency. In December 1933 he was unanimously made honorary president for life.

Nowhere can one see more clearly his vision and his enthusiasm than in the opening years of the Association's life. Originating fresh plans for Association activities or aiding in developing valuable suggestions from others, he was largely instrumental in the formation of plans (1) for a study of the reorganization of secondary school mathematics and the notable work done by the National Committee on Mathematical Requirements under the chairmanship of the late Professor J. W. Young, (2) for the establishment of sections of the Mathematical Association, (3) for the possible development of a journal for articles of an expository type, a plan which took the shape of a subsidy whereby the *Annals of Mathematics* was furnished at half price to Association members, and (4) for a re-definition of the "units" in secondary mathematics, incorporated in the report published by the National Committee which we have mentioned.

Another outstanding opportunity arose during the World War when Gustav Enestrom wrote to Professor David Eugene Smith inquiring as to the possibility of continuing the publication of the *Bibliotheca Mathematica* in America. Professors Smith and Slaught, after careful consideration, decided to appeal to Mrs. Mary Hegeler Carus for aid. The outcome, however, was not the publication of the *Bibliotheca* but of the *Carus Mathematical Monographs* instead. In his article on Mrs. Carus in the MONTHLY for May 1937, Professor Smith states that the initial suggestion for the *Carus Monographs* was probably made to Mrs. Carus by Mr. Slaught. Certain it is that much of the success of this well-known series has been due to his executive ability on the Carus Monograph Committee for the years 1921–1937.

His strong grasp and intimate acquaintance with the possibilities before the Mathematical Association and affiliated bodies is shown in his retiring presidential address given in September 1920 and published in the MONTHLY for

December of that year. He instanced the Association's help in fostering the beginnings of research, in promoting higher standards in teaching mathematics, in inspiring more teachers to active efforts through the sections and the widening circle of the undergraduate mathematics clubs. He also outlined the imperative need of a permanent endowment fund for the heavy cost of printing the MONTHLY and for proper provision for offices and moderate salaries for the secretary-treasurer and the editor-in-chief.

It was very natural that Mr. Slaught should take an active part in shaping the plans for the publicizing and the sale of the Chace edition of the *Rhind Mathematical Papyrus*, beginning in 1926.

The many men and women who in the past forty-five years have been his pupils and associates need not be told of the happy combination of scholarship and fine humor which he possessed. These and the dramatic touch which he gave to his work were evidenced again and again in such writings as *The evolution of numbers—an historical drama in two acts* printed in the MONTHLY for March 1928; in his address at the summer meeting at the Century of Progress on *The lag of mathematics behind literature and art in the early centuries*, printed in the MONTHLY for March 1934; and in his many addresses before groups of mathematics teachers over the country.

Professor Slaught frequently took part in the activities of the Central Association of Teachers of Science and Mathematics, speaking at their annual meetings and sharing in their counsels. In recognition of this he was made honorary life member of the Central Association.

His was again a helpful hand when the National Council of Teachers of Mathematics was organized in 1920. Always sympathetic with progress in the teaching of secondary school mathematics, he continued, in his busy existence, to hearten and to advise. The Council paid him the high honor in January 1937 of electing him honorary president for life. Much to his own delight and to the later pleasure of the Council members, he was enabled in his own home in February 1937 to make a disk record of a speech to be reproduced at the annual meeting of the Council at the Hotel Stevens in Chicago.

I cannot close this account of one great sector of his activities without paying grateful tribute to him for the intelligent direction which he gave to those whom he selected as his assistants. He moved consistently toward bringing as many different people as possible into active work for the MONTHLY and the Association; this is strikingly verified if one merely glances at the extensive lists of associate editors of the MONTHLY and officers of the Association. He was remarkable in his power of spying out new men and women of promise and of guiding them into enthusiastic service. Many of his hopes have not yet been realized for the establishment of the Association on an adequate financial foundation but he more than any other person has seen that the groundwork has been properly laid, and that the Association and its publications have been made highly worthy of support. It will be our hope, as it was his earnest hope, that in the not distant future some benevolent-minded person will crown his life's plans and hopes with a substantial foundation for the journal and the organization to which he so successfully devoted his fine ability and unswerving loyalty.

HERBERT ELLSWORTH SLAUGHT—TEACHER AND FRIEND*

G. A. BLISS, University of Chicago

Most of the members of our Mathematical Association of America are actively engaged in the study and teaching of mathematics. They are inspired by their confidence in the interest and importance of their science, and they realize the unlikelihood of finding as profound satisfaction and stimulation in any other profession. Herbert Ellsworth Slaught was one of our fellow members whose life was most strongly motivated by such impressions. In his case his convictions found expression not only in his own teaching, but also in the sympathy and encouragement which he gave to others in the same pursuit. He was a founder of our Association and of the National Council of Teachers of Mathematics, and an early member and officer in the American Mathematical Society. He was thus influential in the early years of three of the most important mathematical societies in America, and up to the time of his death on May 21, 1937 he continued to be a wise adviser in their councils. My colleague on this program, Professor Cairns, has told you in more detail of Slaught's activities in connection with the affairs of these societies. The purpose of the following paragraphs is to present an account of other events of Slaught's life which show how consistently he was inspired by an unselfish interest in the problems and personalities of those with whom he came in contact, and which explain the cordial friendship with which he was regarded by all who knew him.

The foundation of Slaught's influence in educational work undoubtedly lay in his own effectiveness as an instructor in the classroom. He was lucid and encouraging, and essentially friendly even when criticism was necessary. He enjoyed the mathematics he taught and his classes so much that his students reciprocated by enjoying them too. An undergraduate in the early days of the University of Chicago, registered for a course with Slaught, unexpectedly found that he liked the mathematics and was successful. He registered for another, but to his dissatisfaction was assigned by lot to a second section. He appealed to his dean and to Slaught for a transfer without success, and to his father who held an influential post in the University but who declined to interfere. Finally he appealed to his mother, quite justly as the highest authority of them all. She went undaunted to President Harper and explained to him how unfortunate it would be for her son to lose his new found interest and encouragement. The President succumbed. That the encouragement engendered by Slaught in this student is typical of his successes as a teacher is confirmed by his colleagues, some of whom have themselves been students in his classes, and by the testimony of many letters from former students, tucked away in his correspondence, expressing appreciation of the clarity and friendliness of his teaching, letters which must have given him the greatest satisfaction.

I think that Slaught's unusual sympathy and understanding for students struggling with mathematical or other difficulties must have had its source in

* Presented at the meeting of the Association at Pennsylvania State College, September 7, 1937.

the hardships which he himself cheerfully overcame in securing his own high school and college education. He was born on July 21, 1861 and lived on a farm near Watkins, N. Y., until he was thirteen years old. Up to that time he had had a full boy's share of the farm work to do, in spite of a serious illness which had left him with one leg partially crippled. It was a blessing to the family, though undoubtedly effectively disguised at the time, when the farm was taken away from them in the first half of 1875 because of an unpaid mortgage. In that year Herbert and two older brothers of college age, with an older sister and their invalid mother, drove to Hamilton, N. Y., in a horse and wagon salvaged from the farm, with a cow as trailer. In Hamilton the boys were successful from the start in supporting the family and in completing their educations. They sold milk and carting, had jobs as janitors in college buildings, picked hops, and earned money in many other ways. While still on the farm they had been interested in telegraphy, a marvel of their day as radio is now, and had constructed and practiced with telegraph instruments. Herbert was always very proud of a position as telegraph operator at a summer resort in the Catskills which he held during four of his summer vacations. It was profitable to him financially, but best of all it afforded him a chance to recuperate his rather frail physique after the over strenuous labors of the college year. Characteristic of him was the fact that he made a permanent friend of the proprietor. In time the two older boys became successful physicians, and Herbert graduated from Colgate Academy in 1879, and from Colgate University in 1883, each time as valedictorian of his class. He won a third prize in mathematics, and first and second prizes at different times in Greek. In the latter instance he probably would have had first prize if he had not left unfinished one question on the prize examination in order to keep an engagement with a young woman in whom he was interested, all of which conclusively confirms the general principle that even classicists may at times be human.

It seems to me that Slaught's career as a student was remarkable. It was undoubtedly more strenuous than desirable, but it was of a kind which makes strong men who value highly the training they have received. With his physical and financial handicaps he could never have carried his course through successfully without the time-saving devices for systematic study which he invented and used consistently. He was given an M.S. by Colgate in 1886, and an honorary Sc.D. in 1911.

During his college course Slaught's principal interest had been in the classics, and he hoped to be able to secure a teaching position in that field after his graduation. It was therefore a great surprise to him in the summer of 1883 to learn that one of his professors had recommended him for an instructorship in mathematics at Peddie Institute in Hightstown, New Jersey. He was appointed, and entered upon his first teaching work with enthusiasm and originality. He was greatly encouraged to find that from the start his presentation of mathematics to young students was regarded as unusually successful. After some years of experience he became much dissatisfied with the high school texts in

mathematics which were at that time the most popular. His criticisms came to the attention of the publishers, and irritated them so much that they arranged an interview for Slaught with the author. The outcome was surprising. It was a contract between Slaught and the publishers, strongly recommended by the author, for a book on high school algebra to be written by Slaught. Unfortunately he was not able to carry out this contract because of increasing administrative duties at Peddie. He was made assistant principal in 1886, and was principal during the remainder of his sojourn there from 1889 to 1892. As principal he was popular and successful. He always recalled with much satisfaction his success in meeting the terms of a gift of a student observatory to the Institute. The donor had specified that it must be ready for a June commencement only six weeks away. Fortunately the instruments were already at hand, except for a telescope clock which Slaught somehow constructed out of old telegraph instruments. The dome on a tower of the new library had to be planned and completed. On the Saturday evening before commencement Sunday the scaffolding was removed, and the observatory was open for inspection on commencement day.

The same year that Slaught went to Peddie a young woman from Boston, named Mary L. Davis, had been appointed to the headship of the music department. They became great friends and spent much time together, especially on the croquet ground. In the summer of 1885 they were married, and an eminently successful and happy partnership was begun, terminated only by her death in 1919. By 1892 it became evident to both of them that a choice must be made between the administrative responsibilities of an academy principalship and a possible university career in mathematics and teaching. Adoption of the latter alternative meant uncertainty and serious sacrifice for several years to come, but Mrs. Slaught urged without hesitation what she by that time knew to be in accord with her husband's real interests. He accordingly planned to go to Johns Hopkins University and applied for a fellowship there, but his plan was unexpectedly interrupted. In one of his campaigns for funds for Peddie Institute he had met Mr. Fred T. Gates, private secretary and investigator for John D. Rockefeller. Mr. Gates learned of Slaught's intention to go to Johns Hopkins to study mathematics and promptly arranged an interview for him with President W. R. Harper of the newly founded University of Chicago. President Harper with characteristic decisiveness offered Slaught a two-year fellowship with a guarantee of extra summer quarter teaching to help out with his finances, and Slaught accepted. He thus became one of the first three fellows in the department of mathematics at the University of Chicago.

Slaught was a strong student of mathematics as an undergraduate, but did not take as many courses in that subject as he would have done if he had known that it was to become his specialty. When he was appointed to his fellowship he felt this lack of mathematical experience very keenly, and he set about repairing it with characteristic energy as soon as he and Mrs. Slaught had found a place to settle down with their small daughter in Chicago in the early summer

of 1892. His effort was successful, and from the start the quality of his graduate work justified his appointment to the fellowship. His summer teaching and a few courses which he gave as a fellow convinced the University authorities of his unusual skill in the class room, and during the years from 1894 to 1897 he was successively appointed reader, associate, assistant, and instructor, an illustration of the fact that in those days President Harper believed in giving his young men frequent encouragement, but in rather small doses. Meanwhile Slaught had been working steadily toward his doctorate, delayed by his increasing teaching responsibilities. He received his Ph.D. in the winter quarter of 1898, and his thesis entitled *The cross-ratio group of 120 Cremona transformations of the plane* appeared in the *American Journal of Mathematics* in 1900. It was a difficult and thorough piece of work. He was appointed assistant professor at the University of Chicago in 1900, associate professor in 1908, professor of mathematics in 1913, and professor emeritus in 1931.

In the early days of the University of Chicago President Harper cultivated with especial assiduity the relations of the University with secondary schools in the central west. Slaught turned out to be one of the most effective of the representatives of the University in this work. He had charge of the section on mathematics in the conferences of teachers of secondary schools which were held at the University each summer quarter, and he was an official visitor of secondary schools on many occasions. Many teachers, in other subjects as well as mathematics, came to know him and to appreciate his friendly interest. I think it must have been a consciousness of this growing influence which led Slaught to decide to devote his life to the promotion and improvement of the teaching of mathematics rather than to research. His decision to do so was not an accident. It was reached only after very serious deliberation during a sojourn in Europe in the college year of 1902–03. We all know some at least of the most striking of its consequences. He saved the struggling MONTHLY, and organized two of the most effective societies of teachers of mathematics in this country. We must not forget, in our admiration of these achievements, that he was also one of the small group of enthusiastic mathematicians which in the nineties of last century became the Chicago Section of the American Mathematical Society, and that he was a most efficient secretary of that section for ten years from 1906 to 1916. In 1916, when others doubted, he foresaw the advantages of cooperation between the Society and the proposed Association, now amply demonstrated by years of profitable and interesting experience. His confidence in his own judgment, once carefully formulated, and his championship of friendly cooperation, are characteristic of his other activities as well as these.

Naturally much of the time and thought which Slaught gave so freely to various associations of mathematicians was centered on their journals. He was an editor of a number of them and of the *Carus Mathematical Monographs*, as Professor Cairns has already said to you. In 1906 he became mathematical adviser to a prominent book firm which in 1907 published two high school algebras by Slaught and N. J. Lennes, and in 1910 two more books on plane and

solid geometry by the same authors. Up to the year 1926 numerous revised editions of these texts appeared. Slaught was an author-editor of college texts on trigonometry and algebra in collaboration with E. J. Wilczynski, and he edited a group of five other college texts on mathematics by Lennes and his collaborators. These books have been an effective expression of Slaught's skill and influence as a teacher.

Slaught had a habit of getting himself into difficulties in some good cause, and then getting himself out again by steadfast interest and devotion. He had confidence in the values of visual education and in 1922, by his encouragement and financial support, made possible the first number of the periodical called *The Educational Screen*. He was president of its directorate and staff up to the time of his death, and the editor, Nelson L. Greene, says of him: "His was a rare spirit, a great heart, a rich personality, a genial soul. We of the magazine cannot hope to replace such a President. We shall seek only a successor. Whatever the value of *The Educational Screen* to the visual field during the past fifteen years, and whatever the service it may render in the future, must be considered largely as a gift from him who has gone, to a field that was not his own."

In 1914 Slaught was the leader of a group of members of the Hyde Park Baptist Church in establishing the South Chicago Neighborhood House for the benefit of workers in neighboring steel mills and their families. As usual he started with only the cordial support of his colleagues in the undertaking and his confidence in its value. His judgment of values was immediately justified, but the needs of the House continuously far outstripped its financial resources, as they seem likely to continue to do for a long time to come. As a result of his resourcefulness and steady interest through the years the House has now, however, many interested supporters and a building of its own with well-trained and successful workers. It has over 1000 neighborhood members, and an average monthly attendance of over 10,000 visitors and participants in its various activities.

One of Slaught's lighter interests was the game called roque, a development of croquet. Its name is in fact croquet without the first and last letters. Successful play requires a delicacy of touch similar to that of a billiard player. He and Mrs. Slaught had a sentimental interest in croquet dating from the days when their friendship was developing on the croquet ground of Peddie Institute. When they heard of the new game they went to see it played in New York and were fascinated by it. They had a court built at Peddie, and when they moved to Chicago Slaught interested others in the game and organized the Chicago Roque Club. He was what our students would call a "big shot" in the convention at which the American Roque League was formed, and in fact proposed the compromise which smoothed out seriously conflicting interests of eastern and western clubs and made the organization of the League possible. He was active president of the Chicago Roque Club for twenty years, and afterward honorary life president.

For his colleagues at Chicago the every day work which Slaught did for his Department and his University seemed by far the most important. He was for many years in charge of the entrance examinations of the University in mathematics, and was always influential in their administration. His courses ranged from freshman to beginning graduate grade and were uniformly popular. Successive generations of students found that he made mathematics clear to them and continued to elect his courses with enthusiasm up to the time of his retirement. He was in many situations the ablest representative of our Department in our relations with the rest of the University and our students. His house was frequently a meeting place for parties which broke down formality and promoted the friendliest of relations between the faculty and students in our departmental group. After his retirement he was much interested in the development of our collection of photographs of eminent mathematicians, now approaching one hundred in number, one of the best collections which I know. From 1903 to 1916 he was secretary of the Board of Recommendations of the University. He placed hundreds of our graduates, and employers seeking candidates had confidence in his judgment. Many of the policies which he formulated during those years have been followed steadily by the Board since his retirement as secretary. For twenty-five years before he gave up full time service in 1931 he was secretary of the Association of Doctors of Philosophy of the University, and he was known and beloved everywhere by our alumni.

Throughout his lifetime Slaught was a cheerfully sociable person, loving company and a good story, but always essentially serious in his interests. After he fell and broke his hip in the winter of 1935 he was confined for long months to his bed or room, ever hoping to be out and around again, but actually with strength gradually failing. Some of us at first sent him mystery stories and other light reading to while away the hours, but I doubt if he ever read or enjoyed them. His interest was essentially in his activities of the past and in the visitors who came from time to time to seek advice in carrying them on, or to report to him their consequences and progress. It is impossible that any of us who knew him personally can ever be oblivious to his influence, and future generations cannot escape it, though they may not always be conscious of it. I am hoping that the remarks made here this afternoon, recorded permanently in the annals of the Association, may serve to remind many generations of our successors of his fine enthusiasm and remarkably effective devotion to his science and his University.

HALLEY'S METHODS FOR SOLVING EQUATIONS

HARRY BATEMAN, California Institute of Technology

1. Halley's approximations. About 1694 Edmund Halley [1] became interested in some rules that were formulated by the French mathematician Thomas Fautet de Lagny in his book *Méthodes nouvelles et abrégées pour l'extraction et l'approximation des racines* which was published in Paris in 1692, a second edition appearing in 1697. Halley did not see the book himself but heard of the rules through a friend. Halley proceeded to test the rules and to extend them. He expressed the opinion that de Lagny's rule for finding $(a^5+b)^{1/5}$ is printed incorrectly in the first edition of his book. This rule, according to Halley's method of derivation, is that the root is given approximately by an expression which in modern notation is

$$\tfrac{1}{2}a + [(\tfrac{1}{4}a^4 + b/5a)^{1/2} - \tfrac{1}{4}a^2]^{1/2}. \tag{1}$$

The French book gives $\frac{1}{2}a^2$ in place of $\frac{1}{4}a^2$. Nielsen [2], on the other hand, quotes the approximate expression as

$$[(\tfrac{1}{4}a^4 + b/5a)^{1/2} - \tfrac{1}{4}a^2]^{1/2}.$$

These expressions may be incorrect but we must not infer that they are not correct simply because they differ from Halley's expression. There are many different ways of approximating to roots of equations, some of them better than others. The advantage of Halley's form of the rule is that it is particularly good when the first approximation is close to the root. This may be seen, for instance, by taking $b=0.8080401001$, when the exact value of the root is 2.01 and $a=2$. Even when $b=65.65625$ and the exact value of the root is 2.5 the formula gives a good approximation with $a=2$.

Halley expressed his admiration for the rule (1) in the words, "But the Rule for the Root of a pure Solid, or the fifth power, is of something a higher Enquiry, and does much more perfectly yet do the Business; for it does at least quintuple the given Figures in the Root, neither is the Calculus very large or operose."

Halley seemed indeed to prefer methods of approximation depending on the extraction of square roots to methods depending on the use of rational fractions because in the latter the denominators of the fractions may be very large and the processes of division very laborious while the extraction of the square root proceeds much more easily, as "manifold Experience has taught me." This remark should be of some interest to computers.

The rule of de Lagny that $(a^3+b)^{1/3}$ lies between*

$$a + \frac{ab}{3a^3+b} \quad \text{and} \quad (\tfrac{1}{4}a^2 + b/3a)^{1/2} + \tfrac{1}{2}a, \tag{2}$$

when a and b are positive and† $b \ll a^3$ was checked by Halley, and then ex-

* It should be remarked that the first expression differs from that quoted by Nielsen.

† The notation $b \ll a^3$ means "b is small in comparison with a^3."

tended first to the extraction of $(a^n+b)^{1/n}$ where n is a positive integer and b is either positive or negative but small in comparison with a^n. Halley's rational approximation, which, when $n=2$, is included in a more general one given by Jacob [3], is

$$A = \frac{2na^{n+1} + (n+1)ba}{2na^n + (n-1)b}, \tag{3}$$

while his irrational one is

$$\frac{n-2}{n-1}\,a + \frac{1}{n-1}\,[a^2 + 2ba^{2-n}(1 - 1/n)]^{1/2}. \tag{4}$$

The rational approximation has been rediscovered several times. When Dedekind gave his theory of irrational numbers [4] which involves the idea of a cut or division of the rational numbers into two classes he used an algorithm like that of Halley and de Lagny for the specification of a cut to define $\sqrt{2}$. The approximation

$$A = \frac{4a^3 + 3ab}{4a^2 + b} \tag{5}$$

was recommended by Weill [5] for the extraction of $(a^2+b)^{1/2}$, being regarded as superior to the familiar Newton-Raphson approximation [6].

Halley next extended his methods of approximation so that they might be used for any equation whether algebraic or transcendental. He was familiar with the binomial theorem but not with Taylor's formula for the expansion of an arbitrary function in a power series. His rules, however, are based on the supposition that such an expansion can be made. In modern notation these rules give the following approximations for the solution of $f(x)=0$:

$$A = a - \frac{2f(a)f'(a)}{2f'^2(a) - f(a)f''(a)}; \tag{6}$$

$$A = a - f'(a)/f''(a) + [f'^2(a) - 2f(a)f''(a)]^{1/2}/f''(a). \tag{7}$$

The first of these is given explicitly by Kobald [7] without any reference to Halley's work. He emphasizes the advantage of this particular type of approximation and uses it to solve $x^3-2x-5=0$. With $a=2.1$ he finds $x=2.0945515$, all the figures being correct.

It is likely that if Halley had applied his methods in a general way to transcendental equations he would have been led naturally to Taylor's theorem. It was indeed by the application of Halley's methods to Kepler's problem (an astronomical problem with which Halley was undoubtedly familiar) that Brook Taylor was led to the series which bears his name. In a letter [8] to John Machin written from Bissons near Canterbury on July 26, 1712 he says "While I was thinking of these things I fell into a general method of applying Dr.

Halley's Extraction of roots to all Problems, wherein the Abscissa is required, the Area being given which, for the service that it may be of calculations, (the only true use of all corrections) I cannot conceal." Earlier in the letter he says, "Whence (by Dr. Halley's method of extracting roots, which is in the *Transactions*, and at the end of Sir Isaac Newton's *Algebra*) · · · "

Halley clearly states that his approximations are to be used as a basis of a method of iteration, the computed value A being used in place of a in the next operation. The process of iteration was, of course, known from the writings of Gregory, Dary, Newton and Raphson; but Halley deserves the credit of having found formulas of approximation which usually make the convergence of the process of iteration extremely rapid when the first approximation is fairly good. The theory of the rational approximation of Halley and de Lagny is included in that of the Newton-Raphson approximation

$$A = a - F(a)/F'(a), \tag{8}$$

for the solution of $F(x)=0$, where $F(x)$ in the Halley formula is $f(x)/[f'(x)]^{1/2}$.

By generalizing this connection between the Newton and Halley methods of approximation Halley's rational approximation may be extended to systems of equations. The generalized Newtonian substitution (6) for the solution of

$$f(x, y) = 0, \qquad g(x, y) = 0 \tag{9}$$

is, according to Biermann [9], $a_1=A(a, b)$, $b_1=B(a, b)$,

$$A = a - (gf_2 - fg_2)/J, \qquad B = b + (fg_1 - gf_1)/J, \tag{10}$$

where

$$f=f(a, b),\ g=g(a, b),\ A=A(a, b),\ B=B(a, b),$$
$$f_1 = \frac{\partial f}{\partial a}, \quad f_2 = \frac{\partial f}{\partial b}, \quad g_1 = \frac{\partial g}{\partial a}, \quad g_2 = \frac{\partial g}{\partial b}, \quad J = f_1g_2 - f_2g_1 = \frac{\partial(f, g)}{\partial(a, b)}, \tag{11}$$

and is just one of innumerable substitutions of type $a_{n+1}=A(a_n, b_n)$, $b_{n+1}=B(a_n, b_n)$ which, when $a_0=a$, $b_0=b$, make a_m, b_m converge to a pair of solutions of (9) as $m\to\infty$. If now we put

$$F(a, b) = J^{-1/2}f(a, b), \qquad G(a, b) = J^{-1/2}g(a, b), \tag{12}$$

and apply the generalized Newtonian substitution to $F(x, y)$ and $G(x, y)$, we obtain a generalized Halley approximation which can be iterated just like the generalized Newtonian approximation. If, in particular

$$f(x, y) = p(x) - y = 0, \qquad g(x, y) = q(y) = 0, \qquad b = p(a), \tag{13}$$

the generalized Newtonian and Halley approximations for the solution of (13) are equivalent to the ordinary Newtonian and Halley approximations for the solution of the single equation

$$q[p(x)] = 0. \tag{14}$$

To find the complex roots of $f(z)=0$ we may write

$$f(z) = f(x, y) + ig(x, y), \tag{15}$$

where $f(x, y)$ and $g(x, y)$ are real when x and y are real, and apply the generalized Newtonian or Halley approximations to solve the two simultaneous equations

$$f(x, y) = 0, \qquad g(x, y) = 0. \tag{16}$$

In the case of an equation with real coefficients Craig [10] uses a method in which the roots of $z^2+mz+n=0$ are first approximations to two conjugate complex roots and second approximations are found by means of the equations $m_1=M(m, n)$, $n_1=N(m, n)$, where

$$M = m + [2acn - (ad + bc)m + 2bd]/(c^2n - cdm + d^2),$$
$$N = n + [acmn - bc(m - 2n) - 2adn + bdm]/(c^2n - cdm + d^2),$$
$$f(z) = (z^2 + mz + n)g(z) + az + b,$$
$$f'(z) = (z^2 + mz + n)h(z) + cz + d.$$

There is still another method in which the equation $g(x, y)=0$ is solved by means of uniform functions $x=x(s)$, $y=y(s)$ where s is a real parameter; and the equation $f[x(s), y(s)]=0$ is solved approximately by Halley's method and the approximate value of s used to find x and y. For a rough preliminary location of the roots of $f(z)=0$ a graphical method based on graphs of the curves (16) may be used. This process is slow but sure.

There are of course many algebraic methods as may be seen from Van Vleck's report [11] which lists 150 papers published since 1900, Mignosi's report [12] on the method of Sturm which described the chief advances from the time of l'abbé de Gua (1741), and Lecat's bibliography [13] of 75 papers.

2. The convergence of the iterative process. The convergence of the sequence obtained by iteration of (8) has been studied by Faber [14] who supposes that $F(x)$ is an analytic function such that for a positive number c less than one the inequality

$$|F(z)F''(z)/F'^2(z)| < c \tag{17}$$

is satisfied for all points z in the complex plane that lie within a circle round a with radius R defined by the equation

$$R(1 - c) = |F(a)/F'(a)|. \tag{18}$$

Faber proves that under these conditions the sequence converges to a root of $F(x)=0$ which lies within the circle. This theorem is suitable for the case of complex roots and is valid not only when $F(z)$ is regular in the circle but also when it has algebraic branch points within the circle. Faber finds, however, that if $F(z)$ is of the form $F(z)=(z-x)^kG(z)$ where $G(x)\neq 0$ and $0<2k<1$ the Newtonian sequence does not converge to x. It should be noticed on the other

hand that if $f(z)=(z-x)^k g(z)$, where $g(x)\neq 0$ and if $F(z)$ is defined by the equation $F(z)=f(z)[f'(z)]^{-1/2}$ then $F(z)$ is of the form

$$F(z) = (z-x)^{1/2+(1/2)k}h(z),$$

where $h(x)\neq 0$. Consequently if $k>0$ we have $\frac{1}{2}+\frac{1}{2}k>\frac{1}{2}$, and the Newtonian sequence for the solution of $F(x)=0$ will converge by Faber's theorem. This means that the Halley sequence for the solution of $f(x)=0$ will converge as soon as $k>0$ and the initial point a is sufficiently close to the root x.

For the approximation to a real root x of an equation $F(x)=0$ Faber finds it sufficient to assume the existence of $F'(z)$ and $F''(z)$ in place of the analyticity of $F(z)$. By taking $c=\frac{1}{2}$ in his theorem he obtains the following useful rule: If, for all values of z between a and $a-2F(a)/F'(a)$ we have

$$|F(z)F''(z)/F'^2(z)| < \tfrac{1}{2}, \tag{19}$$

then the Newtonian sequence which starts with a will converge to a root of $F(x)=0$ lying between a and $a-2F(a)/F'(a)$.

Putting $F(z)=f(z)[f'(z)]^{-1/2}$, we find that if for all values of z between a and $a-2f(a)f''(a)/[f'^2(a)-\frac{1}{2}f(a)f''(a)]$ we have $f'(z)>0$ and

$$\frac{3[f(z)f''(z)]^2 - 2f'(z)f'''(z)[f(z)]^2}{4[f'(z)]^4 - 4f(z)f''(z)[f'(z)]^2 + [f(z)f''(z)]^2} < \frac{1}{2}, \tag{20}$$

then the Halley sequence starting with a converges to a root of $f(x)=0$ lying between the limits assigned for z.

If x is a regular point or an algebraic branch point of an analytic function $F(z)$ then if $F(x)/F'(x)=0$ and $F(x)F''(x)/F'^2(x)<1$ there is a certain neighbourhood S of the place x such that Newton's sequence converges to x as soon as a is chosen arbitrarily within S, but if $F(x)F'(x)/F'^2(x)>1$ Newton's sequence can never converge to x. Replacing $F(z)$ by $f(z)[f'(z)]^{-1/2}$ as before, we obtain the corresponding conditions for the convergence of Halley's sequence by replacing z in (20) by x and the right-hand side of (20) by 1.

Faber's rules are easier to apply than the rule of Fouret which states that if $F(a)$ and $F''(a)$ have the same sign and $F''(z)$ does not change sign in (a, x) then the Newtonian sequence converges to x. For this and other developments see G. Fouret, *Nouvelle Annales de Mathématiques* ser. 3, vol. 9, 1890, pp. 567–585. A. Pellet [*Association Française*, vol. 40, 1912, pp. 5–7] writes $u=F(a)/F'(a)$ and denotes the greatest value of $|F'(a+2\theta u)|$, as θ runs from -1 to $+1$, by M. He then shows that if $|F'(a)|^2>2M|F(a)|$, the equation $F(x)=0$ has a root between $a-2u$ and $a+2u$ which may be found by Newton's sequence. The convergence is compared with that of the Newtonian sequence for the smallest root of $Mx^2-2x|F'(a)|+|F(a)|=0$.

The convergence of the Newtonian sequence for solving $F(z)=0$ has been studied by many writers. For some early work, reference may be made to Cauchy's works and to Serret's *Algebra*. The work of Koenigs [15], Isenkrahe

[16], and Schmidt [17] made some points clear, especially for the case of complex roots. The early work of Schroeder [18] and Laguerre [19] has also proved inspiring. Laguerre showed that when $F(z)$ is of degree n the error of the approximation A to a root x is given by the inequality

$$|x - A| < |(n-1)F(a)/F'(a)|.$$

Estimates of the error in Newton's method of approximation have been made also by Huber [20], Romanowsky [21], and Lagrange [22].

Gravé [23] considers the more general approximation

$$A = a - F(a)/QF'(a),$$

where $Q>1$ and $Q>|F(a)F''(a)/F'^2(a)|$. He finds that when these inequalities are satisfied the process of iteration based on the substitution (22) is convergent and leads to a root of $F(x)=0$. This theorem may also be used in the theory of Halley's rational approximation. Further studies of the convergence may be based on the extensive researches on the iteration of rational substitutions that have been made by Fatou, Julia, Montel, Ritt, and others in papers that are too numerous to be listed here.

A discussion of the convergence and an estimate of the error has been made recently for the Newtonian method of solving a system of equations [24]. Ostrowski, to whom the analysis is due, gives also references to early work on the Newtonian method for a single equation and refers to a forthcoming paper of his own on this subject.

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A GENERALIZATION OF THE CIRCLES OF APOLLONIUS

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1. Introduction. If we have a triangle $A_1A_2A_3$, and if the internal and external bisectors of the angles A_i intersect the opposite sides in the points B_i' and B_i'' respectively ($i=1, 2, 3$), the circles on the segments $B_i'B_i''$ as diameters are called the circles of Apollonius. Many properties of these circles are well known [**1**, p. 92; **2**, p. 1259; **3**, p. 148; **4**, p. 294].*

Moreover, since the bisectors of the angles of a triangle divide the opposite sides proportional to the adjacent sides, the product of the three ratios is unity. This suggests the generalization of dividing the sides of a triangle internally and externally in ratios such that the product of the three ratios is unity, and then determining the properties of the circles described on the segments thus

* See references at the end of the paper.

formed as diameters. It is our purpose to discuss these circles, which we regard as generalizations of the circles of Apollonius.

2. Equations of the circles. The equations of such circles may be found in the following manner. Let the side A_iA_j be divided harmonically in the ratio $m_j:m_i$ (m_i and m_j are real numbers) in the points B_k' and B_k''. Let the circumcircle of the triangle be taken as the unit circle. Then the vertices of the triangle may be represented by the complex coördinates

$$z_i = t_i, \qquad |t_i| = 1, \qquad (i = 1, 2, 3).$$

Let $\bar{z}$ be the conjugate of z.

With this notation it is easy to show that the center O_k of the circle on $B_k'B_k''$ as diameter is given by

$$z_k = \frac{t_im_i^2 - t_jm_j^2}{m_i^2 - m_j^2}, \quad (i \neq j \neq k \neq i), \ (i, j, k = 1, 2, 3). \tag{1}$$

Also the vector radius O_kB_k'' is represented by

$$r_k = \frac{m_im_j(t_i - t_j)}{m_i^2 - m_j^2}. \tag{2}$$

Now the equation of a circle with center O_i and radius r_i is given by [5, p. 159]

$$(z - z_i)(\bar{z} - \bar{z}_i) = r_i\bar{r}_i. \tag{3}$$

Hence the equations of the three circles are given by

$$z\bar{z}t_it_j(m_i^2 - m_j^2) - z(t_jm_i^2 - t_im_j^2) - \bar{z}t_it_j(t_im_i^2 - t_jm_j^2) + t_it_j(m_i^2 - m_j^2) = 0. \tag{4}$$

3. Properties of the circles. The circles (4) are coaxal and the equation of the radical axis is*

$$z\sum t_it_jm_k^2(m_i^2 - m_j^2) + \bar{z}s_3\sum t_km_k^2(m_i^2 - m_j^2) = 0, \tag{5}$$

where the sum is formed for $i=1, 2, 3$, and where $i\neq j\neq k\neq i$. Equation (5) shows that the radical axis always passes through the circumcenter of the triangle.

The equation of the line of centers of the system (4) is

$$z\sum t_it_jm_k^2(m_i^2 - m_j^2) - \bar{z}s_3\sum t_km_k^2(m_i^2 - m_j^2) + \sum t_im_j^2m_k^2(t_j^2 - t_k^2) = 0. \tag{6}$$

If in (6) we set $m_2^2 = hm_1^2$, $m_3^2 = h^2m_1^2$, we obtain

$$\begin{aligned} h^2t_1(t_2 - t_3)(z + \bar{z}t_2t_3 - t_2 - t_3) + ht_2(t_3 - t_1)(z + \bar{z}t_3t_1 - t_3 - t_1) \\ + t_3(t_1 - t_2)(z + \bar{z}t_1t_2 - t_1 - t_2) = 0. \end{aligned} \tag{7}$$

* In what follows s_i is a symbol for the elementary functions of the t_i, that is, the t's are the roots of the equation $t^3 - s_1t^2 + s_2t - s_3 = 0$.

Equation (7) represents a one parameter family of lines with h as parameter. If we take the partial derivative of (7) with respect to h we obtain

$$(8)\quad 2ht_1(t_2 - t_3)(z + \bar{z}t_2t_3 - t_2 - t_3) + t_2(t_3 - t_1)(z + \bar{z}t_3t_1 - t_3 - t_1) = 0.$$

If we eliminate h from equations (7) and (8) we obtain

$$(9)\quad \begin{aligned} &t_2^2(t_3 - t_1)^2(z + \bar{z}t_1t_3 - t_1 - t_3)^2 \\ &\qquad = 4t_1t_3(t_1 - t_2)(t_2 - t_3)(z + \bar{z}t_1t_2 - t_1 - t_2)(z + \bar{z}t_2t_3 - t_2 - t_3). \end{aligned}$$

Equation (9) represents a parabola which is the envelope of the family of lines represented by equation (7). It is obvious that the parabola (9) passes through the points A_1 and A_3. Moreover the clinant of the parabola (9) at A_1 is $dz/d\bar{z} = -t_1t_2$. Hence it is tangent to the line A_1A_2 at A_1. It is also tangent to the line A_2A_3 at A_3. It is therefore the Artzt parabola of $A_1A_2A_3$ which passes through A_1 and A_3. The other Artzt parabolas may be found in a similar fashion [1, p. 88].

If now we divide the side A_iA_j harmonically in the ratio $m_i : m_j$ and construct circles on the segments thus obtained as diameters we find the equations for them to be

$$(10)\quad t_it_jz\bar{z}(m_i^2 - m_j^2) - \bar{z}t_it_j(t_jm_i^2 - t_im_j^2) - z(t_im_i^2 - t_jm_j^2) + t_it_j(m_i^2 - m_j^2) = 0.$$

The line of centers of the circles (10) is represented by

$$(11)\quad z\sum m_i^2t_i(t_j - t_k) + \bar{z}s_3\sum m_i^2(t_j - t_k) + \sum m_i^2t_i(t_j^2 - t_k^2) = 0,$$

where the sum is formed for $i = 1, 2, 3$, and where, as before, we have $i \neq j \neq k \neq i$.

The radical axis will be represented by

$$(12)\quad z\sum m_i^2t_i(t_j - t_k) - \bar{z}s_3\sum m_i^2(t_j - t_k) = 0.$$

If in (11) we make the substitution $m_2^2 = hm_1^2$, $m_3^2 = h^2m_1^2$ we obtain

$$(13)\quad \begin{aligned} &h^2t_1(t_2 - t_3)(z + \bar{z}t_2t_3 + t_2 + t_3) + ht_2(t_3 - t_1)(z + \bar{z}t_3t_1 + t_3 + t_1) \\ &\qquad + t_3(t_1 - t_2)(z + \bar{z}t_1t_2 + t_1 + t_2) = 0. \end{aligned}$$

Equation (13) represents a one parameter family of lines and it may be shown that its envelope is an Artzt parabola of the triangle whose vertices are the reflections in the circumcenter of the vertices A_1, A_2, A_3.

4. Special cases. Some interesting special cases of the above theory arise. Let

$$(14)\quad m_i^2 = \frac{(t_j - t_k)^2}{t_jt_k}.$$

The values of m_i in (14) are such that a side of the triangle is divided in the ratio of the lengths of the adjacent sides. If we substitute the values of m_i from (14) in (4), (5), and (6) we obtain

$$z\bar{z}(t_it_j - t_k^2) - \bar{z}(3s_3 - t_ks_2) - z(s_1 - 3t_k) + t_it_j - t_k^2 = 0, \tag{15}$$

$$z(s_1^2 - 3s_2) - \bar{z}(s_2^2 - 3s_1s_3) = 0, \tag{16}$$

$$z(s_1^2 - 3s_2) + \bar{z}(s_2^2 - 3s_1s_3) = s_1s_2 - 9s_3, \tag{17}$$

respectively. Equation (15) defines the circles of Apollonius. Equation (16) defines the Hessian line and (17) the Lemoine axis of the triangle.

If now we substitute the values of m_i^2 from (14) in (10) we obtain

$$t_it_jz\bar{z}(t_it_j - t_k^2) - \bar{z}t_i^2t_j^2(s_1 - 3t_k) - z(3s_3 - s_2t_k) + t_it_j(t_it_j - t_k^2) = 0. \tag{18}$$

The radical axis and line of centers of the set of circles (18) are given by

$$zs_2 - \bar{z}s_1s_3 = 0, \tag{19}$$

$$zs_2 + \bar{z}s_1s_3 = s_1s_3 - 3s_3, \tag{20}$$

respectively. Equation (19) is the equation of the Euler line of the triangle. I have not identified (20) with any of the known lines of the triangle.

If we wish to divide the sides of the triangle in the ratio of the distances of the circumcenter from the other two sides we have

$$m_i^2 = \frac{(t_j + t_k)^2}{t_jt_k}. \tag{21}$$

If we substitute the values of m_i^2 from (21) in (4) we obtain

$$z\bar{z}(t_it_j - t_k^2) + \bar{z}t_k(s_2 + t_it_j) - z(s_1 + t_k) + t_it_j - t_k^2 = 0. \tag{22}$$

The radical axis and line of centers of the system (22) are given by

$$z(s_1^2 + s_2) - \bar{z}(s_2^2 + s_1s_3) = 0, \tag{23}$$

$$z(s_1^2 + s_2) + \bar{z}(s_2^2 + s_1s_3) = s_1s_2 - s_3, \tag{24}$$

respectively.

The lines (23) and (24) intersect in the point

$$z = \frac{s_1s_2 - s_3}{2(s_1^2 + s_2)}. \tag{25}$$

This point is the inverse of the Nagel point of the tangential triangle in the circumcircle. The inverse of the line (24) is a circle concentric with the Spieker circle of the tangential triangle [**4**, p. 226; **6**, p. 87].

If we substitute the values of m_i from (21) in (10) we obtain

$$z\bar{z}t_it_j(t_k^2 - t_it_j) + \bar{z}t_i^2t_j^2(s_1 + t_k) - zt_k(s_2 + t_it_j) + t_it_j(t_k^2 - t_it_j) = 0. \tag{26}$$

The radical axis of the set (26) is the Euler line of the triangle and the line of centers is given by the equation

$$zs_2 + \bar{z}s_1s_3 = s_1s_2 + s_3. \tag{27}$$

If we make the substitution

$$m_i^2 = \frac{t_j^2 + t_k^2}{t_j t_k} \tag{28}$$

in (4) we obtain

$$z\bar{z}(t_i t_j - t_k^2) + \bar{z}t_k^2(t_i + t_j) - z(t_i + t_j) + t_i t_j - t_k^2 = 0. \tag{29}$$

The radical axis and line of centers of the circles (29) are given by

$$z(s_1^2 - s_2) - \bar{z}(s_2^2 - s_1 s_3) = 0, \tag{30}$$

$$z(s_1^2 - s_2) + \bar{z}(s_2^2 - s_1 s_3) = s_1 s_2 - s_3. \tag{31}$$

The lines (30) and (31) intersect in the point

$$z = \frac{s_1 s_2 - s_3}{2(s_1^2 - s_2)}, \tag{32}$$

which is the inverse of the orthocenter of the tangential triangle in the circumcircle [6, p. 87].

If we make the substitution (28) in (10) we obtain

$$z\bar{z}t_i t_j(t_i t_j - t_k^2) + zt_k^2(t_i + t_j) - \bar{z}t_i^2 t_j^2(t_i + t_j) + t_i t_j(t_i t_j - t_k^2) = 0. \tag{33}$$

The radical axis and line of centers of the circles (33) are given by

$$zs_2 - \bar{z}s_1 s_3 = 0, \tag{34}$$

$$zs_2 + \bar{z}s_1 s_3 = s_1 s_2 - s_3. \tag{35}$$

The lines (34) and (35) intersect in the point

$$z = \frac{s_1 s_2 - s_3}{2s_2}, \tag{36}$$

which is the inverse of the circumcenter of the tangential triangle in the circumcircle [6, p. 86].

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AN APPLICATION OF NUMBER THEORY TO GEAR RATIOS

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1. Statement of problem. The problem which is to be discussed arose in connection with the design of a machine for winding induction coils on toroidal cores. This machine demanded a gear train of the type illustrated in Figures 1A and 1B in which gears 2 and 3 are rigidly connected to the common shaft so that they rotate with the same angular velocity, while gears 1 and 4 also have a common axis but are free to rotate about it with different velocities. Obviously, the arrangement is such that when gear 1 is driven at a given angular velocity the motions of the other gears—and in particular that of gear 4—are completely determined. Indeed, if the gears have n_1, n_2, n_3, and n_4 teeth respectively the ratio of the speed of the fourth (driven) gear to that of the first (which drives it) is n_1n_3/n_2n_4.

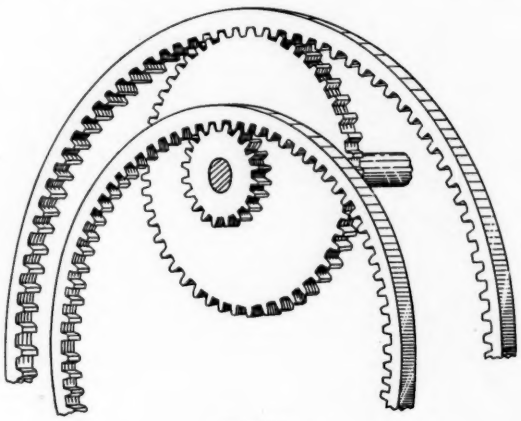


FIG. 1A

Since the n's are necessarily integers this speed ratio is a rational fraction which we may write

$$\frac{n_1n_3}{n_2n_4} = \frac{s}{t}, \tag{1}$$

where, for the sake of simplicity in what is to follow, we may assume that s/t has been reduced to its lowest terms. We shall also assume that $s>t$. This restriction involves no essential loss in generality since the contrary case ($t>s$) can be accommodated by a simple interchange of subscripts.

In designing such a machine, it is the speed ratio s/t which is given. The designer must then choose a set of n's which will satisfy the relation (1) and the further condition that the gears shall be of such diameters as to mesh properly

* The author is indebted to his colleague Dr. T. C. Fry for assistance in the exposition of this paper.

with one another. This latter condition clearly requires that the difference in the radii* of 1 and 2 shall equal the difference in the radii of 4 and 3. However, if the differences in the radii are equal, the differences in the circumferences must also be. But the circumference of any gear is obviously equal to the number of teeth multiplied by the pitch (that is, the distance between the centers of the teeth). Furthermore, in order that gears 1 and 2 may mesh properly they must both have the same pitch, which we may denote by p_{12}, and similarly

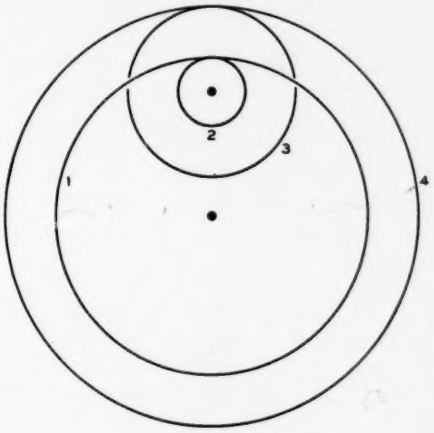


FIG. 1B

gears 3 and 4 must have a common pitch p_{34}. Hence the second condition which the n's must satisfy is

$$p_{12}(n_1 - n_2) = p_{34}(n_4 - n_3). \tag{2}$$

It may be noted incidentally that this equation requires p_{12} and p_{34} to be commensurable. Hence there is some unit of length in terms of which they would turn out to be relatively prime integers. We assume that all measurements are made in terms of this unit; hence that p_{12} and p_{34}, like s and t, are relatively prime numbers.

We may add at this point another observation which will be of service to us later on. Mathematically it grows out of the observation that if p_{12}, n_1, and n_2 satisfy (1) and (2), then $p_{12}' = p_{12}/\alpha$, $n_1' = \alpha n_1$, and $n_2' = \alpha n_2$ also satisfy these equations.

The converse of this statement is, however, the more important one; namely, that if (1) and (2) are satisfied by the numbers p_{12}, p_{34}, n_1, n_2, n_3, n_4 and if n_1

* Theoretically, each gear could be replaced by a circle of such radius that the four circles would roll on one another without slipping. These are known as the "pitch circles" of the gears. Whenever we speak of radii, circumferences, or similar dimensions of the gears or of the distance between teeth ("pitch"), it is always assumed that the measurements are made on these pitch circles.

and n_2 have a common factor α and/or n_2 and n_3 a common factor β, then $p_{12}' = \alpha p_{12}$, $p_{23}' = \beta p_{23}$, $n_1' = n_1/\alpha$, $n_2' = n_2/\alpha$, $n_3' = n_3/\beta$, $n_4' = n_4/\beta$ also satisfy these equations. Mechanically this merely corresponds to shifting from many *fine* teeth to fewer *coarse* ones on some or all of the gears without changing their diameters or relative speeds of rotation. When this process has been carried as far as it can be, we shall say that we have a "prime solution" of our problem. If the pitch ratio p_{34}/p_{12} is arbitrary, this means that n_1 and n_2 must be relatively prime, and also n_3 and n_4. If, on the other hand, the pitch ratio is specified in advance so that the condition $p_{34}'/p_{12}' = p_{34}/p_{12}$ must be satisfied, it will obviously be necessary that $\alpha = \beta$. Under these circumstances then, the solution will be prime provided n_1, n_2, n_3, n_4 have no common factor, though the pairs n_1, n_2 and n_3, n_4 may have.

The solution given here is applicable whether the pitch ratio is specified or not. In the latter case, however, there is a simpler method of solution, based on the fact that if n_1 and n_2 are any factors of ρs and ρt respectively, where ρ is any integer, then the equations

$$n_3 = \frac{\rho s}{n_1}, \qquad n_4 = \frac{\rho t}{n_2}, \qquad \frac{p_{12}}{p_{34}} = \frac{n_4 - n_3}{n_1 - n_2}, \tag{3}$$

define the remaining integers.

2. General solution. Since p_{12} and p_{34} are relatively prime by hypothesis, we obviously may write

$$p_{12}(n_1 - n_2) = p_{34}(n_4 - n_3) = p_{12}p_{34}w, \tag{4}$$

or

$$n_1 = n_2 + p_{34}w, \qquad n_4 = n_3 + p_{12}w, \tag{5}$$

in which case w is an integer whenever (2) is satisfied, and conversely (2) is automatically satisfied by any integer w. Then (1), which is now the only condition that remains, becomes

$$rn_2n_3 + (sp_{12}n_2 - tp_{34}n_3)w = 0, \tag{6}$$

where for simplicity we have written $s - t = r$.

If, now, we write

$$x = rsp_{12}n_2, \qquad y = rtp_{34}n_3, \qquad z = stp_{12}p_{34}w \tag{7}$$

in (6) it reduces to the very simple form

$$xy = (y - x)z. \tag{8}$$

In other words, we have the following:

THEOREM 1. *To every integral solution of* (6) *there corresponds a solution of* (8) *in which x is divisible by rsp_{12}, y by rtp_{34}, and z by $stp_{12}p_{34}$.*

Now (8) is easily solved, for if we denote the greatest common divisor of x and y by a and write

$$x = a\xi, \qquad y = a\eta, \tag{9}$$

so that ξ and η are relatively prime, we find at once from (8) that both ξ and η must be factors of z. Hence, $z = b\xi\eta$. Moreover, by (8), $a = b(\eta - \xi)$.

That is, *every* solution of (8) is of the form

$$x = b\xi(\eta - \xi), \qquad y = b\eta(\eta - \xi), \qquad z = b\xi\eta, \tag{10}$$

where b, ξ, and η are integral, and ξ, η relatively prime. Conversely, whenever b, ξ, and η are integers (10) is an integral solution of (8).

Finally, by (5) and (7) every solution of (6) must be of the form

$$\begin{aligned} n_1 &= \frac{b\xi(s\eta - t\xi)}{rstp_{12}}, & n_2 &= \frac{b\xi(\eta - \xi)}{rsp_{12}}, \\ n_3 &= \frac{b\eta(\eta - \xi)}{rtp_{34}}, & n_4 &= \frac{b\eta(s\eta - t\xi)}{rstp_{34}}. \\ w &= \frac{b\xi\eta}{stp_{12}p_{34}}. \end{aligned} \tag{11}$$

The converse of this statement is not generally true, since the n's are not necessarily integers. However, we readily see that for any pair of values ξ, η there is some b which will make all four n's integral. For example, $b = rstp_{12}p_{34}$ certainly does, but the result of this choice will not necessarily be a prime solution. In order to make it prime, however, we need only divide the n_1 and n_2 thus obtained by their greatest common divisor; and similarly for n_3 and n_4. In other words, we have the following:

THEOREM 2. *To obtain all relatively prime solutions of* (6) *substitute all relatively prime pairs of integers in the equations*

$$\begin{aligned} n_1 &= p_{34}\xi(s\eta - t\xi)/d_1, & n_2 &= tp_{34}\xi(\eta - \xi)/d_1, \\ n_3 &= sp_{12}\eta(\eta - \xi)/d_2, & n_4 &= p_{12}\eta(s\eta - t\xi)/d_2. \end{aligned} \tag{12}$$

If the pitch ratio is arbitrary, choose for d_1 and d_2 the greatest common divisors of the first and second pairs of numerators respectively. If the pitch ratio is given, set both d_1 and d_2 equal to the greatest common divisor of the four numerators.

3. Limitations as to size: pitch ratio specified. So far we have discussed only the question of deriving *all possible* prime solutions of our fundamental equations. This is not quite the problem with which the engineer is usually confronted, however. For one thing, he will not in general be interested in knowing that he has all possible solutions provided he has been able to get one which is technically satisfactory. On the other hand, he might frequently be interested

in knowing how far he should carry his process of solution in order to be sure that none of these "technically satisfactory" solutions had been overlooked. In other words, he would be interested in establishing limits on ξ and η beyond which his search need not be conducted.

To treat questions of this sort in a realistic way is rather difficult, largely because the conditions of "technical convenience" are rather elastic. It may be interesting, however, to choose a condition of a rather realistic sort and see how it can be dealt with mathematically.

We take as this condition the requirement that no gear shall be allowed to have more than N teeth. In the present section we shall apply this limitation to a case in which the pitch ratio is preassigned. In Section 4 we shall deal with the somewhat more usual situation in which it is arbitrary.

We may begin with certain limitations which are inherent in the nature of the physical problem and in the form of equations (12). One such condition obviously is that all the n's must be positive.* Another is that a gear with no teeth is a physical absurdity. A third observation is that replacing ξ and η by their negatives in (12) leaves the set of n's unchanged. Taken together, these observations show that it will be sufficient to consider positive values of ξ, η only, and that when so restricted η must be greater than ξ. That is,†

$$1 \leqq \xi < \eta. \tag{13}$$

In addition to these general restrictions the conditions of the present section require that

$$\begin{aligned} p_{34}\xi(s\eta - t\xi) &\leqq Nd, \\ p_{12}\eta(s\eta - t\xi) &\leqq Nd, \end{aligned} \tag{14}$$

d being the appropriate value of $d_1 = d_2$ when the pitch ratio is specified.

A formula for this denominator d can easily be obtained. An elementary investigation shows that whenever ξ is divisible by a factor of sp_{12} all four n's are also divisible by this factor. The same is true when η is divisible by a factor of tp_{34}, or when $\eta - \xi$ is divisible by a factor of $r = s - t$. Moreover, it is not difficult to show that unless one or the other of these conditions is satisfied $d = 1$. If, then, for each choice of ξ and η we denote by

$$\begin{aligned} &a, \text{ the greatest common divisor of } \xi \text{ and } sp_{12}, \\ &b, \text{ the greatest common divisor of } \eta \text{ and } tp_{34}, \\ &c, \text{ the greatest common divisor of } \eta - \xi \text{ and } r, \end{aligned} \tag{15}$$

* It is not true that negative n's are physically unimportant. On the contrary, they have a rather interesting mechanical interpretation as we shall see in Section 5. They do not, however, lead to structures of the form pictured in Figure 1B, and are therefore of no consequence when we are attempting to design such a device.

† The other condition $s\eta > t\xi$, which is apparent in (12), is automatically satisfied by (13) since $s > t$.

then the common denominator in (12) will be

$$d = abc. \tag{16}$$

How the conditions (13), (14), (15), and (16) may be applied in a particular problem can best be illustrated by an example. Suppose, therefore, that it is desired to design a set of gears which will give a speed ratio $s/t = 7/3$ and which will have a pitch ratio $p_{12}/p_{34} = 5/3$. In this case we readily see from (15) that a may have any of the values 1, 5, 7, 35; b any of the values 1, 3, 9; and c any of the values 1, 2, 4. Hence there are 36 possible values for d. This means that the inequalities (14) may take any one of 36 different forms, each of which must be treated as a special case. We will discuss two of these 36 cases: namely, $d = 1 \cdot 1 \cdot 1$ and $d = 7 \cdot 9 \cdot 2$.

In the first case (13) and (14) reduce to

$$1 \leqq \xi < \eta, \qquad 5\eta(7\eta - 3\xi) \leqq 50, \tag{17}$$

the first inequality of (14) being, in this case, redundant since $p_{34}\xi$ is necessarily

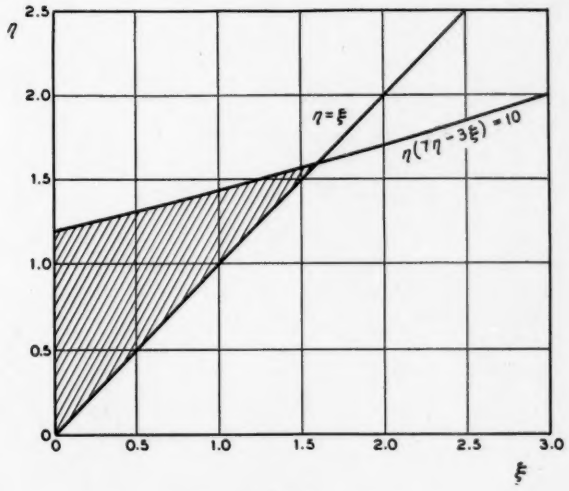


FIG. 2

less than $p_{12}\eta$. Interpreted diagrammatically, however, these inequalities show that all admissible values of ξ, η must be represented by points above (not on) the line $\eta = \xi$, and on or below the hyperbola $\eta(7\eta - 3\xi) = 10$; that is, within the shaded region shown in Figure 2. But this region contains no integral points for which $\xi \geqq 1$; hence there are no possible solutions with $d = 1$.

For the case $d = 7 \cdot 9 \cdot 2$ we must satisfy the inequality

$$1 \leqq \xi < \eta, \qquad 5\eta(7\eta - 3\xi) \leqq 6300. \tag{18}$$

Furthermore, from (15) it is obvious that ξ must be a multiple of 7 (but not of 5, since otherwise a would be 35 instead of 7); η must be a multiple of 9; and $\eta-\xi$ must be even (but not a multiple of 4, since otherwise c would be 4). And as always ξ and η must be relatively prime.

In finding admissible values of ξ, η we can again make use of Figure 2 provided we interpret the scales to represent $\xi/\sqrt{126}$ and $\eta/\sqrt{126}$. In this case values of ξ which are multiples of 7 and values of η which are multiples of 9 are represented by the dotted lines shown in Figure 3 and are the only ones which intersect within the admissible region. As these are relatively prime and their difference is a multiple of 2, but not of 4, they constitute an admissible pair. Substitution in (12) leads to $n_1=7$, $n_2=1$, $n_3=5$, $n_4=15$, which is a satisfactory solution mathematically. Mechanically it is not so good, for a gear of a single tooth is impossible, and even the non-prime solution obtained by multiplying by 3 (the largest integer for which $n_4<50$) is also mechanically undesirable.

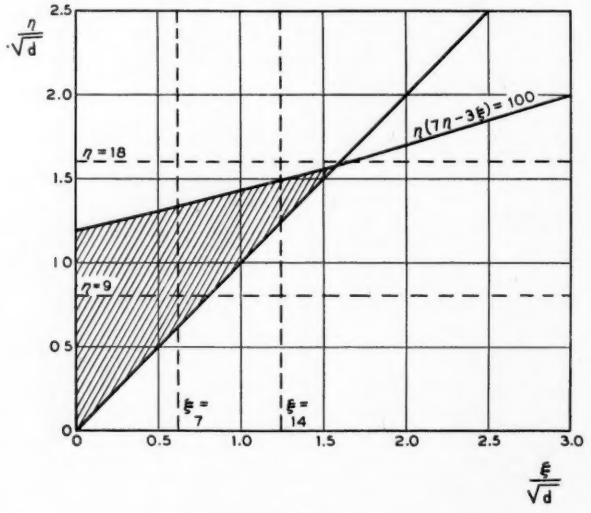


FIG. 3

The remaining of the 36 cases are just as easy to deal with, and in the end it is found that the only combinations of ξ and η which can give prime solutions with no n greater than 50 are the three shown in the first two columns of Table 1. They lead to the solutions displayed in the remaining columns of the table.

4. Limitations as to size: pitch ratio arbitrary. When the pitch ratio is arbitrary the general equations (12) are capable of considerable simplification since the first two obviously have the common factor $p_{34}\xi$ which may be re-

moved, and the last two have the common factor $p_{12}\eta$. Instead of (12) we may therefore write

$$(19)\qquad \begin{aligned} n_1 &= (s\eta - t\xi)/d_1, & n_2 &= t(\eta - \xi)/d_1, \\ n_3 &= s(\eta - \xi)/d_2, & n_4 &= (s\eta - t\xi)/d_2. \end{aligned}$$

The wording of Theorem 2 then applies to these equations without change. In order to solve these equations when the magnitude of the n's is limited, we must again know something about the divisors d_1 and d_2. Again, however, an elementary investigation shows that

$$(20)\qquad d_1 = ac, \qquad d_2 = bc,$$

where

$$(21)\qquad \begin{aligned} &a \text{ is the greatest common divisor of } t \text{ and } \eta, \\ &b \text{ is the greatest common divisor of } s \text{ and } \xi, \\ &c \text{ is the greatest common divisor of } r \text{ and } \eta - \xi. \end{aligned}$$

We shall illustrate the method of solution in this case by means of the same example as before; that is, we shall assume that a speed ratio 7/3 is demanded and that none of the gears may have more than 50 teeth; p_{12} and p_{34} are now arbitrary. As before, the inequality (13) must be satisfied, but instead of (14) we now have

$$(22)\qquad \begin{aligned} s\eta - t\xi &\leqq d_1 N, \\ s\eta - t\xi &\leqq d_2 N. \end{aligned}$$

We also find from (21) that a may be either 1 or 3; that b may be 1 or 7; and that c may be 1, 2, or 4. There are thus 12 cases which must be considered. Again we shall deal with only two; namely those for which a, b, c are 1, 1, 1 and 3, 1, 2.

For the first of these, (22) reduces to

$$(23)\qquad 7\eta - 3\xi \leqq 50,$$

and obviously requires that the point ξ, η lie on or below the line $7\eta - 3\xi = 50$ (Figure 4). As in Section 3, (13) requires that ξ, η lie above (not on) the line $\eta = \xi$. Moreover, we see from (21) that ξ must not be a multiple of 7, nor η of 3, nor must $\eta - \xi$ be even; otherwise d_1 and d_2 would not both be unity. And as always, η must be prime to ξ. Running through all integral pairs (ξ, η) which lie within the permissible region of Figure 4, we find that the only ones which do not violate one or the other of these conditions are those shown in the first two columns of Table 2. They lead to the n's given in the next four columns which represent possible solutions of the present problem.

Since the pitch ratio is now arbitrary, however, it is necessary to compute it as well as the number of teeth before the design can be called complete. This is easily done by substituting (19) in (2), thus getting

$$\frac{p_{12}}{p_{34}} = \frac{d_1\xi}{d_2\eta}. \tag{24}$$

In interpreting (24), of course, we must regard the right-hand member as being reduced to its lowest terms. From this equation the p's shown in Table 2 are easily computed.

For the case $a=3$, $b=1$, $c=2$, (22) becomes

$$7\eta - 3\xi \leqq 100. \tag{25}$$

Comparison of this with (23) shows that Figure 4 again defines the permissible region provided the scale readings are now interpreted as values of $\xi/2$ and $\eta/2$ instead of ξ and η as before. Moreover, we see from (20) that η must be a multiple of 3, ξ must *not* be a multiple of 7, and $\eta-\xi$ must be a multiple of 2 but not

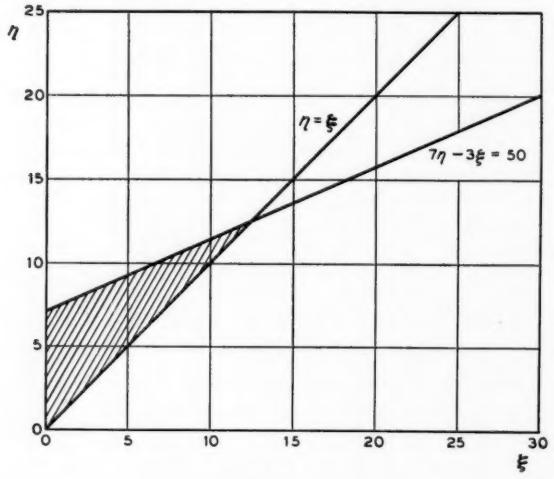


FIG. 4

of 4. It is easily found that the only integer pairs ξ, η which lie within the admissible region of Figure 4 and satisfy these additional requirements are those shown, together with the n's and p's to which they lead, in Table 3.

A complete appraisal of the remaining ten cases shows that there are just 132 different prime solutions of our problem in which no n exceeds the allowable upper limit 50. A great many of these are not mechanically desirable for one reason or another. Among them, however, are combinations which would satisfy almost any conceivable technical requirements.

It may be interesting to note that the time required to consider the 12 cases, including not only the selection of the admissible values of ξ, η but also the computation of the n's and p's, was slightly less than three hours.

5. Mechanical significance of negative integers. The mechanical significance of negative values of the n's may be deduced by considering the equations which introduced them. From the relation

$$\frac{n_1 n_3}{n_2 n_4} = \frac{s}{t} \tag{26}$$

we see that if one or three of the n's are negative, then s/t is negative, and the relative sense of rotation of the gears 1 and 4 is reversed.

If n_1 or n_2 is negative, the other being positive, then gears 1 and 2 are tangent externally instead of internally. Similarly, if n_3 or n_4 is negative, the other being positive, then gears 3 and 4 are tangent externally instead of internally. If both n_1 and n_2 are negative, the arrangement of gears is the same as if both were positive, except that gears 3 and 4 are interchanged. If both n_3 and n_4 are made negative, gears 1 and 2 are interchanged. If one of n_1, n_2 and one of n_3, n_4 are negative, both pairs of gears are tangent externally.

TABLE 1

ξ	η	n_1	n_2	n_3	n_4
1	3	9	3	35	45
5	9	4	1	7	12
7	9	7	1	5	15

TABLE 2

ξ	η	n_1	n_2	n_3	n_4	p_{12}	p_{34}
1	2	11	3	7	11	1	2
1	4	25	9	21	25	1	4
3	4	19	3	7	19	3	4
3	8	47	15	35	47	3	8
5	8	41	9	21	41	5	8
9	10	43	3	7	43	9	10

TABLE 3

ξ	η	n_1	n_2	n_3	n_4	p_{12}	p_{34}
1	3	3	1	7	9	1	1
13	15	11	1	7	33	13	5
19	21	15	1	7	45	19	7

ADVICE TO THE GRADUATE ASSISTANT

A. D. CAMPBELL, Syracuse University

At Syracuse University we face the task of helping our young graduate assistants to make as few mistakes as possible during their first years of teaching. We have found the following measures and advice very helpful.

We have a young teacher visit the classes of older teachers and observe, and we ask older teachers to visit his classes and criticize. We have conferences of all those teaching a given course and we frequently give the same test (made out at such a conference) to all the sections that meet at the same hour. We tell a young teacher not to talk too much in class, not to assign too long lessons, not to let the class sit back and listen (or sleep with or without their eyes shut), not to lecture, not to mark on class work, not to get off the subject in class, not to count homework too much, not to hide his work at the blackboard by his body, not to explain all the next assignment, not to give his own proofs of formulas or theorems but to use the proofs in the textbook, not to call the roll but to glance over a prepared seating-chart and not necessarily to seat the students alphabetically, not to try to explain the fourth dimension or Einstein's Theory to the class, not to copy any other teacher's method slavishly, not to have a set method but to be flexible in his teaching. We have him read Professor Seidlin's little book on the teaching of college mathematics, published by Columbia University Press. We tell him he should be human and humorous and not too serious. He should keep his watch on the table and, for most of the meetings of the class, divide the hour into halves keeping the second half for sending all the class to the board. He should talk in words of few syllables and have several different ways of wording any explanation.

We tell the young teacher to read the text and plan his assignments ahead of time even to picking out definite problems for the homework, to be sure he can work all the problems, to put examples on the board slowly and fill in all the details, to be neat in his work at the board, to face the class and to speak clearly and not too fast. He should let the students hand in their homework at the end of the hour (allowing them to put on any corrections made in class, if they desire), he should glance rapidly through this homework to see what sort of errors the students are making, then return the homework at the next meeting of the class. He should tell the class what topics the course covers, that he does not grade on a curve, how the final grade is made out (two-thirds, say, on the tests and one-third on the final examination with the homework to raise or lower this grade by its satisfactoriness or unsatisfactoriness). If a test proves too hard he should boost all the grades, in other ways he should temper justice with mercy. He should show his tests to an older teacher before he gives them.

He should try to answer questions on the value of the course. He should assume he has forgotten all the elementary mathematics he ever knew, also that the students can not do such simple things as add fractions or remove parentheses. He should sometimes explain more difficult parts of the next as-

signment and (say) work a problem or two, but his main purpose is to have the students read the text. He can tell them that one thing to learn in college is how to obtain information from a book or a magazine. He can threaten to put on a test a problem or so from those worked out in the text. He should not hesitate to go over a difficult point several times. He should emphasize especially the problems that call for setting up formulas and equations from given data.

A young teacher should realize that he is bright and quick in mathematics or he would not be teaching, but that the average student is painfully slow and dull. A lesson plan of the course helps a young teacher. He must cover the topics outlined in this plan. He should compel the weak students to come for a little outside help very early in the course, have them work at the board then before his eyes and so find out what are their weaknesses.

A young teacher should call for volunteers in the class to put on the board and to explain any problems or proofs that were too hard for some in the class, he should look upon the class-hour as partly a time for discussion and the answering of questions and partly a laboratory period when the whole class work examples at the board. When reviewing he should assign for homework a list of problems covering the topics reviewed. Again and again, early in the semester, the young teacher should take time to be sure that every student in the class understands a problem or proof that is under discussion at the moment, by asking each student in turn if he understands it and then having the whole class reproduce the problem or proof on the blackboard.

One or two simple problems worked slowly by the young teacher at the board and perhaps repeated several times will do more good than several complicated problems worked quickly. The students must feel they are not to be afraid to ask questions in the class and so to clear up their difficulties then rather than afterward. The interest of the brighter students can be kept up by having them explain examples or proofs at the board, or by giving them extra and perhaps harder problems to work, or by letting them help the slower students at the board (if they do it quietly).

Above all, the young teacher should feel that teaching is a natural, friendly, helpful procedure requiring patience, much repetition, and no haste. He should make his students feel he is interested in them personally, even to enquiring after a student who is ill or to having the college boys over for a smoker. Knowledge of his subject, enthusiasm, friendliness, common sense, and personality are what go to make up a good teacher.

Note by the Editor. If there are experienced teachers who consider other advice to graduate assistants preferable to that which Professor Campbell has given, the editor will be glad to receive their comments. E. J. M.

QUESTIONS, DISCUSSIONS, AND NOTES

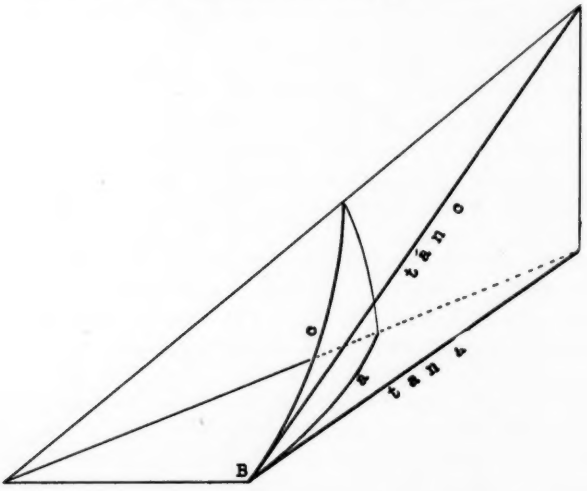
EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

NAPIER'S RULES

W. R. RANSOM, Tufts College

Napier's rules for the spherical right triangle have often been presented as a mere mnemonic device, without proof. Napier's own proof by means of the Pentagramma Mystica (formed by producing the sides of the triangle and intersecting them by the great circles whose poles are the vertices of its acute angles) seems not to have been reproduced in any American textbook. What follows is based upon Napier's device, but employs the notation of substitution groups in order to discover the "circular parts" in a natural manner.



If any one of the formulas for a spherical right triangle is known, for example,

I $$\cos B = \tan a \div \tan c,$$

another formula can be obtained from it by interchanging the legs, a and b, and at the same time interchanging the angles, A and B.

This operation (interchanging) will be named T and will be represented by the notation, $T = (ab)(AB)$. The operation T converts I into the formula

II $$\cos A = \tan b \div \tan c; \quad \text{symbolically} \quad (T)\text{I} = \text{II}.$$

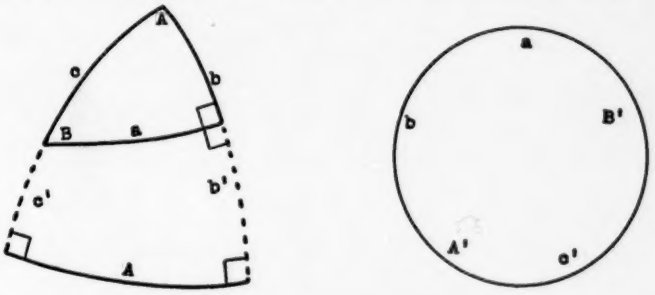
Another operation, Q, can be found as follows. Produce the sides, b and c, of the triangle to meet the great circle whose pole is the vertex of the angle A. A trirectangular spherical quadrilateral is formed with sides equal to a, b', A, c', the *prime* meaning *complement*, $b' = 90° - b$, and so on. With this notation, formula II can be written in the form

II′ $$\cos A = \tan b \tan c'.$$

In the trirectangular spherical quadrilateral A and b' may be interchanged, provided a and c' are interchanged at the same time. By doing this we get the formula

III $$\cos b' = \tan A' \tan a.$$

This operation will be named Q and it will be represented by the notation, $Q = (ac')(Ab')$. We have $(T)\mathrm{I} = \mathrm{II}$, and $(Q)\mathrm{II} = \mathrm{III}$, and we may now write



$(QT)\mathrm{I} = \mathrm{III}$, and consider (QT) as an operation. To find a convenient symbol for the operation (QT) we note that:

T changes a to b, then Q changes b to A′;
T changes A′ to B′, then Q leaves B′ unchanged;
T changes B′ to A′, then Q changes A′ to b;
T changes b to a, then Q changes a to c′;
T leaves c′ unchanged, then Q changes c′ to a.

We therefore represent (QT) by the symbol $QT = (aA'B'bc')$, which means that QT changes each letter to the letter following, and changes the last to the first, thus making a cyclic advance of letters.

Finally, we will define the operation S which repeats the operation QT:

the first QT changes a to A′, then the second changes A′ to B′;
the first QT changes B′ to b, then the second changes b to c′; and so on.

Thus we find that S may be represented by the symbol

$$S = (aB'c'A'b).$$

Any formula which is true for a spherical triangle will give another true formula if transformed by S. The quantities a, B', c', A', b, taken in this cyclic order, as they appear in S, are what Napier calls the "circular parts," and we

have shown that if any formula for the right spherical triangle is expressed in terms of these parts, a cyclic advance of the circular parts will yield another true formula. Formula I can be written

$$\text{I}^\circ \qquad \sin B' = \tan a \tan c';$$

and referring to the ring of circular parts, in which B' is found between the adjacent parts, a and c', we can express I° as follows:

sine of middle part = product of tangents of adjacent parts.

We have shown that this is true for a cyclic advance of the circular parts, and hence any one of the circular parts may be taken as middle.

To prove the other part of Napier's rule we will eliminate c' from I° and (S)I°; that is, from $\sin B' = \tan a \tan c'$, and $\sin c' = \tan B' \tan A'$.

Multiplying these equations together and then dividing by $\tan c' \tan B' \cos B'$ gives

$$\cos c' = \tan a \tan A' \div \cos B'. \tag{1}$$

Eliminating $\sin c'$ and $\cos c'$ by squaring and adding (S)I° and (1), and then solving for $\tan^2 A'$, we have

$$\tan^2 A' = \cos^2 B' \div (\sin^2 B' + \tan^2 a'), \tag{2}$$

whence

$$\begin{aligned} \sec^2 A' &= (1 + \tan^2 a') \div (\sin^2 B' + \tan^2 a') \\ &= \sec^2 a' \div (\sin^2 B' + \tan^2 a'). \end{aligned} \tag{3}$$

Dividing (2) by (3) and taking square roots, we obtain

$$\sin A' = \cos B' \cos a'.$$

This gives us the rule

sine of middle part = product of cosines of opposite parts.

An Extension of a Theorem of Leibniz

J. A. Joseph, Lawrence College

The Theorem of Leibniz

$$D^n(u \cdot v) = \sum_{j=0}^{\infty} \binom{n}{j} D^j u \cdot D^{n-j} v, \tag{1}$$

where $\binom{n}{j}$ is a binomial coefficient, is proved by induction for all non-negative, integral values of n. If we put $n = -1$ and interpret D^{-i} as the ith repeated integral with respect to x, we get

$$D^{-1}(u \cdot v) = \sum_{j=0}^{\infty} \binom{-1}{j} D^j u \cdot D^{-1-j} v = \sum_{j=0}^{\infty} (-1)^j D^j u \cdot D^{-1-j} v. \tag{2}$$

Equation (2), due to Kronecker,* written in finite form with a remainder,

$$D^{-1}(u\cdot v) = \sum_{j=0}^{k-1}(-1)^j D^j u\cdot D^{-1-j}v + R_k, \tag{3}$$

where $R_k=(-1)^k D^{-1}[D^k u\cdot D^{-k}v]$. Equation (3) is obtained by repeated integration by parts, and is a very useful but little used formula. In this paper we obtain a formula for the nth repeated integral of a product, $D^{-n}(u\cdot v)$, which (a) except for a remainder has exactly the same form as (1) for negative integral values of n, and (b) reduces to Kronecker's equation (3) for $n=+1$. This formula is

$$D^{-n}(u\cdot v) = \sum_{i=0}^{k-1}\binom{-n}{i}D^i u\cdot D^{-n-i}v + R_k, \tag{4}$$

where

$$R_k = \sum_{i=0}^{n-1}(-1)^k\binom{k+i-1}{i}D^{-n+i}[D^k u\cdot D^{-k-i}v].$$

Equation (4) is true for $n=1$, after Kronecker. Assuming it to be true for an integral value of n, we prove it true for the case $n+1$. Integrating (4) we have

$$\begin{aligned}
D^{-n-1}(u\cdot v) &= \sum_{i=0}^{k-1}\binom{-n}{i}D^{-1}[D^i u\cdot D^{-n-i}v] + D^{-1}R_k \\
&= \sum_{i=0}^{k-1}(-1)^i\binom{n+i-1}{i}\Big\{\sum_{j=0}^{k-1-i}(-1)^j D^{j+i}u\cdot D^{-1-j-n-i}v \\
&\qquad + (-1)^{k-i}D^{-1}[D^k u\cdot D^{-k-n}v\Big\} + D^{-1}R_k \\
&= \sum_{i=0}^{k-1}(-1)^i\binom{n+i-1}{i}\Big\{\sum_{p=i}^{k-1}(-1)^{p-i}D^p u\cdot D^{-1-n-p}v \\
&\qquad + (-1)^{k-i}D^{-1}[D^k u\cdot D^{-k-n}v]\Big\} + D^{-1}R_k \\
&= \sum_{i=0}^{k-1}\binom{-n-1}{i}D^i u\cdot D^{-n-1-i}v \\
&\qquad + (-1)^k\sum_{i=0}^{n}\binom{k+i-1}{i}D^{-n-1+i}[D^k u\cdot D^{-k-i}v],
\end{aligned} \tag{5}$$

having made use of the fact that

$$\sum_{i=0}^{k-1}\binom{n+i-1}{i} = \binom{n+k-1}{n}, \qquad n>0. \tag{6}$$

The final form of (5) is the same as that of (4) if we place in the latter $n+1$ for n. Hence the theorem is proved for all positive, integral values of n.

* Über eine bei Anwendung der partiellen Integration nützliche Formel, 1884.

Of course, equation (4), as an indefinite n-fold integral is determined except for a polynomial of degree $n-1$ with arbitrary coefficients.

RECENT PUBLICATIONS

EDITED BY TOMLINSON FORT, Lehigh University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Carl Friedrich Gauss—Inaugural lecture on astronomy and papers on the foundations of mathematics. Translated and edited by G. W. Dunnington. Baton Rouge, Louisiana State University Press, 1937. 11+91 pages. $1.00.

Einstein's Theory of Relativity. By A. J. Katsh. (In Hebrew), Tel Aviv, 1937. 80 pages.

Scripta Mathematica Forum Lectures. Addresses by C. J. Keyser, D. E. Smith, E. Kasner, and W. Rautenstrauch. New York, Yeshiva College, 1937. 94 pages. $1.00.

Arithmetic for Teacher-training Classes. By E. H. Taylor. Revised edition. New York, Henry Holt and Company, 1937. 7+432 pages. $1.70.

Preview of Mathematical Analysis. Equations, Probability, Calculus. By A. Freilich, H. H. Shanholt, and J. P. MacCormack. New York, Silver Burdett Company, 1937. 6+137 pages. 60 cents.

Introduction to Mathematics. A survey emphasizing mathematical ideas and their relations to other fields of knowledge. By H. R. Cooley, D. Gans, M. Kline, and H. E. Wahlert. Boston, Houghton Mifflin Company, 1937. 18+634 pages. $3.25.

Manual of Mathematics and Mechanics. By G. R. Clements and L. T. Wilson. New York, McGraw-Hill Company, 1937. 7+266 pages. $2.50.

Introduction to College Algebra. By W. L. Hart. New York, D. C. Heath and Company, 1937. 5+246+24 pages. $1.84.

The New Applied Mathematics. By S. J. Lasley and M. F. Mudd. Revised edition. New York, Prentice-Hall, 1937. 23+544 pages. $1.60.

The Calculus. By R. D. Carmichael, J. H. Weaver, and L. La Paz. Revised edition. Boston, Ginn and Company, 1937. 16+384 pages. $3.00.

Postulates for Assertion, Conjunction, Negation, and Equality. By E. V. Huntington. Proceedings of the American Academy of Arts and Sciences, vol. 72, number 1. Boston, American Academy, 1937. 44 pages.

Das Parallelenproblem und seine Lösungen. By M. Zacharias. Leipzig and Berlin, B. G. Teubner, 1937. 44 pages. RM 1.20.

Mach's Empirio-pragmatism in Physical Science. By C. B. Weinberg. New York, Albee Press, 1937. 4+122 pages. $1.00.

Introduction to the Theory of Fourier Integrals. By E. C. Titchmarsh. Oxford, Clarendon Press, 1937. 10+390 pages. $6.00.

REVIEWS

Portraits of Eminent Mathematicians, with Brief Biographical Sketches. By David Eugene Smith. New York, Scripta Mathematica, 1936. 12 portraits. $2.50.

This portfolio of portraits of men distinguished for their contributions to mathematics is a splendid collection which aside from its intrinsic interest provides a good example of modern methods of photogravure. The men included in the collection are Archimedes, Copernicus, Vieta, Galileo, Napier, Descartes, Newton, Leibniz, Lagrange, Gauss, Lobachevsky, Sylvester. Each portrait is accompanied by a short biographical or descriptive sketch. These accounts are designed to give the lay reader a fair idea of the reason why the person in question is important in the history of mathematics. For the most part this object has been successfully accomplished, though it would seem that, even in such limited space, a more impressive account could be given in some cases, such as those of Galileo and Newton.

In any case, this portfolio would be a valuable addition to the equipment of any classroom where either secondary or collegiate mathematics is taught. It is to be hoped that, as intimated in the preface, further portfolios will follow this one, and that they will include more of the mathematicians of the 19th century—and why not many of the 20th also?

R. B. McClenon

Mechanics. By W. F. Osgood. New York, The Macmillan Company, 1937. 15+495 pages. $5.00.

This text, by one of the foremost teachers of mathematics of his generation, is the outgrowth of some forty years' experience with elementary and advanced courses in mechanics. The author attempts to give an introductory treatment, with sufficient material for a two-year course, combining a discussion of physical phenomena with mathematical rigor.

Thus, the early chapters which treat the statics and dynamics of particles and rigid bodies by elementary methods may be read by a second year college student concurrently with the latter part of his course in elementary calculus. Most of the material and problems of the long chapter on mechanics in Osgood's *First Course* or *Introduction to the Calculus* has been incorporated in the present book. The later chapters on variational principles and the methods of Lagrange and Hamilton-Jacobi, assume the mathematical maturity of the advanced calculus level.

A brief treatment of vector analysis is placed in the appendix. This avoids the long and, to the student, uninteresting digression too often found at the beginning. Vector methods are used throughout the text when convenient, but not exclusively. The Gibbs notation is used, as one would expect unless he had suffered through the Hamiltonian S's and V's of Osgood's *Advanced Calculus*.

The topics are well selected, and they are treated clearly and carefully. However, the attempt to be both physical and mathematical is not completely

successful. In fact, it is not always clear whether we are dealing with the empirical science in which any experimental results may be used, or with the mathematical science in which we adopt certain assumptions once for all and deduce logical consequences from them. For example, on p. 81 there is an apology for a valid mathematical conclusion not agreeing exactly with the experimental situation. But, of course, the book is more suited to pure mathematicians who use physical language than to those who wish to apply mechanics to practical problems.

The problems are well graded and numerous. Those few which ask the student to construct his own problem from some general situation, as a runner or a racing shell, are probably too difficult for the average reader. Occasionally the author goes out of his way to avoid the useful, as when he introduces simple harmonic motion by the particle moving under gravitation along a diameter of the earth. This is, to be sure, an amusing exercise, but it is of negligible importance compared to small elastic vibrations.

The treatment of change of units on p. 77 is confusing, and less simple than the use of conversion factors, as

$$s' \text{ yds.} = s' \text{ yds. } (3\text{ft./yd.}) = 3s' \text{ ft.},$$

which is condemned in the footnote. Similarly, the discussion of absolute units of force and mass on pp. 55 and 119 will not appeal to teachers who prefer to use $F=(W/g)A$ along with $F=mA$. The author's comparison of absolute units with radian measure and natural logarithms is a good one, and for computational purposes weights will probably remain as useful as degrees and common logarithms. On p. 101 we find the term centrifugal force elaborately explained away.

In spite of the many minor points which have been mentioned as not to the reviewer's taste, the book is an excellent text in theoretical mechanics. The teacher seeking a somewhat abridged treatment of the field covered by the first two volumes of Appell's *Mécanique rationelle*, will find no better text in English.

PHILIP FRANKLIN

Plane Trigonometry. By A. M. Harding and G. W. Mullins. Revised edition. New York, The Macmillan Company, 1937. 7+172 pages. $1.60.

Trigonometry. By J. W. Branson and J. O. Hassler. New York, Henry Holt and Company, 1937. 7+198+73 pages. $1.75 (with tables).

Plane Trigonometry. By H. A. Simmons and G. D. Gore. New York, John Wiley and Sons, 1937. 8+201+81 pages. $2.00 (with tables).

All three of these texts are designed for a three semester-hour course (i.e., roughly 48 hours) for college freshmen, although they are sufficiently flexible to allow the individual teacher to select items for a shorter course if desired. Simmons-Gore makes definite suggestions as to the distribution of lessons for both a 48-hour and a 32-hour course. As for tables, Harding-Mullins has none, and the other two both have five place tables (with proportional parts) of logarithms of numbers and of trigonometric functions, and four place tables of

natural functions and radian equivalents. In addition, Branson-Hassler has a table of squares of numbers, and Simmons-Gore has squares and square roots, natural logarithms to five places, and a table of certain important constants—chiefly multiples and powers of π—and their common logarithms to seven places. The problems in all three texts are plentiful and good, and answers are given to the odd-numbered problems only.

For those readers who are acquainted with the first edition of the Harding-Mullins text, it will be sufficient to note that in the revised edition appears a new chapter on logarithms, double the previous number of problems, and a brief discussion of approximate measure and significant digits. (In the opinion of the reviewer, this discussion is almost too brief to be adequate; the same criticism applies to Branson-Hassler. Simmons-Gore, on the other hand, has a clear, detailed discussion of the problem.) The same arrangement of material as before is maintained, and is somewhat unusual in that the solution of oblique triangles is introduced directly after that of right triangles. This is made possible by material introduced in the first chapter—a commendable chapter, by the way, which nicely ties up the subjects of geometry and trigonometry.

The Branson-Hassler arrangement is somewhat the same in that, after a discussion of the right triangle, the order is as follows: reduction formulas, introduction to oblique triangles (laws of sines and cosines), trigonometric analysis, and oblique triangles concluded (law of tangents). It is the authors' hope that this arrangement will enable the student to remove deficiencies in earlier work while keeping up with the class in work not requiring too much previous material. The text includes a short chapter on complex numbers and a 34-page introduction to spherical trigonometry.

In the Simmons-Gore text the solution of right triangles and a good development—mainly through problems—of their important applications are introduced early in the work. Then, preceding logarithms and the solution of the general triangle, comes all of the trigonometric analysis except inverse functions and equations, which are postponed to the end. A section on the complete solution of the problem of triangulation is included. The discussions and proofs throughout this volume are exceptionally clear and possess a welcome degree of rigor.

E. P. NORTHROP

Numbers and Numerals. A Story Book for Young and Old. By David Eugene Smith and Jekuthiel Ginsburg. (Contributions of Mathematics to Civilization, edited by W. D. Reeve, No. 1.) New York, Bureau of Publications, Teachers College, Columbia University, 1937. 10+52 pages. 25¢.

During the last twenty-five years there has been a growing recognition that familiarity with the history of mathematics increases the power of a teacher to give students insight into the nature and meaning of mathematics. But thousands who are now teaching elementary mathematics are not likely to have opportunity to study the history of mathematics at any college or university

where there are facilities for teaching it well. In some way the history of mathematics must be brought to these teachers if they are to have it at all.

It is probably with some such idea in mind that the fifty-page booklet under review has been issued by the *Mathematics Teacher* free to all members of the National Council of Teachers of Mathematics and for sale at an almost negligible price to others. It is the first number in a contemplated series of inexpensive monographs designed to exhibit the contributions of mathematics to civilization.

The work contains eight chapters, but the two chapters, From "Numbers to Numerals" (III) and "From Numerals to Computation" (IV), include twenty-eight of the fifty pages of subject matter. Three pages each are devoted to "Learning to Count" (I), "Naming the Numbers" (II), and "Story of a Few Arithmetic Words" (VIII). Of the remaining thirteen pages, five are given to "Mystery of Numbers" (VI) and four each to "Fractions" (V) and "Number Pleasantries" (VII). There are many excellent illustrations, nearly all of which have appeared somewhere in Dr. Smith's previous publications.

As quite often happens in works which attempt to put much into little space, some things are stated positively without the amplification or qualifying conditions which the authors would doubtless prefer to add if space permitted. For example, we find on page seventeen the statement " . . . our modern numerals were generally known in Europe at least as early as the year 1000." Readers not already familiar with the history of elementary mathematics are likely to identify "our modern numerals" with the forms they are now using and to interpret the "generally known" as meaning that the numerals were generally used for computation, which interpretation is certainly more than the authors intended to imply. The Hindu-Arabic numerals were familiar to travelers and traders long before they were generally taught in European schools or commonly used for computation in Europe. Also, some of the forms we now use are not known to occur anywhere as early as the year 1000. The authors are well aware that the man who was probably most influential in introducing the numerals to the scholars of Europe was not born until the second half of the twelfth century (Cf. Smith-Karpinski, *Hindu-Arabic Numerals*, p. 128) and they would doubtless agree with Mr. G. F. Hill when, speaking from a little different point of view, he says in his *Development of the Arabic Numerals in Europe* (p. 11), "The numerals are first found in MSS. of the tenth century but they can not be said to have been at all well known until the beginning of the thirteenth."

Elementary mathematics is under condemnation in many a school today because of the formal and deadening way in which it is being taught there. One way of contributing to its salvation is to lead students to an insight into and understanding of mathematics through natural (even if frequently vicarious) experiences suggested by the study of the evolution of mathematics. As a missionary tract to carry this kind of gospel to many a class-room where it has never been proclaimed before, but where it is greatly needed, this little monograph should have the blessing of all friends of mathematical instruction.

U. G. MITCHELL

Men of Mathematics. By E. T. Bell. New York, Simon and Schuster, 1937. 21+593 pages. $5.00.

The reader of *Men of Mathematics* lays down the book after a first reading with a feeling of profound satisfaction that here is a fascinating set of short stories (it is more than a series of biographies), for frequent perusal in parts or as a whole, a book of reference for historical use, and a means of forming friendships with men already more or less well known.

The work does not claim to be a history of mathematics but to show what "sort of human beings the men were who created *modern* mathematics," "the importance for modern mathematics of the man's work" and "the human appeal of the man's life and character." We are made to appreciate, for example, the contrast between Gauss and Cauchy. The former "never published trivialities. When he put out anything it was not only finished in itself but was also so crammed with ideas that his successors were enabled to apply what Gauss had invented to new problems." With the latter, it was his "habit of rapid publication and his gift for effective teaching which really got rigor in mathematical analysis accepted." We are impressed with the author's broad outlook on many historical matters; thus in a statement about Euler he says: "In his day, he systematized and unified vast tracts cluttered with partial results and isolated theorems, clearing up the ground and binding up the valuable things by the easy power of his analytical machinery"; and again in a statement relating to royal academies he states: "Mathematics owes an undischargeable debt to Frederick the Great of Prussia and Catherine the Great of Russia for their broadminded liberality. They made possible a full century of progress in one of the most active periods in scientific history."

The uncanny insight into the special role played by each mathematician presented and the vivid translation of this insight into words is shown by the chapter headings. A few of the twenty-nine headings chosen at random are: The Prince of Amateurs—Fermat; Master of all Trades—Leibniz; Analysis Incarnate—Euler; A Lofty Pyramid—Lagrange; The Copernicus of Geometry—Lobatchewsky; Genius and Stupidity—Galois; Invariant Twins—Sylvester, Cayley; Master and Pupil—Weierstrass, Sonja Kowalewski; The Last Universalist—Poincaré; Paradise Lost?—Cantor.

This book may be read with appreciation by intelligent laymen who are unfamiliar with the field of mathematics. One hopes that it may serve to revise their notions of mathematicians. The author expresses these ideas when he refers to "the mere mathematician of conventional literature, written for the most part by people who may indeed have known some pedantic college professor of mathematics but who never in their lives saw a real mathematician in the flesh." One of many indications of the interest that mathematics offers is given in connection with Johannes Bernoulli I and his son: "After all, if rational human beings get excited about a game of cards, why should they not blow up over mathematics which is infinitely more exciting?"

Time and again, phrases are used whose aptness is extraordinary. They are

superlatively illuminating as in: " . . . felt about as much freedom as a skylark trying to soar up through an atmosphere of cold glue." Just because these occur so often, the reader may feel certain that others might have been toned down.

The illustrations are exceptionally fine both from the standpoint of the original portraits and from that of reproduction.

This reviewer feels compelled in all humility to use with regard to this work a quotation which the author gives from Jacobi: "It is above my praises as it is above my own works."

LAO G. SIMONS

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 462 East Foster Ave., State College, Pa.

BOOKS FOR CLUBS

This list is continued from the previous issue.

5. *Mathematics for the Million.* By L. Hogben, London, George Allen; N. Y., Norton, 1936, 647 pages. $2.00.

   A work of very exceptional interest by a distinguished professor of social biology in the University of London. It is intended for the intelligent reader and there is emphasis on things historical. There are 12 chapters at the end of which are exercises for solution. The titles of some chapters are as follows: Mathematics the mirror of civilization; First steps in measurement *or* Mathematics in prehistory; Euclid without tears *or* What you can do with geometry; The collectivization of arithmetic *or* How logarithms were discovered; The arithmetic of growth and shape *or* What calculus is about; Statistics *or* The arithmetic of human welfare. Reviewed in this MONTHLY, vol. 44, p. 591.

6. *Brush up your Wits.* By H. Phillips (Caliban). London, Dent, 1936, 24+116 pages.

   About one hundred mathematical recreations mostly original.

7. *Caliban's Problem Book.* By H. Phillips, S. T. Shovelton, and G. S. Marshall. London, T. de la Rue, 1933, 10+330 pages.

   Many ingenious inferential problems.

8. (a) *Amusements in Mathematics*, 1917, 7+258 pages. (b) *The Canterbury Puzzles and other Curious Problems*, 1932, second ed., 1919, 254 pages. (c) *Puzzles and Curious Problems*, 1932, 195 pages. By H. E. Dudeney.

   All three exceptionally interesting volumes published in London.

9. *Mathematical Snack Bar. A Collection of Notes and Results.* By N. Alliston. Cambridge, 1936, 6+155 pages.

   Many new results in geometry and theory of numbers worked out.

10. *Handbook of the Collections illustrating Time Measurement.* Part I, *Historical Review.* Part II, *Descriptive Catalogue.* By F. A. B. Ward. Board of Education, Science Museum. London, 1936–37.

    Over 400 time-measuring devices are described. Richly illustrated.

11. *Time and Timekeepers including the History, Construction, Care, and Accuracy of Clocks and Watches.* By W. I. Milham. N. Y., 1923, 20+609 pages.

    Admirable work.

CLUB REPORTS

1936–37

Pi Mu Epsilon, Washington Square College, New York University

Monthly meetings were held with topics including: Summability methods; Mathematics in education; Theory of aggregates. The Fourth Annual Pi Mu Epsilon Interscholastic Mathematics Contest was held. 103 high schoo's entered, 92 of which sent full teams of 4 students each. The highest score was made by the team from the Boys High School of Brooklyn, N. Y., which was awarded the large cup. The team placing second was from Abraham Lincoln High School of Brooklyn. Four smaller cups were awarded to the teams making the highest score in their sections. The society also conducted a mathematical model contest for high school students. Seventeen schools took part, and plaques were awarded for the three best models. The first award was made to the Bloomfield High School of Bloomfield, N. J., for a circular slide rule; the second award was made to Stratford High School of Stratford, Conn., for a model illustrating the principle of the sundial; and the third award was made to Edison Technical High School of Mount Vernon, N. Y., for a gear shift box. This contest was a new activity of the society this year. The annual dinner at which Dr. D. A. Flanders spoke on "The pathology of geometry" closed the year.

Mathematics Club, Boston University

President, W. Earley; Vice-President, Mary Hickey; Secretaries, Ruth Dular, S. Keto; Faculty Adviser, Professor E. B. Mode. Meetings were held twice each month. Prizes were awarded for the best papers presented by students. The prize for graduate students, a copy of L. Hogben's *Mathematics for the Million*, was won by Eleanor Martin. The prize for undergraduates, a slide rule, was won by John Robinson. A supper and a theater party were the principal social events. A joint meeting held with the Mathematics Clubs of Wellesley College and of Tufts College gave "opportunity to exchange ideas and get acquainted with other mathematically minded students. It is expected to be an annual event." Topics discussed at regular open meetings included: Magic squares; Photo-electric number sieve; Mathematical prodigies; Trisection of an angle; Perfect and amicable numbers; Indeterminate equations. The guest speaker of the year was Professor T. E. Mergendahl, Tufts College, who spoke on "Check and double check."

Pi Mu Epsilon, University of Colorado

This young chapter held meetings every three weeks at which mathematical papers were given. In addition it held two initiation banquets, and a steak fry concluded the year's social functions.

Pi Mu Epsilon, University of Toledo

Director, Professor W. Dancer; Vice-Director, S. Boyk; Treasurer, K. Whelan; Secretary, Blanche Fishler. The club welcomed Professor J. H. Weaver, Ohio State University, as a guest speaker. His subject was "The value of mathematics." Topics discussed at the regular meetings included: Linear coördinates; Complex variables; Effects of mathematics on philosophy and

vice versa; Mathematics in the curriculum; Mathematics in a liberal education. A debate, "Resolved that imaginary numbers are really imaginary," was the novel feature of the year's program. Social functions included an initiation banquet and a buffet supper.

Mathematics Club, Wellesley College

President, Grace Mandeville; Vice-President, Evelyn Wicoff; Senior Executive, Doris Gasteiger; Junior Executive, Ruth Hawkes; Secretary, Gloria Sharp. This club enjoyed the hospitality of Boston University at a joint meeting with the club from Tufts College. The club has many mathematical songs, copies of which have been sent to the Stunt Library. Most meetings are enlivened by some of these songs. Topics discussed at regular meetings included: Review of *Mathematical Recreations;* Number tricks and their algebraic justification; Four-color map problem; Magic squares; Number systems with special reference to a system with base twelve; Imaginary numbers and their realness; Mathematical meetings at Oslo.

Mathematics-Physics Club, College of Saint Teresa

President, Marion Claridge; Vice-President, Rita Leinfelder. The mathematical subjects of this joint club's program included: Incidents in the lives of Archimedes, Thales, and Pythagoras; Mechanical devices and instruments for curve constructions, etc.; The perpetual calendar. The club adopted a song, "Trig an' Me." The December program included a geometrical proof that there is a Santa Claus, the geometrical construction of the Christmas star and a mathematical Christmas tree. A mathematical play, "The Naughty Little Three," completed the year's program.

Pi Mu Epsilon, Duke University

Director, H. Flowers; Vice-Director, C. Kruser; Secretary, Patricia Gibson; Treasurer, W. Pono. Members of the chapter furnished all the talks at regular meetings. After the initiation banquet Professor A. A. Bennett, Brown University, spoke on "The mathematics of the inexact."

Pi Mu Epsilon, The State College of Washington

Director, Virginia Patton; Vice-Director, P. Clement; Secretary, G. Grafton; Treasurer, E. C. Colpitts; Corresponding Secretary, Dr. C. A. Isaacs. The chapter holds bi-monthly meetings together with The Mathematics Club at which speakers are largely drawn from the student body. At the initiation banquet Dr. Paul Anderson, department of physics, was the guest speaker.

Kappa Mu Epsilon, University of New Mexico

President, L. York; Vice-president, Clara M. Matthew; Secretary, O. Keller; Corresponding Secretary, Dr. H. D. Larsen; Treasurer, C. Barker. The topics discussed at monthly meetings included: Vibration problems; Asymptotic development of a new type of power series; Infinity. The club annually pays the expenses of a member who reads a paper before the Southwestern Section of the Mathematical Association of America. The club also sponsors the travelling lecturer nominated by this Section. This lecturer visits several institutions in the Southwest. This year the club felt honored in the choice of one of their members, Professor C. V. Newson, as the lecturer.

Mathematics Club, Albion College

President, Doris Pugh; Vice-President, R. Jordan; Secretary-Treasurer, C. Anderson. This club has a roll call which is answered differently each month. For example, once by "an interesting event in the life of a mathematician," another time by "a contribution made to trigonometry by a noted mathematician." Topics discussed included: Graphical method of computing permutations and combinations; Fundamental ideas in parallel nomographs; Theory of the planimeter; A shortening of Horner's method; The cubic equation; An original pendulum problem; Geometric representation of certain trigonometric functions. A symposium on teaching mathematics in the high schools helped the club to maintain relations with its alumni. The club accepted membership in the National Kappa Mu Epsilon, becoming the Michigan Alpha chapter.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 309. *Proposed by N. A. Court, University of Oklahoma.*

The orthogonal projections of any point of a sphere upon three diameters determine a plane, the orthogonal projections of whose pole, for the sphere, upon the same diameters, determine a plane which touches the sphere.

E 310. *Proposed by V. Thébault, Le Mans, France.*

In a certain system of notation there exists a four-place number of the form $cbba$ which is the square of bb. Show that the numbers, b and $a^2+4(a-1)^2$ are each perfect squares. Determine the base of such a system and the values of a and b, knowing that a is also a perfect square.

E 311. *Proposed by J. S. Robberson, Dallas, Texas.*

The quadrant, AOB, of a circle varies in size and position, but keeps the segment, QA, of one bounding radius, fixed. Find the locus of the point P on the arc AB, if it divides that arc in the same ratio as Q divides the radius AO.

E 312. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

If the scalene triangle ABC has its external angle bisectors at B and C equal, show that $(s-a)/a$ is the geometric mean of $(s-b)/b$ and $(s-c)/c$.

E 313. *Proposed by J. Rosenbaum, Bloomfield, Connecticut.*

Find four rational numbers such that their sum, the sum of their squares, and the sum of their cubes, each equals 2.

E 314. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

Find the locus of the center of a variable sphere which cuts each of two (or three) fixed planes in a circle of constant size.

SOLUTIONS

E 276 [1937, 246]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

The inside dimensions of a rectangular box with a lid on it, are three feet, four feet, and five feet. A post in the form of a right circular cylinder nine inches in diameter just fits diagonally in the box, touching all six inner faces. How long is the post? (Note that the axis of the post, if prolonged, would miss the corners of the box.)

Solution by E. P. Starke, Rutgers University.

Let the coördinate axes be taken parallel to the edges of the box, with the origin at its center. With the inch as a unit, let $x=30$, $y=24$, and $z=18$ be the faces of the box meeting in the corner into which an end of the post is fitted. The midpoint of the axis of the post is at the origin. Let $P(x_0, y_0, z_0)$ be the end of the axis, and let $2a$ be its length. Then $x_0^2+y_0^2+z_0^2=a^2$, and

$$x_0x + y_0y + z_0z = a^2 \tag{1}$$

is the equation of the plane in which the end of the post lies. Now the lines determined by (1) and $x=30$, $y=24$, $z=18$ respectively, are to be tangent to the circumference of the end; hence each line must lie at a distance of $9/2$ from P. By the method of Osgood and Graustein's *Analytic Geometry* (p. 514) or otherwise, these distances are easily computed to be

$$\frac{a(30-x_0)}{\sqrt{a^2-x_0^2}}, \qquad \frac{a(24-y_0)}{\sqrt{a^2-y_0^2}}, \qquad \frac{a(18-z_0)}{\sqrt{a^2-z_0^2}}.$$

If the square of each is equated to $81/4$ and the resulting equations are cleared of fractions and added, there results $(30-x_0)^2+(24-y_0)^2+(18-z_0)^2=81/2$. That is, the distance from P to the corner is $9/\sqrt{2}$. Thus a is less than the diagonal, $30\sqrt{2}=42.43\cdots$ but greater than $30\sqrt{2}-9/\sqrt{2}=36.06\cdots$. Solving the three relations just obtained for x_0, y_0 and z_0 in terms of a, we have

$$\begin{aligned}(4a^2+81)x_0 &= 120a^2-9a\sqrt{(4a^2-3519)},\\ (4a^2+81)y_0 &= 96a^2-9a\sqrt{(4a^2-2223)},\\ (4a^2+81)z_0 &= 72a^2-9a\sqrt{(4a^2-1215)}.\end{aligned}$$

Now let $f(a)\equiv(x_0^2+y_0^2+z_0^2)-a^2$, in which x_0, y_0, and z_0 are to be replaced by their expressions in terms of a. To find the solution of $f(a)=0$, for $36.06<a<42.43$, it seems simpler to employ interpolation. If we take $a=36$, we find $x_0=27.027$, $y_0=20.282$, $z_0=13.846$, and thus $f(36)=37.55$. Similarly, $f(37)=-44.14$. Then again, $f(36.4)=4.92$, and $f(36.5)=-3.20$. Hence, correct to the second decimal place, $a=36.46$ inches. Consequently, the post is slightly over six feet eleven inches in length.

Also solved by C. L. Weaver and the proposer.

E 277 [1937, 330]. *Proposed by V. Thébault, Le Mans, France.*

A sphere (S) of constant size so moves as to always bear a fixed point P on its surface. If (C) is another sphere, fixed in size and position, and Q is the orthogonal projection of P on the radical plane of (S) and (C), prove that Q moves on the surface of a sphere.

Solution by C. E. Springer, University of Oklahoma.

Let the variable sphere (S) have the equation $(x-u)^2+(y-v)^2+(z-w)^2=r^2$ with the point P at the origin, so that

$$u^2 + v^2 + w^2 = r^2. \tag{1}$$

Take for (C) the equation $(x-d)^2+y^2+z^2=R^2$. The radical plane of (S) and (C) has the equation

$$2[(u - d)x + vy + wz] = R^2 - d^2, \tag{2}$$

and the line

$$\frac{x}{u - d} = \frac{y}{v} = \frac{z}{w} \tag{3}$$

through P, orthogonal to the radical plane, meets the latter in the point Q, whose locus, found by eliminating u, v, and w from (1), (2), and (3), is

$$4(r^2 - d^2)(x^2 + y^2 + z^2) = 4xd(R^2 - d^2) + (R^2 - d^2)^2.$$

Also solved by C. Grace Shover, E. P. Starke, and the proposer.

E 278 [1937, 330]. *Proposed by Fred Discepoli, New York.*

Prove the non-existence in the decimal system of a palindromic, four-digit square.

Solution by C. W. Trigg, Cumnock College, Los Angeles.

A palindromic four-digit number is a multiple of 11, and if it is a square, $N^2 \equiv 0 \pmod{11^2}$ so $N \equiv 0 \pmod{11}$. Now $1000 < N^2 < 10000$, so that $31 < N < 100$. There are only seven multiples of 11 in this range, and none of them has a palindromic square.

Also solved by W. E. Buker, Mary L. Constable, Daniel Finkel, E. P. Starke, and the proposer.

E 279 [1937, 330]. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

Given two sides, construct a parallelogram whose angles equal the angles between its diagonals.

Solution by C. Grace Shover, Carleton College.

Let the two sides of a parallelogram, whose angles equal the angles between its diagonals, be of lengths S and s, and the two diagonals D and d, where $s<S$ and $d<D$. Let A and B be the angles that D makes with s and S respectively. Then the angle between d and S is also A.

Now from the Law of Sines,

$$D/S = \sin (A + B)/\sin A, \tag{1}$$

$$S/s = \sin A/\sin B = \tfrac{1}{2}D/\tfrac{1}{2}d = D/d, \tag{2}$$

so that $S/D = s/d$, and

$$\sin (A + B)/\sin A = 2s/d. \tag{3}$$

Substitute (2) in the right member of (3) and then substitute the result in the right member of (1). We obtain $D=S\sqrt{2}$.

Construction: We find D from the diagonal of a square with side S. The construction of the parallelogram then follows from the construction of a triangle with three given sides.

Also solved by Daniel Finkel, Samuel Kramer, J. Milkman, E. P. Starke, H. Tate, and the proposer.

E 280 [1937, 330]. *Proposed by J. W. Cell, North Carolina State College.*

Prove that the determinant

$$\begin{vmatrix} a & a+d & a+2d & \cdots & a+(n-1)d \\ a+d & a+2d & a+3d & \cdots & a \\ a+2d & a+3d & a+4d & \cdots & a+d \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a+(n-1)d & a & a+d & \cdots & a+(n-2)d \end{vmatrix}$$

has the value, $d^{n-1}n^{n-2}S_n(-1)^{n(n-1)/2}$, where S_n is the sum of the elements forming the arithmetic progression in the first column.

Solution by John Williamson, Johns Hopkins University.

The determinant D, whose value is to be found, is a circulant, and as such has the value

$$D = (-1)^{(n-1)(n-2)/2}b_1b_2b_3\cdots b_n,$$

where $b_i=a+(a+d)w_i+(a+2d)w_i^2+\cdots+(a+[n-1]d)w_i^{n-1}$, and $w_1=1$, $w_2, w_3, \cdots w_n$ are the nth roots of unity. [See Muir, *Theory of Determinants*, vol. 2, page 406.]

But $b_1=S_n$, and, if $i\neq 1$, we have

$$\begin{aligned} b_i &= w_id(1+2w_i+3w_i^2+\cdots+[n-1]w_i^{n-2}) \\ &= w_id\,\frac{1-nw_i^{n-1}+(n-1)w_i^n}{(1-w_i)^2} \\ &= \frac{dn(w_i-1)}{(1-w_i)^2} = \frac{-nd}{1-w_i}. \end{aligned}$$

Since $\prod_{i=2}^{n}(1-w_i)=n$, we have

$$b_1b_2b_3\cdots b_n = S_n(-1)^{n-1}d^{n-1}n^{n-2},$$

and therefore

$$D = (-1)^{n(n-1)/2}d^{n-1}n^{n-2}S_n.$$

Also solved by C. Grace Shover, C. E. Springer, E. P. Starke, Samuel Zuckerman, and the proposer.

E 281 [1937, 330]. *Proposed by W. B. Clarke, San Jose, California.*

Let the incircle of triangle ABC touch sides a, b, and c at points D, E, and F respectively. Call the incenter, I. With A as center and AE as radius, swing an arc to cut DI produced, inside triangle ABC, at P. Similarly, let arcs centered at B and C cut EI and FI, inside the triangle, at Q and R. Let AP, BQ, and CR meet sides a, b, and c at points J, K, and L, respectively. Now prove or disprove the following: (1) AJ, BK, and CL are concurrent; (2) Triangles AJB and AJC, BKA and BKC, CLA and CLB have their incircles equal each to each in pairs.

Solution by D. L. MacKay, Evander Childs High School, New York.

Let PM, $\|BC$, meet the altitude AG at M. Then $BG=(a^2-b^2+c^2)/2a$, $AP=AE=s-a$, $BQ=BD=s-b$, $CD=CR=s-c$, and $PM=DG=BD-BG=(b-c)(s-a)/a$. Hence, from the right triangle APM, $AM=2[(s-a)/a]\sqrt{(s-b)(s-c)}$. Now $AJ/AP=AG/AM$, and $AJ=\sqrt{s(s-a)}$, and similarly, $BK=\sqrt{s(s-b)}$, and $CL=\sqrt{s(s-c)}$. Also, from the similar triangles AGJ and APM, $GJ=PM\cdot AJ/AP=[(b-c)/a]\sqrt{s(s-a)}$. Hence

$$BJ = PG + GJ = \frac{a^2 - b^2 + c^2 + 2(b-c)\sqrt{s(s-a)}}{2a}$$

and

$$CJ = BC - BJ = \frac{a^2 + b^2 - c^2 - 2(b-c)\sqrt{s(s-a)}}{2a},$$

with corresponding formulas obtainable by cyclic permutation for CK, AK, AL, and BL. Since $BJ\cdot CK\cdot AL \neq CJ\cdot AK\cdot BL$, it follows that AJ, BK, and CL are not concurrent.

Assume now that AJ' divides triangle ABC so that triangles ABJ' and ACJ' have equal incircles. Let incircle (X) of triangle ABJ' touch BJ', AB, and AJ' at E', F', and G' respectively, and incircle (Y) of ACJ' touch $J'C$, AC, and AJ' at H', K', and L' respectively. Finally set $AJ'=x$, $J'E'=J'G'=z$, $AK'=AL'=w$, $XE'=YH'=y$, and $ID=r$. Then $BD=s-b$, $CD=s-c$, $AG'=AF'=x-z$, $BF'=BE'=c-x+z$, $CK'=CH'=b-w$, $J'L'=J'H'=x-w$. From triangles $XE'B$ and IDB, $y/r=(c-x+z)/(s-b)$, so that

$$rx + (s-b)y - rz = rc. \tag{1}$$

Since $DC=BC-BJ'$, $x-w+b-w=a-c+x-2z$, and

$$z - w = s - b - c. \tag{2}$$

Since $\Delta ABJ'+\Delta ACJ'=\Delta ABC$ we have

$$y(s + x) = rs. \tag{3}$$

From triangles IDC and $YH'C$, we have $y/r=(b-w)/(s-c)$, so that

$$(s-c)y + rw = br. \tag{4}$$

Solving for x, we get $x=\sqrt{s(s-a)}$, and since there is only one such line as AJ', and as $AJ=\sqrt{s(s-a)}$, AJ and AJ' are one and the same line. In like manner BK and CL divide triangle ABC so that triangles ABK and CBK, and triangles ACL and BCL have equal incircles respectively.

It may be of interest to note that the product of the lines AJ, BK, and CL is equal to the product of the perimeter and area of the triangle ABC.

Also solved by the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3851. *Proposed by V. Thébault, Le Mans, France.*

Let $N=123\cdots(n-3)(n-2)n$ be a number written in the system with the base $n+1$, where the digits in that system are placed in increasing order of magnitude omitting 0 and $(n-1)$. Form the product $P=N\cdot L$, where $L=\alpha\beta$ has two digits such that the sum of the digits $\gamma=\alpha+\beta$ is a number less than n and prime to n. The product may be written with n distinct digits taken from the $n+1$ figures 0, 1, 2, 3, $\cdots$, n and suitably arranged; the digit missing will be $n-\gamma$.

3852. *Proposed by V. Thébault, Le Mans, France.*

Let I, I_a, I_b, I_c, I_d be the center of the inscribed sphere of radius r, and the centers of the spheres (I_a), (I_b), (I_c), (I_d) escribed in the truncated trihedrals with the vertices A, B, C, D of a tetrahedron $ABCD$; let O, O_a, O_b, O_c, O_d be the circumcenters of the tetrahedrons $ABCD$, $IBCD$, $ICDA$, $IDAB$, $IABC$. Produce the segments OO_a, OO_b, OO_c, OO_d by the lengths $O_aA'=O_bB'=O_cC'=O_dD'=r/2$. (1) Prove that the circumspheres of $A'BCD$, $B'CDA$, $C'DAB$, $D'ABC$ are respectively tangent to the pairs of spheres (I), (I_a); (I), (I_b); (I), (I_c); (I), (I_d). (2) Examine the cases where (I) is replaced by each of the escribed spheres corresponding to pairs of opposite edges, when they exist.

Dedicated to N. A. Court who has studied the tetrahedron so extensively.

3853. *Proposed by N. A. Court, University of Oklahoma.*

Two tangent spheres (A), (D) are each touched by the spheres (B) and (C). The two lines joining an arbitrary point of (A) to the points of contact of this sphere with (B) and (C) meet the latter two spheres again in two points coplanar with the two points of contact of (D) with (B) and (C).

3854. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

Prove that

$$(x + \sqrt{y})^4 + (x - \sqrt{y})^4 = z^4$$

has no solution in which x, y, and z are positive integers.

3855. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

Solve in positive integers

$$(x^{1/2} + y^{1/2})^4 + (x^{1/2} - y^{1/2})^4 = z^4.$$

3856. *Proposed by J. H. M. Wedderburn, Princeton University.*

If the fundamental units h_{ij} $(i, j = 1, 2, \cdots, n)$ of an algebra have the law of combination $h_{ij}h_{pq} = k_{jp}h_{iq}$, where k_{jp} is a scalar, the algebra is associative; and a necessary and sufficient condition that it shall be equivalent to the algebra of matrices of order n is that the matrix $K = \|k_{jp}\|$ is non-singular, or that it contains an identity element.

3857. *Proposed by V. Thébault, Le Mans, France.*

In a triangle the minor auxiliary circle of the Brocard ellipse is tangent to the nine-point circle.

Note. The Brocard ellipse for a triangle is tangent to its sides and has for its foci the Brocard points of the triangle.

SOLUTIONS

3762 [1935, 626]. *Proposed by W. P. Udinski, University of Texas.*

Given n urns numbered 1, 2, 3, $\cdots$, n. The jth urn contains w_j white balls and b_j black balls. One of the urns is selected at random and, without replacement, $w+b$ balls are drawn from it. Of this total w balls are found to be white and b black.

Show that the probability that the selected urn be the jth one is given by the formula

$$P_j = W_j / \sum_{i=1}^{n} W_i,$$

where

$$W_i = (w_i + b_i - w - b)!(w + b)! \frac{(w_i)!}{w!(w_i - w)!} \frac{(b_i)!}{b!(b_i - b)!}$$

with the usual conventions that

$$\frac{1}{(w_i - w)!} = 0 \text{ if } w > w_i, \quad \text{and} \quad \frac{1}{0!} = 1.$$

It evidently may happen that for some value of j, $P_j = 1/n$. Hence our increased knowledge is not always sufficient to increase our ability to predict

an event. This particular example (for simple cases) is thus often used to illustrate a possible "faux pas" in reasoning on this subject. See, Poincaré, *Probabilités*, page 26, "Problème des trois coffrets." See also Czuber, *Wahrscheinlichkeitsrechnung*, vol. 1, art. 16. Is the above statement true if each of the urns is known to contain only white balls but not all the same number?

Solution by E. G. Olds, Carnegie Institute of Technology.

The probability of choosing the jth urn is $1/n$. Then the probability of drawing w white balls and b black balls is

$$p_j = \frac{(w+b)!(w_j+b_j-w-b)!w_j!b_j!}{(w_j+b_j)!w!(w_j-w)!b!(b_j-b)!}.$$

This result does not agree with the proposer's W_j. Applying Bayes's theorem, the probability that the selected urn was the jth is given by

$$P_j = p_j / \sum_{i=1}^{n} p_i.$$

In answer to the last question in the problem, it seems apparent that, when each of the urns contains only white balls, p_j is unity and $P_j = 1/n$. The information that all of the urns do not contain the same number of white balls does not aid in prediction.

Solved also by the proposer.

3764 [1936, 49]. *Proposed by J. S. Frame, Brown University.*

Prove that

$$\sinh^{n+1} nx + \cosh^{n+1} nx < \cosh^{n}(n+1)x,$$

for all real positive values of x, and for all real values of n greater than unity. Show that the inequality is reversed when $0<n<1$ and $x>0$; whereas equality holds either if $n=0$ or 1, or if $x=0$.

Solution by the Proposer.

(1). For any positive integer k, and any $n>1$, the binomial coefficients

$$\binom{1+1/n}{2k}$$

are all positive.

(2). If $a>b>0$, $y>0$, we have

$$ae^{-by} - be^{-ay} < a - b,$$

the equality occurring when and only when either $a=b$, $b=0$, or $y=0$.

Proof. With the given conditions $(1-e^{-ay})/ay < (1-e^{-by})/by$, since $(1-e^{-u})/u$ is a decreasing function of u. Hence $b-be^{-ay} < a-ae^{-by}$.

(3). If we set

$$f(t, n) = \frac{(\sinh t)^{1+1/n} + (\cosh t)^{1+1/n}}{[\cosh (n+1)t]^{1/n}}$$

then we find by direct substitution that

$$f\left(\frac{1}{2}\log \operatorname{ctnh} t, \frac{1}{n}\right) = 1/[f(t, n)]^n, \qquad 0 < t < \infty.$$

(4). We next expand in a series the denominator of $f(t, n)$:

$$[\cosh (n+1)t]^{1/n} \cdot 2^{1/n} e^{-(1+1/n)t} = (1 + e^{-2t(n+1)})^{1/n}$$

$$= \sum_{k=0}^{\infty} \binom{1/n}{2k-1} e^{-2t(n+1)(2k-1)} + \binom{1/n}{2k} e^{-2t(n+1)2k}$$

$$= \sum_{k=0}^{\infty} \binom{1+1/n}{2k} e^{-2t(2k)} \left[\frac{2kn}{n+1} e^{-2t(2kn-n-1)} - \frac{(2kn-n-1)}{n+1} e^{-2t(2kn)}\right].$$

Applying (2), with $a=2kn$, $b=2kn-n-1$, $y=2t$, we find that this last bracket is less than unity for $k>0$, $t>0$, $n>1$; for $k=0$, it is unity. Hence

$$[\cosh (n+1)t]^{1/n} \cdot 2^{1/n} \cdot e^{-(1+1/n)t} < \sum_{k=0}^{\infty} \binom{1+1/n}{2k} e^{-2t(2k)}$$

$$= \tfrac{1}{2}(1 - e^{-2t})^{1+1/n} + \tfrac{1}{2}(1 + e^{-2t})^{1+1/n};$$

$$[\cosh (n+1)t]^{1/n} < (\sinh t)^{1+1/n} + (\cosh t)^{1+1/n} \text{ for } t > 0,\ n > 1.$$

This proves that $f(t, n)>1$, for $t>0$, $n>1$. It follows from (3) that $f(x, 1/n)<1$, for $x=\frac{1}{2}\log(\operatorname{ctnh} t)>0$, $n>1$.

(5). We now write

$$\phi(x, n) = f\left(nx, \frac{1}{n}\right) = \frac{(\sinh nx)^{n+1} + (\cosh nx)^{n+1}}{[\cosh (n+1)x]^n}.$$

Then by (4), we have

$$\phi(x, n) < 1, \quad \text{for} \quad n > 1, \quad x > 0;$$

$$\phi(x, n) > 1, \quad \text{for} \quad 0 < n < 1, \quad x > 0;$$

$$\phi(0, n) = \phi(x, 0) = \phi(x, 1) = 1.$$

This completes the proof of the theorem.

It is of some interest to note that the substitution $z=e^{-2x}$ reduces the inequality to the algebraic form:

$$(1 - z^n)^{n+1} + (1 + z^n)^{n+1} < 2(1 + z^{n+1})^n,$$

valid for $0<z<1$, $n>1$, in which the inequality is reversed for $0<n<1$, and the equality holds on all the boundaries. In this form the problem was proposed by J. A. Clarkson.

3765 [1936, 49]. *Proposed by R. P. Agnew, Cornell University.*

The number x, $0<x<1$, is not representable in the form $k/2^n$ where k and n are positive integers. For each n there is a unique integer k_n such that $(k_n-1)/2^n<x<k_n/2^n$. Show that for an infinite set of values of n, x is a point inside the middle third of this interval, i.e., for an infinite set of values of n

$$(3k_n - 2)/(3\cdot 2^n) < x < (3k_n - 1)/(3\cdot 2^n).$$

Solution by S. B. Townes, Norman, Oklahoma.

This theorem is not completely true, for if x is an irreducible fraction of the type $r/2^s\cdot 3$, r a positive integer and s either a positive integer or zero, then for all values of $n\geqq s$ one of the inequality signs must be replaced by equality.

In addition, excluding these values of x, let x be expressed as a radix fraction with 2 as a base, $x=\cdot a_1a_2\cdots$, each of the a's being either 0 or 1.

After a finite number of terms, the remaining terms will not consist exclusively of 1's nor of 0's, nor will 0's and 1's alternate indefinitely; hence at least one of the triples 011 and 100 must occur an infinite number of times.

Let

$$x = .a_1a_2\cdots a_n011\cdots,$$

or

$$x = .a_1a_2\cdots a_n100\cdots;$$

then

$$\frac{k_n}{2^n} = .a_1a_2\cdots a_n111\cdots 1\cdots$$

and

$$\frac{3k_n - 2}{2^n\cdot 3} = .a_1a_2\cdots a_n0101\cdots 01\cdots,$$

$$\frac{3k_n - 1}{2^n\cdot 3} = .a_1a_2\cdots a_n1010\cdots 10\cdots.$$

Thus the given inequalities are satisfied an infinite number of times, i.e., whenever one of the triples 011 or 100 appears.

Solved also by W. B. Campbell.

3766 [1936, 50]. *Proposed by Morris E. Levenson, New York.*

Evaluate

$$\int_0^\infty e^{-x}\log^2 x\,dx.$$

I. *Solution by J. F. Locke, State Teachers College, Memphis, Tennessee.*

From the inequalities

$$0 \leqq e^{-t} - (1 - t/n)^n \leqq t^2 e^{-t}/n, \qquad 0 \leqq t \leqq n, \tag{1}$$

it follows without difficulty that

$$\begin{aligned}
\int_0^\infty e^{-x} \log^2 x dx &= \lim_{n\to\infty} \int_0^n (1 - x/n)^n \log^2 x dx \\
&= \lim_{n\to\infty} \left\{ \sum_{i=0}^n (-1)^i \frac{{}_nC_i}{n^i} \int_0^n x^i \log^2 x dx \right\} \\
&= \lim_{n\to\infty} n \left\{ \log^2 n \sum_{i=0}^n (-1)^i \frac{{}_nC_i}{i+1} \right. \\
&\qquad \left. - 2 \log n \sum_{i=0}^n (-1)^i \frac{{}_nC_i}{(i+1)^2} + 2 \sum_{i=0}^n (-1)^i \frac{{}_nC_i}{(i+1)^3} \right\} \qquad (2) \\
&= \lim_{n\to\infty} \frac{n}{n+1} \left\{ \log^2 n - 2 \log n \sum_{i=1}^{n+1} \frac{1}{i} + 2 \sum_{i=1}^{n+1} \frac{1}{i} \sum_{j=1}^{i} \frac{1}{j} \right\} \\
&= \lim_{n\to\infty} \frac{n}{n+1} \left\{ \log^2 n - 2 \log n \sum_{i=1}^{n+1} \frac{1}{i} + \left(\sum_{i=1}^{n+1} \frac{1}{i} \right)^2 + \sum_{i=1}^{n+1} \frac{1}{i^2} \right\},
\end{aligned}$$

where the last step is easily verified by induction. Hence

$$\int_0^\infty e^{-x} \log^2 x dx = \gamma^2 + \pi^2/6,$$

where γ is Euler's constant. A proof of (1) is given in Whittaker and Watson's *Modern Analysis*, 4th Ed., p. 242. The reduction of the finite sums in (2) may be found in the solution of 3701 [1936, 196–197].

II. *Solution by H. Tate, McGill University, Montreal, Canada.*

The integral

$$\Gamma(a) = \int_0^\infty x^{a-1} e^{-x} dx$$

is known to converge and to possess the following properties when $a > 0$:

$$\Gamma'(a) = \int_0^\infty x^{a-1} e^{-x} \log x dx,$$

$$\Gamma''(a) = \int_0^\infty x^{a-1} e^{-x} \log^2 x dx,$$

$$D^2 \log \Gamma(a) = \frac{1}{a^2} + \frac{1}{(a+1)^2} + \frac{1}{(a+2)^2} + \cdots.$$

[See *Cours d'Analyse Infinitésimale*, 1925, de la Vallée Poussin, Tome II, page 78, §57, and page 82, §60.] Hence

$$\Gamma''(1) = \int_0^\infty e^{-x} \log^2 x dx,$$

$$D^2 \log \Gamma(1) = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots = \frac{\pi^2}{6}.$$

But

$$D^2 \log \Gamma(a) = \frac{\Gamma(a) \cdot \Gamma''(a) - [\Gamma'(a)]^2}{[\Gamma(a)]^2}.$$

When $a = 1$, $\pi^2/6 = \Gamma''(1) - C^2$, since $\Gamma(1) = 1$, where C is Euler's constant. Hence

$$\int_0^\infty e^{-x} \log^2 x dx = \frac{\pi^2}{6} + C^2.$$

Solved also by D. H. Ballou, N. A. Hall, G. E. Raynor, S. B. Townes, and the proposer.

Editorial Note. The two inequalities (1) in solution I are true if n and t are any real numbers such that $n \geqq 1$ and $|t| \leqq n$. The first inequality is obvious from the fact that $y = e^x$ lies above its tangent at $x = 0$. The second is easily seen to be true by considering

$$f(t) = \frac{t^2}{n} - 1 + \left(1 - \frac{t}{n}\right)^n e^t.$$

At $t = 0$ both $f(t)$ and $f'(t)$ are zero. If $f'(t) = 0$ at any other point within the interval, then at such points

$$f(t) = \frac{(t-1)^2}{n} + 1 - \frac{1}{n} > 0.$$

Since neither $f(-n)$ nor $f(n)$ is negative, $f(t)$ can vanish only at $t = 0$, if $n > 1$; but also at $t = n$, if $n = 1$. At all other points we must have $f(t) > 0$, and this proves the second inequality.

Since the integral of the problem converges, it suffices to use positive integral values of n for its evaluation. The method of I may be used for other positive integral powers of $\log x$ in the integrand, but it appears that no such simple form for the result is known when the exponent is odd and greater than unity.

The remaining solutions used differentiation of the Gamma function expressed as an integral as in II. Raynor mentioned that the integral can be evaluated for other positive integral exponents of $\log x$ in the same manner. Townes gave a second solution using

$$\Gamma(x+1) = \lim_{n=\infty} \frac{n! n^x}{(1+x)(2+x)\cdots(n+x)}.$$

3767 [1936, 50]. *Proposed by N. A. Court, University of Oklahoma.*

Construct four spheres such that the sphere of similitude* of any two shall pass through the centers of the remaining two spheres.

Solution by the Proposer.

Let (A), (B), (C), (D) be the required spheres, centers A, B, C, D. If the sphere of similitude of the spheres (A), (B) passes through C, the bisectors of the angles formed by the lines CA, CB will meet the line AB in the centers of similitude of the spheres (A), (B). Similarly for the point D. Thus the edge AB of the tetrahedron $DABC$ is met in the same two points by the bisectors of the two face angles ACB, ADB. Likewise for the other edges of $DABC$. Consequently the points A, B, C, D must be so chosen that the tetrahedron $DABC$ shall be isodynamic. (Ibid. p. 281, ex. 2.)

If the points A, B, C, D are taken so as to satisfy this condition, the four spheres having these points for centers and having the squares of their radii proportional to the products of the triads of edges of the tetrahedron $ABCD$ passing through the respective vertices constitute a solution of the problem. (Ibid. p. 286.)

3768 [1936, 50]. *Proposed by N. A. Court, University of Oklahoma.*

Construct a tetrahedron given in position the midpoints of the six edges.

Solution by Roy MacKay, Junior College, Portales, New Mexico.

A necessary and sufficient condition that the six points be the midpoints of the edges of a tetrahedron is that the ${}_6C_2$ line segments determined by the given points form an octahedron in which the centroid bisects each of the three diagonals. Consequently, opposite faces are symmetric triangles with respect to the centroid. Hence if five of the points are given the sixth is uniquely determined. Mark consecutive faces of the octahedron which concur at one vertex with the letters A, B, C, D and denote the opposite faces by A', B', C', and D', respectively. The planes of A, C, B', D', and those of B, D, A', C' determine two tetrahedrons symmetric with respect to their common centroid.

To show that the given points are the midpoints of the edges of these tetrahedra it is sufficient to show that the vertices of the triangle A are the midpoints of the edges on one face of the first tetrahedron, since identical arguments can be used for other edges.

The planes A and C contain opposite sides of a parallelogram AC' and $A'C$ and therefore determine a line AC parallel to each of these. Similarly AB and AD are respectively parallel to the lines AB' and AD'. That is the edges on one face of the first tetrahedron are lines through the vertices of the triangle A parallel to its opposite sides. Hence the vertices of triangle A are midpoints of the edges AC, AB', and AD' of the tetrahedron $ACB'D'$.

Solved also by W. B. Campbell, J. M. Feld, G. E. Raynor, and S. Vatriquant.

* See, for instance, Modern Pure Solid Geometry by the proposer, 1935, The Macmillan Company, p. 153.

Corrected statement of 3768 [1936, 50, 378]. *Proposed by N. A. Court, University of Oklahoma.*

Construct a tetrahedron, given in position the midpoints of four of the edges. (Correction of an error in editing.)

Solution by J. Rosenbaum, Bloomfield, Conn.

Case I. The given points P_1, P_2, P_3, P_4 lie in a plane. If the four points do not form a parallelogram, no solution is possible. If they form a parallelogram the construction is obvious; and there is an infinite number of solutions.

Case II. The four points do not lie in a plane. Construct the triangle ABC such that P_1, P_2, P_3 are the midpoints of its sides. Join a vertex of this triangle, such as A, to P_4, and produce AP_4 its own length to D. Then $ABCD$ is a required tetrahedron. It is clear that there are twelve solutions, each tetrahedron having a volume equal to eight times that of the tetrahedron $P_1P_2P_3P_4$.

Editorial Note. The twelve tetrahedrons may be paired by a formulation of the above construction for II so that pairs have the same six midpoints of edges.

Solved also by the proposer.

3769 [1936, 110]. *Proposed by Don Wallace, Charlottesville, Va.*

If the medians of a triangle be extended to meet the circumcircle in points forming a second triangle, then the centroid of the given triangle is that point for which the product of the distances to the vertices of the second triangle is an extreme.

Solution by C. M. Sparrow, University of Virginia.

Let $A_1A_2A_3$ be the first triangle, with centroid at O. Let B_1, B_2, B_3 be the midpoints of the sides, and $C_1C_2C_3$ the second triangle. Let r_1, r_2, r_3 denote the lengths of OA_1, OA_2, OA_3; s_1, s_2, s_3 those of OC_1, OC_2, OC_3. By an elementary property of the circle r_ks_k is constant. Let θ_1, θ_2, θ_3 be the angles between s_2, s_3; s_3, s_1; s_1, s_2, respectively.

If $s_1s_2s_3$ is an extreme, so also is $\log s_1+\log s_2+\log s_3$. But this is the logarithmic potential at O of equal charges or masses at C_1, C_2, C_3. The theorem to be proved is therefore equivalent to proving that O is a point of equilibrium of forces $1/s_1$, $1/s_2$, $1/s_3$ directed toward C_1, C_2, C_3; that is, of forces r_1, r_2, r_3 toward these points. By Lamé's theorem

$$\sin\theta_1 : \sin\theta_2 : \sin\theta_3 = r_1 : r_2 : r_3$$

and the theorem is proved if we establish this relation. O is the centroid of the triangle $B_1B_2B_3$; hence the triangles B_2OB_3, B_3OB_1, B_1OB_2 are of equal area. Also $OB_k = r_k/2$. Hence

$$r_2r_3\sin\theta_1 = r_3r_1\sin\theta_2 = r_1r_2\sin\theta_3;$$

dividing by $r_1r_2r_3$ we get the required relation.

Solved also by C. E. Springer, S. Vatriquant, and the proposer.

3770 [1936, 110]. *Proposed by N. A. Court, University of Oklahoma.*

Given four spheres, the spheres orthogonal to one of them and belonging to the coaxal net determined by the remaining three form a coaxal pencil. The four lines of centers of the four pencils thus obtained form a hyperbolic group of lines.

Note. The corresponding property of three circles was discussed in *Nouvelles Annales de Mathématique*, 1852, p. 198, and 1858, p. 79. Q. 195.

In regard to coaxal pencils see, for instance, N. A. Court, *Modern Pure Solid Geometry*, Macmillan, 1935, p. 200, ex. 4.

Solution by the Proposer.

Let (R) be the orthogonal sphere of the four given spheres (A), (B), (C), (D). The spheres of the coaxal pencil (a) orthogonal to (A) and belonging to the coaxial net determined by (B), (C), (D) are also orthogonal to the sphere (R). Thus any sphere (M) of the coaxial pencil (a) has its center in the radical plane σ of the two spheres (A) and (R), i.e., the line of centers a of (a) lies in σ. But (R) and (A) are orthogonal; hence σ is the polar plane of the center A of (A) with respect to (R).

Now a obviously lies in the plane BCD determined by the centers B, C, D of the spheres (B), (C), (D); consequently a is the line of intersection of the face BCD of the tetrahedron $ABCD$ with the polar plane of the opposite vertex with respect to the fixed sphere (R). Similarly for the analogous lines b, c, d. Hence the four lines a, b, c, d form a hyperbolic group, according to a theorem due to Michel Chasles, see Gergonne's *Annales de Mathématiques*, vol. XIX, 1818–1819, p. 76 and *Aperçu historique* . . . , p. 692. Or see H. F. Baker, *Principles of Geometry*, vol. III, p. 41, ex. 7.

Editorial Note. We shall give a proof of the theorem in the references which is more compact. Let T and T' be two polar reciprocal tetrahedra with respect to a quadric surface S, i.e., the polar of the vertex D of T is the plane of the face $A'B'C'$ of T', etc. Let $B'C'$ cut the plane of DBC in L, $C'A'$ cut DCA in M, and $A'B'$ cut DAB in N. We shall prove first that L, M, N lie in a straight line. If D, B, B' lie in a straight line, then B', L, N coincide and the theorem is obvious. If we exclude this case, DBB' determines a plane cut by the straight line $MA'C'$ in W. Let DB cut the plane of $A'B'C'$ in B_1. The polar planes of C' and A' are, respectively, $DABN$ and $DBCL$; the polar plane of M is $DBB'W$; that of W is DBM. We have then a pencil of planes with the axis DBB_1 which is projective with its section by the plane of $A'B'C'$, i.e., with the pencil $B_1\,(N, L, W, M)$, and also with the range of points C', A', M, W, and with the pencil $B'(L, N, M, W)$. Hence

$$B_1(N, L, W, M) \barwedge B'(N, L, W, M).$$

But B_1, B', W are in a straight line, hence L, M, N also lie on a straight line δ'.

Denote by α the intersection of the faces DBC and $D'B'C'$; and denote the other similar intersections by β, γ, δ. Obviously, δ' cuts δ, and it cuts α, β, γ in L, M, N, respectively. Hence there are four straight lines, α', β', γ', δ', each of which cuts each one of α, β, γ, δ. Such sets of four straight lines may be called a hyperbolic tetrad. If two lines of the tetrad intersect, say α and β meet in P, the four planes DBC, $D'B'C'$, DCA, $D'C'A'$ meet in P. Hence DC and $D'C'$ meet in P. Moreoever, $A'B'$ and AB meet in a point Q and also γ and δ meet in Q. The points P and Q are conjugates. Hence if no two corresponding edges of T and T' lie in a plane, then no two of the set α, β, γ, δ meet in a point.

It is easily shown that AA', BB', CC', DD' is a hyperbolic tetrad. For the conjugate of δ' is a straight line δ_1 passing through D. The polar plane of L passes through δ_1, A', A. Hence δ_1 cuts AA'; similarly it cuts BB' and CC'; obviously it cuts DD'. Thus each of the straight lines α_1, β_1, γ_1, δ_1 cuts each of AA', BB', CC', DD'. If any two of the latter intersect, say AA' and BB', then AB and $A'B'$ intersect in Q. It then follows that CD and $C'D'$ meet in P. Hence the other two, CC' and DD', meet in a point.

The altitudes of a tetrahedron give an interesting example of a hyperbolic tetrad. See the solution of 3697 [1936, 190–193].

A QUOTATION FROM E. T. BELL

"It may be emphasized once for all that a close description of how mathematics handles the problem of continuous change is neither possible nor desirable in a first approach. Further, to turn aside for a moment if I may be allowed to vent a heretical opinion, I strongly disbelieve in ever giving either engineers or mathematical physicists a rigorous course in the calculus—the mathematics of continuous change. By rigorous I mean going right down to the ambiguous logical roots of the number system. Any professional mathematician who is not also an analytical bigot knows that the foundations of analysis are in a terrible mess. And any scientifically literate mathematician who follows what mathematical physicists are doing with this analysis, and who criticizes them for their bold use of their dangerous mathematical machinery, proclaims himself a mathematical bigot of the first magnitude." E. T. Bell, *The Handmaiden of the Sciences*, pp. 62–63. Reynal & Hitchcock, New York, 1937.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending new items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

Several Benjamin Peirce Instructorships at Harvard University for the year 1938–39 are open to men who have the degree of Ph.D., or its equivalent. Applications should be sent to the Chairman of the Division of Mathematics.

A Mathematical Colloquium, similar to those held in 1926, 1930, and 1934, will be held in St. Andrews, Scotland, from July 4 to July 15, 1938, under the auspices of the Edinburgh Mathematical Society. Short courses of lectures will be given by prominent mathematicians on topics of pure and applied mathematics. On their way to the Colloquium, members will have the opportunity of attending a meeting of the Royal Society of Edinburgh on July 4, 1938 at Edinburgh. For particulars concerning membership of the Colloquium, apply to the Hon. Secretary, Edinburgh Mathematical Society, 16 Chambers St., Edinburgh 1.

The University of Notre Dame announces a symposium on the algebra of geometry and related subjects to be held on February 11 and 12, 1938. Among those who will participate are A. A. Albert, Emil Artin, Garrett Birkhoff, E. V. Huntington, Georges Lemaître, Karl Menger, John von Neumann, Oystein Ore, and M. H. Stone.

Professor R. B. Allen of Kenyon College is on leave of absence for the current academic year.

Dr. C. H. Fischer has been promoted to an assistant professorship at Wayne University.

John Gaffney and W. A. Longacre, of the Michigan College of Mining and Technology, have been promoted to assistant professorships.

Dr. Edwin Oberg has been appointed an instructor at the University of Iowa.

At the annual dinner in New York City of the Radio Club of America on October 29 Dr. L. A. Hazeltine, professor of mathematics at Stevens Institute of Technology, was presented with the Armstrong Medal and citation for 1937 "in recognition of his outstanding contributions to the art and science of radio communication."

I. F. Keller, of the University of Iowa, has been appointed to a professorship at North Central College, Naperville, Illinois.

Canon G. Lemaître will be at the University of Notre Dame during the spring term of the current year. Dr. Kurt Gödel, who in an earlier issue of this MONTHLY was reported to have joined the staff at Notre Dame, will not be there in the immediate future.

The William Lowell Putnam Mathematical Competition

For a period of at least three years beginning with the spring of 1938 an intercollegiate mathematical competition is to be held annually designed to stimulate a healthful rivalry in the undergraduate work of mathematical departments in colleges and universities of the United States and Canada. This is made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund, left by Mrs. Putnam in memory of her husband, a member of the Harvard class of 1882. It is hoped that such a competition will further the spirit of intercollegiate scholastic rivalry, in the possibility and value of which Mr. Putnam thoroughly believed. The competition is open only to undergraduates who have not received a degree.

The Mathematical Association of America stands sponsor to the plan, which is operated in its name.

The general regulations governing the contest and award are as follows:

1. The Association will issue annually in the American Mathematical Monthly a general invitation to the colleges and universities of the United States and Canada to participate in the contest.

Any college or university desiring to compete will send to the Secretary of the Association (or to the person selected by him and approved by the President) the names of the three persons who are to be the mathematical team of the institution in question for the year. Any college or university desiring to have individual undergraduates compete will likewise inform the Secretary to this effect; such persons may be asked to take the examination at some near-by institution where the examination is being given. The names of the approved candidates are to be forwarded by the head of the department of mathematics and must be in the hands of the Secretary, or his delegate, not later than April 1. This person will see that the head of the department is notified of the receipt of the list and that each member of the team or each individual candidate is assigned a numeral which alone is to appear on the examination book by way of identification.

If three candidates are presented from a college or university, they are to constitute a team. The examination can be given at any place where a team or at least three candidates can be assembled; exceptions to this rule may be made by the Secretary, or his delegate, in case of unusual circumstances.

2. The examinations will be constructed so as to test not only technical competence but also originality. Examination questions will be taken from the fields of calculus (elementary and advanced) with applications to geometry and mechanics, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid analytic geometry). The groundwork of knowledge required for the examination is such as would usually be met by two years (twelve semester hours) in the calculus, a half year's work (three semester hours) each in higher algebra and differential

equations, and a year's work (six semester hours) in analytic geometry. The student will be allowed some choice in the questions.

The President of the Association will appoint a committee of three (not necessarily members of the Association) which will select two examiners, mathematicians qualified and willing to undertake the preparation of the examination, and a reader qualified to grade the examination books; in view of the special relation of the founder of the Fund to Harvard University, one member of the committee shall be a member of the Division of Mathematics at Harvard University. The examiners will furnish the Secretary, or his delegate, sealed copies of the examination by April 1.

3. The date of the examination will be announced by the Secretary in the MONTHLY not later than March 1 each year. The examination shall consist of two periods of exactly three hours each and shall be given from nine to twelve and from two to five on the announced day, under the official supervision of the department of mathematics in the institution. The sealed envelope containing the examination shall be opened by an authorized representative of the department at the beginning of the examination period, and immediately after the close of the examination period the books shall be sent through the person in charge of the examination by registered mail to the Secretary of the Association, or his delegate, and will be forwarded by him to the reader. The reader will grade the books and return them listed in order of rank, within a calendar month after the examination date.

The Secretary, or his delegate, will then determine the first three teams in order of merit, in the following manner: the rank of any team shall be defined as the sum of the ranks of the contestants on the team. In case of ties among the teams, all of those tying shall be counted as far as necessary to obtain at least three winning teams.

4. There shall be awarded to the departments of mathematics of the institutions who have the first three winning teams the sums of $500, $300, $200 in the order of their rank. When there are ties, these sums shall be equally divided among the teams which tie for the respective positions. If books are purchased with part or all of the money, suitable bookplates indicating that the books have been obtained as a result of winning the William Lowell Putnam Mathematical Competition may be obtained from the Secretary of the Association.

There shall be awarded to each member of the three winning teams and to each of the first five contestants a suitable medal indicating his success in the Competition and, in addition, to each member of the three winning teams a prize of fifty dollars, thirty dollars, or twenty dollars, according to the rank of his team, and to each of the first five contestants a prize of fifty dollars.

5. The names of the institutions with the three winning teams in the order of their rank, the names of the members of these teams, the names of the five persons whose positions in the examination are highest (without indicating their relative position), and the names of all institutions which entered teams (like-

wise without indicating their relative positions), shall be published in the MONTHLY. In case of ties the list may be increased as far as necessary.

6. For further encouragement of the Competition, there will be awarded at Harvard University* an annual $1000 William Lowell Putnam Prize Scholarship to one of the first five contestants, this scholarship to be available for one of the two following academic years, at the option of the recipient.

The records will be available to all institutions interested, and the Association is sponsoring this plan in the hope that other institutions may find it possible to offer scholarships to winners in the contest.

7. For purposes of record, all the examination books will be sent to the office of the Association each year.

8. The conditions of the plan herein formulated are subject to such modification as seems desirable to the Association and the trustees of the Fund.

Note. The Secretary of the Mathematical Association, or his delegate, will announce the date of the examination and details as to the places of the examination, the means of obtaining application blanks, and so on, in the near future, both through the AMERICAN MATHEMATICAL MONTHLY and by circular letters to the departments of mathematics of the colleges and universities in the United States and Canada.

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* Or at Radcliffe College, in the case of a woman.

THE ANNUAL MEETING OF THE KENTUCKY SECTION

The twentieth annual meeting of the Kentucky Section of the Mathematical Association of America was held in joint session with the Kentucky Academy of Science at the University of Louisville, Louisville, Kentucky, on Saturday, May 1, 1937. W. L. Moore, Chairman, presided at both sessions.

The attendance was twenty-eight, including the following sixteen members of the Association: N. B. Allison, P. P. Boyd, M. C. Brown, L. W. Cohen, A. R. Fehn, Georgia M. Haswell, Charles Hatfield, W. R. Hutcherson, Fritz John, C. G. Latimer, F. Elizabeth LeStourgeon, W. L. Moore, D. W. Pugsley, J. H. Simester, Guy Stevenson, C. W. Williams.

Professor C. G. Latimer gave the principal paper. The physicists of the Academy had luncheon with the Section members at the Puritan Apartments. Professor D. E. South, University of Kentucky, was chosen as Chairman, and Professor A. R. Fehn as Secretary for the year 1937–38. Voluntary contributions were made to cover the expense of the Secretary's correspondence.

The following six papers were read:

1. "Continuity" by Professor L. W. Cohen, University of Kentucky.
2. "On inequalities" by Professor Fritz John, University of Kentucky.
3. "Echoes from the annual meeting of the National Council" by Dawn Gilbert, Western Kentucky State Teachers College, introduced by the Chairman.
4. "Note on the reduction of quadratic forms" by C. W. Williams, University of Kentucky.,
5. "On the correspondence between ring ideals and field ideals in a quadratic field" by K. L. Palmquist, University of Kentucky, introduced by the Chairman.
6. "Recent developments in quaternion arithmetic" by Professor C. G. Latimer, University of Kentucky

Abstracts of some of the papers follow, the numbers corresponding to the numbers in the list of titles.

2. In his paper Dr. John showed how various classical inequalities, *e.g.*, those of Hölder and Minkowski, can be easily derived from the geometric theorem that the center of mass of a set of points with positive masses lies in the convex extension of the set.

3. Echoes from the annual meeting of the National Council given by Dawn Gilbert, the state representative of the Council, presented the crisis in mathematics as due to a failure to recognize the change in the high school population since 1908, the failure to provide courses in mathematics that would meet the needs of the majority of ninth grade pupils, and a failure to provide courses for the brighter pupils who do not need their courses in mathematics diluted. Revision of the present course of study, reform in teaching methods and in aims, better trained teachers, and interest in mathematics created by illustrative material, projects, and stories of the great discoveries in elementary mathematics were suggested as ways of meeting this crisis.

4. A very elegant transformation, which reduces a given quadratic form to the sum of squares, is due to Jacobi. The applicability of this transformation, however, depends on the non-vanishing of certain principal minors of the matrix of the form. Mr. Williams shows in this paper, that upon application of a very simple transformation, if necessary, the method of Jacobi is applicable in every case.

5. Let n be a positive rational integer and 1, θ be a basis of the integral set G of a quadratic field. The set R of numbers in the form $a+bn\theta$, where a, b range over G, is a ring in G. Mr. Palmquist employs Bachmann's proof to show the existence of a one-to-one correspondence between the regular ring ideals in R and the field ideals in G which are prime to n. This correspondence is preserved under multiplication.

A. R. FEHN, *Secretary*

THE TWELFTH ANNUAL MEETING OF THE PHILADELPHIA SECTION

The twelfth annual meeting of the Philadelphia Section of the Mathematical Association of America was held at Haverford College, Haverford, Pennsylvania, on Saturday, November 27, 1937, Professor H. H. Mitchell presiding.

The attendance was fifty-seven, including the following thirty-seven members of the Association: J. A. Benner, William Beverley, H. W. Brinkmann, L. H. Bunyan, P. A. Caris, J. W. Clawson, D. R. Davis, J. E. Davis, Arnold Dresden, H. F. Fehr, Tomlinson Fort, Michael Goldberg, H. S. Grant, W. L. Graves, E. H. C. Hildebrandt, J. R. Kline, V. V. Latshaw, D. H. Lehmer, W. F. Long, F. L. Manning, H. H. Mitchell, Richard Morris, W. R. Murray, C. A. Nelson, C. O. Oakley, F. W. Owens, Helen B. Owens, T. S. Peterson, G. E. Raynor, J. B. Reynolds, C. A. Shook, W. M. Smith, E. P. Starke, R. M. Walter, A. H. Wheeler, A. H. Wilson, C. R. Wilson.

At the business meeting the following officers were elected for next year: Chairman, L. L. Smail, Lehigh University; Secretary, P. A. Caris, University of Pennsylvania; Program Committee, W. M. Smith, C. A. Nelson, G. A. Hedlund. It was agreed to hold the next meeting at Ursinus College, Collegeville, Pa., on Saturday, November 26, 1938.

The following papers were presented:

1. "Farey series" by Professor H. S. Grant, Rutgers University.

2. "Some multiple perspective relationships" by Professor F. W. Owens, Pennsylvania State College.

3. "On the Bernoulli numbers and the Von Staudt-Clausen theorem" by Professor Hans Rademacher, University of Pennsylvania, introduced by Professor Mitchell.

4. "Stellated polyhedra, illustrated by models" by A. H. Wheeler, North High School, Worcester, Mass.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. Professor Grant stated that there are three well known properties exhibited by the terms of a Farey series of order n: (1) If a/b and c/d are two consecutive terms, then $ad-bc=\pm 1$. (2) $b+d \geqq n+1$. (3) If a/b, e/f, c/d are three consecutive terms, then $e/f=(a+c)/(b+d)$. The third property, that actually stated by Farey, is an immediate consequence of the first. Many proofs have been given establishing the first property. It was the purpose of this paper to lay emphasis on the third property, and to show that by it we can generate a Farey series. It was emphasized further the fact that the properties of Farey series are essentially those of integers, and effect a generalization. Three advantages of this approach are: (1) The generating of a Farey series by the most natural operation of arithmetic, addition. (2) The fact that we can write down immediately the successive terms of any Farey series in order of magnitude. (3) The establishing of a correspondence between the terms of a Farey series and the points of rational sub-division on a linear segment in euclidean space.

2. Professor Owens's paper was mainly expository in character, giving some of the main results of Rosanes, Schröter, Hess, Clebsch, and others, with some new material, including the theorem: It is impossible to construct three distinct triangles, real or imaginary, for which each triangle is in six-fold perspective with each of the others.

3. Professor Rademacher stated that the general binomial coefficient of the jth order $\binom{x}{j}=[x(x-1)\cdots(x-j+1)]/j!$ can be regarded as an integral-valued polynomial of the jth degree. Each integral-valued polynomial can be expressed as a linear combination of binomial coefficients by means of integral coefficients. These coefficients A_{qj} were studied in some detail for the expansion of the qth power of x. The formula thus obtained can be used for the summation of consecutive qth powers. The resulting formula yields the $(q+1)$th Bernoulli polynomial and hence an expression for the Bernoulli numbers, in which the coefficients A_{qj} appear. The Von Staudt theorem is then a simple and direct consequence of a congruence property of those A_{qj}.

4. Mr. Wheeler took his audience into an interesting and little explored field. He showed by means of paper models some very unusual transformations of solids. Archimedean solids were derived by dissecting actual models of the regular solids and removing parts. From a cube were obtained four congruent octahedra and a cubo-octahedron inscriptible in the original cube. Another cube was separated into four congruent regular octahedra and two orthogonally intersecting regular tetrahedra, forming a Keplerian "Stella Octangula." A cube was turned "inside-out" to produce two congruent tetrakaidekahedra, or Archimedean solids (number IV Bruckner). A cube was cut into eight congruent cubes whose parts were hinged together in such a way that when rearranged in a different order there were produced two congruent rhombic dodecahedra of the second species. One of these was then dissected, and from it were obtained two congruent rhombic dodecahedra of the first species, and when one of these

was dissected and turned "inside-out" there were recovered two of the eight cubes into which the original cube was subdivided. Other dissections and transformations were shown, and in addition there were exhibited groups of intersecting solids and also new higher forms of the regular pentagonal dodecahedron and the regular icosahedron. In addition to the paper models, Mr. Wheeler showed photographs of many forms of stellated polyhedra.

P. A. CARIS, *Secretary*

THE FOUNDATION OF ARITHMETIC*

N. J. LENNES, University of Montana

The purpose of the present brief paper is to give a somewhat critical analysis of the foundation of arithmetic in terms of the theory of sets. It is desired, on the one hand, to make this matter quite intelligible for those of limited formal education in mathematics and with little or no information about more recent work in the foundation of mathematics, and on the other hand to suggest a mode of treatment of the foundation of arithmetic, and hence of analysis, that can easily be made rigorous and as complete as that which had been developed on the usual axiomatic basis. Indeed, the question is raised as to whether the treatment proposed here is not more closely in consonance with natural ways of thinking than is the treatment usually adopted. In fact it seems safe to assert that such a simple idea as $3+4=4+3$ could never occur to a human being as we know him except as an abstraction from his dealing with groups of things, that is, from his experience with sets. We shall deal with these considerations more in detail a little later.

1. General background. Gradually, at various stages in the development of mathematics, there arose a desire to use purely deductive reasoning. The question at issue came to be, not whether a proposition is "true," but whether it is a logical consequence of certain other propositions. This question is clearly in the foreground in Euclid's *Elements*. At that early date it had led to a definite, and it must be said highly successful, effort to state a minimum body of propositions from which all the others should follow by logical necessity. Much later a similar effort was made in the field of pure numbers and the operations on them, that is, in the field of arithmetic. It was only in the first decades of the nineteenth century that the five identities,

$$a+b=b+a,\ a+b+c=a+(b+c),\ ab=ba,\ abc=a(bc),\ a(b+c)=ab+ac,$$

came to be stated explicitly as fundamental axioms of arithmetic. When these propositions are admitted for all positive integral values of a, b, c, they can be proved for all other real numbers provided these are introduced either by appropriate definitions or by axioms giving their relations to the integers.

As "pure mathematicians" we are not supposed to inquire into the "validity"

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or "truth" of these axiomatic propositions. Indeed we do not inquire into the meaning of "validity" or "truth" as related to the axioms. But the ideas of numbers, and of addition and multiplication, did not descend from the skies; they grew up down here on this earth, the legitimate off-spring of human intelligence and of experience. Man learned to work with numbers and to make them serve him in most vital ways long before he came upon the highly sophisticated (might we say almost artificial?) ways of thinking of those who have worked in the foundations of mathematics.

The theory of sets (sets of points) furnishes a natural opportunity for a re-examination of the way in which these fundamental propositions are related to the experiences on which historically and psychologically they are based. The ideas involved are extremely simple: We have, for instance, two baskets A and B, each partly filled with apples. We may empty basket A into B, or we may empty basket B into A. Will the resulting basket of apples have the same set of apples in each case and will the number of apples be the same? Do the answers require an axiom or a proof? Especially, does the first question require an axiom? If so, we need axioms for the simplest elements of the theory of sets. In any event no axioms for sets are here attempted. It is simply recognized that historically our ideas of arithmetic arose out of experience with groups of things, sets if you will. That is how the child now learns about numbers. It may not be amiss to make our logical development follow this course that we all have trod in our first learning. For this purpose we will begin with the most elementary of arithmetic.

2. Early beginnings. Beyond question numbers had their inception historically in the counting of "things." Counting as the word is here used, includes the recognition of the singleness or multiplicity of objects, or events, that is, of things, whether this recognition arises from "counting" or in some more direct way.

We can only conjecture details. For the purpose of communication, and possibly also for remembering, words indicating degree of multiplicity came into use, the equivalents of "one," "two," "three," and so on. No doubt such words were at first entirely concrete, as "two men," "three women," "four horses," etc. Gradually there arose, as abstractions, the ideas of oneness, twoness, threeness, etc. dissociated from any particular object or thing. These words came to be arranged in order, one following another, and in this way man equipped himself with a series of number words. The counting of *any* group of objects then must have come quite easily. Possibly there was a direct recognition of the number of objects in very small groups. In fact it is quite likely that for purposes of conduct this may have preceded even the very simplest "counting." It is likely that dogs recognize oneness, twoness, or threeness for the purposes of adjusting themselves to the exigencies of life.

The next step in developing the use of numbers was, no doubt, the "adding" of groups of things. "If one has one group of three things and adds to it a group of four things, how many things will there be in the combined group?" The ob-

vious way to find the answer is to count the number of things in the combined group. Another way, nearly as obvious, is to remember that there are three in the first group and then to count "four, five, . . . " until the objects in the second group are used up.

There follows then a process of abstraction resulting in $3+4=7$, which is applicable to the adding of all groups of objects (or events) containing these numbers. At this point a question arises: We start with 3 and count forward 4 steps. But suppose we start with 4 and count forward 3 steps. How will the results compare? This results in $3+4=4+3$. That is, we have the beginning of the commutative law of addition. If, however, we think of the groups of three and four objects as taken together to form one group, it is evident that the "order" in which the constituent groups are taken has no effect on the resulting group, since this will consist of the objects of the two "added" groups, the idea of "order" being in no sense involved. It is at this point that the substance of this paper really begins. A digression now becomes necessary.

3. Theory of sets. In connection with the study of real functions there was developed during the last century an extensive theory of "sets of points." This theory was at once made to cover sets of any kind of "things" (objects or events). Words such as class, assemblage, menge, etc. are used to represent the same concept as the word set.

A set consists of a collection of things (elements) so described that it is thereby determined whether or not a given object belongs to the set. Thus we speak of the set of all integral numbers, or the set of all rational numbers, or the set of all transcendental numbers. Again all numbers between two given numbers form a definite set, as do also all integers between any two numbers.

In working with sets the idea of one-to-one correspondence is fundamental. Two sets are said to be in one-to-one correspondence if all the elements of one set are paired with all the elements of the other so that no element in either set is used more than once. Thus in analytic geometry the set of all real numbers is in one-to-one correspondence with the set of all points on a line.

In this way we can say that two sets are equal though we have no information about the "number" of elements in either set. Two sets are said to have the "same number," or to be "equal," if they are capable of being put in one-to-one correspondence. This connects very directly and closely with the commonsense view of things. Thus, if in a crowd of men every man wears a hat, we know that the number of men is equal to the number of hats, though we do not count either set (the set of men or the set of hats).

In ordinary counting we set out number-words in one-to-one correspondence with the objects that are counted. This leads at once to a correspondence between adding and multiplying ordinary numbers on the one hand and adding and multiplying sets on the other.

4. Finite sets. Consider the integers 0, 1, 2, $\cdots$ up to some fixed integer as 100. This set of integers has the following properties:

1. There is a first integer in the series.

2. Every integer, except the first, has one immediately preceding it and every integer, except the last, has one immediately following it.

3. There is a last integer in the set.

All sets whose elements are capable of this type of arrangement are said to be *finite*. All other sets are said to be *infinite*. Thus the set of integers (positive or negative) between any two integers is finite, while the set of all positive integers is infinite since there is no last one among them. The set of all integers, negative and positive, is infinite since it has neither a first nor a last (when taken in its natural arrangement). We will not consider here the possibility of arranging infinite sets in the form of the arrangement found in the set of all positive integers when in the natural order, since in this paper we are dealing only with finite sets.

Suppose we have two finite sets $[a]$ and $[b]$ that are capable of being put into one-to-one correspondence. Then, no matter how the elements in $[a]$ and $[b]$ are arranged in order, the two sets will be in one-to-one correspondence preserving this order. That is, the order of the elements will put the sets into one-to-one correspondence. This is easily seen when we consider the permuting of the elements of either set. We note that counting the number of elements in a set is a process of putting these elements into one-to-one correspondence with the integers that we use in counting.

5. Addition; the commutative and associative laws of addition. For the purpose of further discussion of sets we will denote a set by a symbol like $[a]$ in which the a's, distinguished as $a_1, a_2, \cdots$, or in any other way, denote the elements of the set. Consider any two sets $[a]$ and $[b]$ that have no common elements. Then the *sum* $[a]+[b]$ of these sets is a set $[a, b]$ whose elements are the elements of $[a]$ and also of $[b]$. That is, every element of $[a]$ and also of $[b]$ is an element of $[a, b]$ and every element of $[a, b]$ is an element of $[a]$ or of $[b]$.

Let a be the number of elements in $[a]$ (obtained by ordinary counting) and b the number of the elements in $[b]$ obtained in the same way. Then the fundamental property of addition in arithmetic is that $a+b$ is the number of the elements in $[a, b]$. Now consider the two sums $a+b$ and $b+a$ and the corresponding sets $[a, b]$ and $[b, a]$. The two sets contain exactly the same elements and hence can be set into one-to-one correspondence with the same set of positive integers, no matter in what order the elements of $[a, b]$ are arranged. It follows, therefore, that $a+b=b+a$, which is the *Commutative Law of Addition.*

Consider now the three finite sets $[a]$, $[b]$, $[c]$, no two sets having an element in common, and the corresponding numbers a, b, and c. The set $[a, b, c]$ is the sum of these three sets. By an argument strictly analogous to the above we can now show that $[a]+[b]+[c]$ may be set into correspondence with the integers up to $a+b+c$ and that the order in which the sets, and hence the numbers, are taken makes no difference in the sum. That is, $a+b+c=a+(b+c)$, which is the *Associative Law of Addition*. We may note that $a+b+c$ is taken to mean that we start with a, add b to it, and then add c to the sum, while in $a+(b+c)$ we add c to b and then add this sum to a.

6. Multiplication; the commutative and associative law of multiplication. Consider again the two finite sets $[a]$ and $[b]$. The product of these sets is denoted by $[ab]$, which is a set whose elements consist of all pairs of elements, one part of each pair being an element of $[a]$ and one part an element of $[b]$. To illustrate, let $[a]$ consist of the elements a_1, a_2, a_3, and $[b]$ consist of the elements b_1, b_2, b_3, b_4. Then the elements of the product set are the pairs a_1, b_1; a_1, b_2; a_1, b_3; a_1, b_4; a_2, b_1; a_2, b_2; a_2, b_3; a_2, b_4; a_3, b_1; a_3, b_2; a_3, b_3; a_3, b_4. It is obvious that this consists in taking 4 three times, that is, in multiplying 4 by 3. In this multiplication it is agreed that a *pair* as a, b involves no element of order. That is a, b and b, a are the same element. Hence it follows that $[a]\times[b] = [b]\times[a]$. It is an immediate consequence that using the corresponding numbers a and b we have $ab=ba$, which is the *Commutative Law of Multiplication.*

Consider now the three sets $[a]$, $[b]$, $[c]$. The product of these sets is a set $[a, b, c]$, whose elements consist of *all* triples of elements formed by taking one element from each of the given sets. Thus, suppose the given sets consist of the elements $a_i; b_j; c_k$. Then the elements of the product sets are $a_ib_jc_k$ where i, j, k take all values from to 1 to a, b, c respectively. It is now evident that the elements in the products $[a]\times[b]\times[c]$ and $[a]\times([b]\times[c])$ are exactly the same. That is, these sets are identical and hence correspond to the same number. That is, $abc=a(bc)$, which is the *Associative Law of Multiplication.*

7. The distributive law of multiplication with respect to addition. Again consider the sets $[a]$, $[b]$, $[c]$. The expression $[a]\times([b]+[c])$ represents the set obtained by adding the sets $[b]$ and $[c]$ and multiplying the sum by $[a]$. The elements of this product set consist of all pairs, one part of each pair consisting of an element of $[a]$ and the other part consisting of an element of $[b]$ or of $[c]$. It is now evident that exactly the same set of pairs as the above is obtained from the products $[a]\times[b]$ and $[a]\times[c]$. Hence we have $[a]\times([b]+[c]) = [a]\times[b] + [a]\times[c]$. It follows that for the corresponding numbers a, b, c we have $a(b+c)=ab+ac$, which is the *Distributive Law of Multiplication with Respect to Addition.*

8. The arithmetic of irrational numbers. The above treatment of the operations of arithmetic on integers has obvious extension to rational fractions. The addition of fractions is included by reducing them to a common denominator, using $a/b=ma/mb$, and then adding integral numerators. A somewhat similar remark applies to multiplication of fractions.

The arithmetic of irrational numbers is not quite so obvious. Certainly, numbers such as $\sqrt{2}$, π, or e are not direct abstractions from sets of objects. A clear-cut definition of such numbers can be made to arise out of integers only by means of infinite sets (possibly sequences) and the concept of limits.

Consider a bounded infinite sequence of numbers (rational numbers for our purpose) $a_1, a_2, a_3, \cdots, a_n, \cdots$ such that any number in the sequence is less than the one that follows it. The "boundedness" of the sequence means that there is some fixed number that is greater than every number in the sequence.

Since the sequence is infinite there is no last number in it. Without changing the essential nature of the sequence some numbers in it may be equal to (but not greater than) the one following. Such a sequence is called an *increasing bounded sequence.*

An axiom of continuity, which brings all irrational numbers into our system and thus completes it into what is called the number continuum, may then be stated as follows:

Every increasing bounded sequence has a least upper bound.

That is, there is a smallest number that is greater than every number of the sequence. Consider now the two increasing bounded sequences,

$$a_1, a_2, a_3, \cdots, a_n, \cdots,$$

$$b_1, b_2, b_3, \cdots, b_n, \cdots,$$

of which A and B are the least upper bounds. For the purpose of simplicity in defining multiplication let all numbers in each sequence be positive.

The sum $A+B$ is then defined as the least upper bound of the sequence,

$$a_1 + b_1, a_2 + b_2, a_3 + b_3, \cdots, a_n + b_n, \cdots,$$

and the product AB is defined as the least upper bound of the sequence,

$$a_1b_1, a_2b_2, a_3b_3, \cdots, a_nb_n, \cdots.$$

That these sequences are increasing and bounded, and that hence by our axiom each of them has a least upper bound, is obvious.

Relations of order among the numbers of this enlarged system can now be established fairly easily. It is obvious that the five fundamental identities of arithmetic considered earlier for integers and fractions hold in the enlarged system of numbers when these definitions of addition and subtraction are used.

The above is only a mere outline of a complete treatment of the arithmetic of irrationals. But no serious difficulties are encountered in carrying it out, and such treatment is a not too difficult exercise for one beginning work in the foundations of mathematics.

SOME RECENT RESULTS IN THE PROBLEM OF THREE BODIES*

H. E. BUCHANAN, Tulane University

1. Introduction. The classical mechanics is not so much in vogue nowadays as it was some years ago. The study of quantum mechanics and the theory of relativity have claimed the major part of the attention of mathematical physicists and astronomers, but the Newtonian mechanics has by no means been completely neglected. The famous problem of three bodies has received a share of attention, and some interesting results have been obtained in the last ten years. I shall endeavor to recount some of these results in so far as they relate to the problem of three finite masses.

2. A brief survey of earlier results. To give a background to the questions taken up later, let us first summarize briefly certain earlier work on the three body problem.

The problem of three bodies can be stated very simply. Given three rigid spherical bodies, having masses, m_1, m_2, and m_3, homogeneous in concentric layers, subject to no forces except their own attractions under the Newtonian law of gravitation. Let them be in arbitrary positions and moving with arbitrary velocities at any time, t_0. We wish to find the orbit each will trace and the position of each in its orbit at any future time.

The first serious attack on the three body problem dates back to Newton. He gave general theorems on the motion of the center of mass, and the first discussion of perturbations, in Book I, Section XI of the *Principia*, which was published in 1687, just two hundred and fifty years ago. From Newton to the present year the list of those who have contributed something toward the solution of the three body problem contains the names of many of the world's most famous mathematicians. Among them are: Euler, Lagrange, Laplace, Jacobi, Gauss, Poincaré, Hill, F. R. Moulton, Levi-Civita, and Sundman.

The first particular solutions of the problem of three bodies were found by Lagrange in 1772 [1]. These are the celebrated straight line and equilateral triangle solutions. In the straight line solutions the bodies always remain on a straight line and revolve with uniform angular velocity in circles or ellipses about their common center of mass. In the equilateral triangle case they remain at the vertices of an equilateral triangle which rotates uniformly about the center of mass. The size of the triangle may change as it rotates so that each body may describe an ellipse with a focus at the center of mass of the three bodies.

3. Rotating axes. The fact that in each of these particular Lagrangian solutions the bodies may move in circles with constant angular velocity probably suggested to later writers that the positions of members of the system might

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well be given by reference to a set of axes rotating with uniform angular velocity. If the rectangular coördinates $\bar{\xi}_i$, $\bar{\eta}_i$, $\bar{\zeta}_i$ be transformed by the equations

$$\begin{aligned}\bar{\xi}_i &= \xi_i \cos \omega t - \eta_i \sin \omega t,\\ \bar{\eta}_i &= \xi_i \sin \omega t + \eta_i \cos \omega t,\\ \bar{\zeta}_i &= \zeta_i,\end{aligned}$$

ω being the angular velocity, we have the following eighteenth order system of differential equations:

$$\begin{aligned}\frac{d^2\xi_i}{dt^2} - 2\omega \frac{d\eta_i}{dt} &= X_i(\xi_i, \eta_i, \zeta_i),\\ \frac{d^2\eta_i}{dt^2} + 2\omega \frac{d\xi_i}{dt} &= Y_i(\xi_i, \eta_i, \zeta_i),\\ \frac{d^2\zeta_i}{dt^2} &= Z_i(\xi_i, \eta_i, \zeta_i), \qquad (i = 1, 2, 3),\end{aligned} \tag{1}$$

where the center of gravity is at the origin and where X_i, Y_i, Z_i are functions of ξ_i, η_i, ζ_i.

4. Solutions and integrals. Before we proceed further we should make clear the meanings of the terms "solution" and "integral." Suppose that at any instant of time, t_0, it is known that $\xi_i = a_i$, $\eta_i = b_i$, $\zeta_i = c_i$, $d\xi_i/dt = d_i$, $d\eta_i/dt = e_i$, $d\zeta_i/dt = f_i$, $i = 1, 2, 3$. Then by a solution of equations (1) we shall mean a set of equations expressing ξ_i, η_i, ζ_i and their derivatives as functions of t, satisfying equations (1) identically in t and reducing to $\xi_i = a_i, \cdots, d\zeta_i/dt = f_i$, $(i = 1, 2, 3)$, at $t = t_0$.

The function, $F(\xi_i, \eta_i, \cdots, d\zeta_i/dt, t)$, is an integral of (1) if it reduces to a constant for every value of t when any solution of (1) is substituted for the variables.

5. Integrals of equations (1). Equations (1) have 10 known integrals. These and the theorems to which they lead were known to Euler. At the present time it seems highly probable that there are no more integrals of a simple type. Bruns [2] and Poincaré [3] have made important contributions in this connection. Six of these integrals are linear functions of ξ_i, η_i, and ζ_i, and their derivatives. They can be used to eliminate ξ_2, η_2, and ζ_2 and thus reduce the order of the system to the 12th. We shall suppose that this has been done.

6. Particular solutions. Equations (1) are satisfied by $\xi_i = a_i$, $\eta_i = \zeta_i = 0$, which requires that the three bodies remain on the ξ_i-axis. They can also be satisfied by $\xi_i = a_i$, $\eta_i = b_i$, $\zeta_i = 0$, where (a_i, b_i) are the coördinates of the vertices of an equilateral triangle in the $\xi_i\eta_i$-plane.

This is an easier way of getting the straight line and triangle solutions than

the method of Lagrange, but it has the disadvantage of not giving the elliptical orbits.

Obviously one important question which we may ask is whether or not these Lagrangian positions are stable; that is, if three bodies, large or small, should ever be found approximately in the proper positions on a straight line with proper velocities would the system oscillate near these positions for a considerable time or would it rapidly break up. Liouville, in 1845, answered this question for the case in which the three bodies are the sun, the earth, and the moon, assumed placed in the straight line positions; he showed that the position was unstable.

7. Stability of the straight line solutions. The method of determining the stability is as follows: In equations (1) let $\xi_i = a_i + x_i$, $\eta_i = y_i$, $\zeta_i = z_i$, then expand the right members as power series in x_i, y_i, and z_i. Drop all terms except the linear ones from the right members. The resulting system of linear, homogeneous differential equations are called the equations of variation. The z_i equations happen to be independent of the x_i and y_i equations in all the cases we shall consider except one which will be mentioned at the proper time. We have, then, for $(i=1, 2, 3)$,

$$(2)\qquad \begin{cases} \dfrac{d^2x_i}{dt^2} - 2\omega \dfrac{dy_i}{dt} = A_{i1}x_1 + A_{i3}x_3 + B_{i1}y_1 + B_{i3}y_3, \\ \dfrac{d^2y_i}{dt^2} + 2\omega \dfrac{dx_i}{dt} = C_{i1}x_1 + C_{i3}x_3 + D_{i1}y_1 + D_{i3}y_3, \end{cases}$$

$$(3)\qquad \frac{d^2z_i}{dt^2} = E_{i1}z_1 + E_{i3}z_3.$$

These equations may be solved by setting $x_i = K_i e^{\lambda t}$, $y_i = L_i e^{\lambda t}$, $z_i = M_i e^{\lambda t}$. In order that the resulting equations (2) may have a solution different from $K_i = L_i = 0$, it is necessary that their determinant vanish. This gives the following equations for determining λ:

$$(4)\qquad \begin{vmatrix} \lambda^2 - A_{11}, & -A_{13}, & -2\omega\lambda - B_{11}, & -B_{13} \\ -A_{31}, & \lambda^2 - A_{33}, & -B_{31}, & -2\omega\lambda - B_{33} \\ 2\omega\lambda - C_{11}, & -C_{13}, & \lambda^2 - D_{11}, & -D_{13} \\ -C_{31}, & 2\omega\lambda - C_{33}, & -D_{31}, & \lambda^2 - D_{33} \end{vmatrix} = 0,$$

$$(5)\qquad \begin{vmatrix} \lambda^2 - E_{11}, & -E_{13} \\ -E_{31}, & \lambda^2 - E_{33} \end{vmatrix} = 0.$$

Equation (4) is of the eighth degree in λ, while (5) is a quadratic in λ^2. Liouville [4] found one real root for equation (4). This gave rise to terms of the type $x_i = K_i e^{ht}$, $y_i = L_i e^{ht}$, where h is real and positive; and hence each of the coördinates increases without limit as t increases. He accordingly remarked that if the creator had placed the Moon at one of these straight line positions with

reference to the Earth and Sun it would not have stayed there on account of the perturbations of the other planets.

If Liouville had been able to find all the values of λ he would have found $\lambda = 0, 0, \pm i\omega, \pm i\rho_1, \pm\rho_2$, from equation (4) and $\lambda = \pm i\omega, \pm i\nu$, from equation (5), where ρ_1, ρ_2, and ν are real numbers.

8. Stability of the equilateral triangle positions. The characteristic equation for the equilateral triangle positions can be set up in a similar manner. Again the z_i equations are independent of x_i and y_i equations so that the values of λ must be obtained from solving an eighth degree equation and a fourth degree equation. All these values of λ have been found [6]. If a certain condition on the masses is satisfied, they are $\lambda = 0, 0, \pm i\omega, \pm i\nu, \pm i\rho$, from the x_i and y_i equations, while $\lambda = \pm i\omega$ from the z_1 equation and $\lambda = \pm i\omega$ from the z_3 equation. Hence the only non-periodic terms in the complete solution are those coming from the double zero exponent. Hence the oscillations are of the type

$$x_i = A_{01} + A_{11}t + \text{periodic terms},$$
$$y_i = B_{01} + B_{11}t + \text{periodic terms},$$
$$z_i = \text{periodic terms}.$$

If we exclude the possibility of incommensurability of the periods of the periodic terms then two conditions are enough to insure periodicity, if these two conditions are chosen to make A_{11} and B_{11} vanish.

It may be well here to compare these results with the so-called restricted problem of three bodies in which one of the masses is infinitesimal. In that case, if a certain condition on the masses is satisfied, all the characteristic exponents are pure imaginaries and apparently the equilateral triangle positions are stable. Such a conclusion is not warranted, however, for there is no truly infinitesimal body.

9. The Trojan planetoids. Returning to the problem of the finite masses, we observe that the condition on the masses which must be satisfied in the equilateral triangle case in order that all the characteristic exponents be zero or purely imaginary, is

$$\omega^4 - 27(m_1m_2 + m_1m_3 + m_2m_3) \geqq 0.$$

If m_1 is the mass of the Sun, m_3 is the mass of Jupiter and m_2 is the mass of one of the Trojan planetoids, the above condition is satisfied, but their position is not stable on account of the terms $A_{0i}+A_{1i}t$ which arise from the double zero values of λ. There are 10 such planetoids now known and for some years they have been near the Trojan equilateral triangle points, five in the forward position and five trailing sixty degrees behind the line of Jupiter and the Sun. One cannot help but wonder whether, since none of the planetoids are exactly at the equilateral triangle point, and since there are constant perturbations by the other planets, they will not in time depart from those positions. There are several

conjectures as to why they remain there. One possible explanation is that there must be one of the largest accumulations of loose material in our solar system near these points and these planetoids are embedded in this so that the actions of the larger planets are nullified. Another possible explanation is that they are inside a surface of zero relative velocity and cannot escape. One remembers that

it was in this way that Hill showed that our Moon cannot escape from the Earth. Some of the surfaces of zero velocity near these triangle points are shaped roughly like the adjacent figure. If a planetoid were inside such a surface it could not escape but could vary quite considerably in its distance from the triangle point.

10. Periodic orbits. The question arises as to whether or not there may be special initial conditions for which there are periodic orbits near the straight line and equilateral triangle positions. This problem was solved by F. R. Moulton [7] and some of his students for an infinitesimal body. His results have been extended to the case where all the bodies are finite in papers by H. E. Buchanan which appeared in the *American Journal* in 1923, 1925, 1927, and 1928 and to which reference has already been made, one type, in each case, being the ellipses which Lagrange found.

So far we have mentioned only particular solutions of equations (1). These equations have been solved in general by Sundman [8], who started from some results of Levi-Civita and gave a complete formal solution in series convergent for all real values of t. F. R. Moulton also outlines a method for obtaining a complete formal solution in his book on *Differential Equations* [9]. Their work is of the highest importance from a mathematical standpoint, but its practical value from the point of view of the astronomer is not very great. Their solutions give no properties of the motion, no shapes of the orbits, no proof of periodicity or of non-periodicity and they have very little prospect of ever being applied.

11. The helium atom. Now we turn to another type of problem. In 1927, U. Crudeli [10] published a twenty-page article on the stability of the equilateral triangle positions in the so-called neutral helium atom. The helium atom is made up of a central mass with a positive charge of $2e$ and two equal small masses with charges of $-e$ each. The equilateral triangle positions are quite different from those of the ordinary three body problem. The triangle rotates about its center of gravity but the nucleus and electrons move in circles whose planes are perpendicular to the plane of the triangle. Crudeli investigated the stability of these positions and found one real root of the characteristic equation,

which was of eighteenth degree. His conclusion was that the positions were unstable. In 1931 and 1933 in two papers in this MONTHLY [11] I discussed the stability of the triangle and straight line positions for the helium atom after showing that such particular solutions exist for finite values of the masses. In both of these papers only the forces due to the electrical charges were considered. All the characteristic exponents were found; they are of the form:

(a) in the triangle case, $\lambda = 0,\ 0,\ \pm i\omega,\ \pm(\alpha \pm i\beta),\ \pm i\nu,\ \pm i\rho$;

(b) in the straight line case, $\lambda = 0,\ 0,\ \pm i\omega,\ \pm i\nu,\ \pm\rho,\ \pm i\sigma,\ \pm i\omega$,

the last two coming from the z_i equations which are independent of the x_i, y_i equations. Hence both positions are unstable.

12. The characteristic exponents. It is rather remarkable that the exponents $0,\ 0,\ \pm i\omega$ appear again in both cases. Their appearance in four different cases leads one to suspect a common underlying reason. Poincaré showed, about forty years ago, that the existence of an integral of equations (2), independent of t and satisfying certain conditions on its partial derivatives, required that one characteristic exponent be zero. There are two such integrals in the three body problem and also two in the helium atom, so in all four cases one could predict that the zeros must occur. This fact is not of much help in solving the characteristic equation, for the lowest degree in λ^2 in any of the four problems is four. The knowledge that there is a zero root in λ^2 would reduce the degree to the third and this is practically impossible to solve with the complicated coefficients which occur.

In a paper which appeared in the *Duke Journal* [12] in 1935 it was proved that if the equations of variation have an integral whose minimum period is $2\pi/\omega$ then one characteristic exponent must be either $+i\omega$ or $-i\omega$. There are two such integrals in each case, hence the characteristic exponents $\pm i\omega$ must appear. This knowledge is sufficient to reduce the degree of the characteristic equation to the second in most of the cases.

13. The generalized helium atom. The characteristic equations in the four cases mentioned above had already been solved before the results of the paper in the *Duke Journal* of December 1935 were known. However, two [13] interesting applications have occurred since then. If, in the helium atom, we consider the gravitational forces as well as the forces due to the electrical charge, then there are straight line solutions and isosceles triangle solutions. The equations of variation from these positions have four integrals of the same type as before. Consequently the characteristic equations must have the roots 0, 0, and $\pm i\omega$. Without this knowledge it would, I think, have been impossible to solve the characteristic equation for these cases since the difficulty of the algebra is at least double that for the four simpler cases. With these new results at hand the characteristic exponents for these more complicated cases have been found and the stability of the straight line and isosceles triangle positions for the general-

ized helium atom have been completely discussed. One of the papers in which this was done has not yet been published.*

14. Conclusion. A mathematician never needs to apologize for studying any problem, however useless. The helium atom we have discussed has long ago passed from the mind of the physicist as a possible explanation of the interior mechanism of any atom. But the results of the above papers have recently been used in studying the stability of certain positions in the problem of four bodies. The positions in the four body problem to which I refer were established by MacMillan and Bartky [**14**] in 1932. One position in the four body problem is that in which the bodies are at the vertices of a square. When the equations of variation from this position are set up, as indicated above, their characteristic equation must have the four roots 0, 0, and $\pm i\omega$.

Many valuable contributions have not been mentioned because they did not seem to be a part of the particular tale I have been telling. The problem of three bodies is not yet solved as thoroughly and completely as we wish but after all it is only about two hundred and fifty years old!

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KASNER'S INVARIANT AND TRIHORNOMETRY

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This paper contains a new proof of a theorem of Kasner on plane conformal geometry. The theorem, it is shown at the end, holds not only in the plane but on any curved surface. It follows that the theorems of "trihornometry" apply to horn angles on any surface.

1. Introduction. Conformal geometry studies those properties of geometrical objects which cannot be changed by any conformal transformation. A conformal transformation can be defined as an analytic homeomorphism between a region

* Miss Adrienne Rayl solved this problem while a student in the author's class in Celestial Mechanics.

in a first plane and a region in a second plane, which turns every curvilinear angle in the first region into an angle of equal magnitude in the second region. The word curve in this paper is taken to mean oriented analytic curve.

Let x, y and X, Y be rectangular cartesian coördinates in the first and second planes respectively. Consider a conformal transformation T which maps some region r lying in the first plane onto a region R in the second plane. Then T will be given by a pair of equations,

$$X = \phi(x, y), \qquad Y = \psi(x, y), \tag{1}$$

where the functions ϕ and ψ are analytic in the region r, obey the Cauchy-Riemann equations,

$$\phi_x = \psi_y, \qquad \phi_y = -\psi_x, \tag{2}$$

and have a positive Jacobian,

$$J \equiv \phi_x\psi_y - \phi_y\psi_x,$$

throughout r.

That T can be written in this way may be derived from the well known fact that T can be represented by

$$X + iY = f(x + iy),$$

where f is an analytic function of the complex variable $x+iy$ in r, with a derivative f' which is different from zero throughout r. For (1) is merely the separation of f into real and imaginary parts, and (2) holds because f is analytic. Using (2) we can write the Jacobian as

$$J = \phi_x^2 + \psi_x^2 = \phi_y^2 + \psi_y^2. \tag{3}$$

Since

$$f' = \frac{df}{d(x+iy)} = \frac{\partial f}{\partial x} = \frac{\partial f}{\partial (iy)} = \phi_x + i\psi_x = \frac{1}{i}(\phi_y + i\psi_y)$$

and $f' \neq 0$, ϕ_x and ψ_x (or ϕ_y and ψ_y) are not both zero. Hence by (3), $J>0$.

Let c_1 and c_2 be a pair of curves in the region r which pass through a common point p in a common direction, and suppose that the curvatures of c_1 and c_2 at p are distinct. The figure which the two curves form is termed a horn angle of first order contact. Let P be the point of R which corresponds to p under the transformation T, and let C_1 and C_2 be the curves in R which are the images of c_1 and c_2 under T.

The symbols γ_1 and γ_2 will stand for the curvatures at p of c_1 and c_2 respectively, while $d\gamma_1/ds_1$ and $d\gamma_2/ds_2$ will represent the values at p of the first derivative of curvature with respect to arc-length. The symbols Γ_1, Γ_2, $d\Gamma_1/dS_1$, $d\Gamma_2/dS_2$ will be used for the corresponding quantities for C_1 and C_2 at P. By assumption, $\gamma_2 - \gamma_1 \neq 0$.

The quantity

$$M_{12} \equiv \frac{(\gamma_2 - \gamma_1)^2}{\dfrac{d\gamma_2}{ds_2} - \dfrac{d\gamma_1}{ds_1}}$$

is called the conformal measure of the horn angle formed by the two curves c_1 and c_2. Its value may be finite or ∞, where $\infty = a/0$, $a \neq 0$.

Now the theorem in question says first, that

$$\frac{(\gamma_2 - \gamma_1)^2}{d\gamma_2/ds_2 - d\gamma_1/ds_1} = \frac{(\Gamma_2 - \Gamma_1)^2}{d\Gamma_2/dS_2 - d\Gamma_1/dS_1}, \tag{4}$$

so that M_{12} has the same value in the capital letters as in the small letters; and second, that $\Gamma_2 - \Gamma_1$ has the same sign as $\gamma_2 - \gamma_1$. We may state this formally as follows:

THEOREM: *If a horn angle of first order contact is subjected to any conformal transformation, the conformal measure remains invariant, and the sign of the difference of curvatures also remains invariant.**

The reason for speaking of M_{12} as the measure of the angle is made clear by a conjectured converse theorem: if two horn angles of first order contact agree in the value of M_{12} and in the sign of $\gamma_2 - \gamma_1$, either one can be turned into the other by a conformal transformation. Thus the real number M_{12}, together with the sign of the curvature difference, really measures the horn angle in the sense of conformal geometry; for as soon as M_{12} and sgn $(\gamma_2 - \gamma_1)$ are known, the pair of curves which compose the angle are determined to within an arbitrary conformal transformation of the whole figure. While this converse theorem is probably true, it has been established only in part. It will not be discussed here.

2. Proof of the theorem. Our proof that M_{12} and sgn $(\gamma_2 - \gamma_1)$ are invariant is as follows:

Let c be any curve in r, and let C be its image in R. Let s be an arc-length parameter on c such that the sense of increasing s is the same as the sense of orientation of c, and let S be a similar parameter on C. As s increases, S will increase, for it is understood that the orientation of C is that derived from c by the conformal transformation T. We wish to calculate dS in terms of ds.

Differentiating (1), we have

$$dX = \phi_x dx + \phi_y dy, \qquad dY = \psi_x dx + \psi_y dy. \tag{5}$$

Since

$$dS^2 = dX^2 + dY^2, \qquad ds^2 = dx^2 + dy^2,$$

we easily find, making use of (2) and (3), that

* E. Kasner, Conformal geometry, Proceedings of the International Congress of Mathematicians, vol. 2, 1912, p. 81; also The two conformal invariants of fifth order, to appear shortly.

$$dS = \sqrt{J}ds. \tag{6}$$

This is the desired relation. The positive square root is taken because as already explained, $dS/ds > 0$.

Next, let θ denote the angle of inclination of the tangent line to c, and Θ the corresponding angle for C. Then

$$\tan\Theta = \frac{dY}{dX} = \frac{\psi_x dx + \psi_y dy}{\phi_x dx + \phi_y dy}$$

in view of (5). Dividing above and below by $\phi_x dx$ and using (2),

$$\tan\Theta = \frac{\psi_x/\phi_x + \tan\theta}{1 - (\psi_x/\phi_x)\tan\theta}.$$

Recalling the formula for the tangent of the sum of two angles, we see that

$$\Theta = \theta + \tan^{-1}\psi_x/\phi_x. \tag{7}$$

Of course everything so far is well known. Equation (6) simply says that a conformal transformation multiplies the element of arc by a constant factor at any one point, while (7) amounts to the definition of a conformal transformation, namely, that it preserves angles.

Let γ and Γ stand for the curvature at corresponding points of c and C respectively. By definition

$$\Gamma = d\Theta/dS, \qquad \gamma = d\theta/ds. \tag{8}$$

Our next step will be to calculate Γ in terms of γ, θ, x, and y.

Differentiating (7) and simplifying, we have

$$d\Theta = d\theta + \frac{(\phi_x\psi_{xx} - \psi_x\phi_{xx})dx + (\phi_x\psi_{xy} - \psi_x\phi_{xy})dy}{\phi_x^2 + \psi_x^2}. \tag{9}$$

From the first and second forms of J in (3) we obtain respectively

$$\tfrac{1}{2}J_y = \phi_x\phi_{xy} + \psi_x\psi_{xy}, \qquad \tfrac{1}{2}J_x = \phi_y\phi_{xy} + \psi_y\psi_{xy}.$$

From (2),

$$\phi_{xy} = -\psi_{xx}, \qquad \phi_{xx} = \psi_{xy}.$$

We now recognize the coefficients of dx and dy in (9) as $-\frac{1}{2}J_y$ and $\frac{1}{2}J_x$ respectively. Dividing (9) by (6) we have, in view of (8) and (3),

$$\Gamma = J^{-1/2}\gamma - \frac{1}{2}J^{-3/2}J_y\frac{dx}{ds} + \frac{1}{2}J^{-3/2}J_x\frac{dy}{ds}.$$

The abbreviation

$$m \equiv 1/\sqrt{J}$$

is evidently worth while. Since

$$dx/ds = \cos\theta, \qquad dy/ds = \sin\theta, \tag{10}$$

our result is that

$$\Gamma = m\gamma + m_y \cos\theta - m_x \sin\theta. \tag{11}$$

*This simple formula is the law of transformation of curvature in conformal mapping.**

We note that (6) now becomes

$$dS = \frac{1}{m}\, ds. \tag{12}$$

Since M_{12} involves also the derivative of curvature, we must differentiate again:

$$d\Gamma = m d\gamma + (m_x dx + m_y dy)\gamma + (m_{xy}dx + m_{yy}dy)\cos\theta - m_y \sin\theta\, d\theta - m_x \cos\theta\, d\theta - (m_{xx}dx + m_{xy}dy)\sin\theta.$$

Dividing by (12) and employing (10) and (8), we find that this reduces to

$$d\Gamma/dS = m^2 d\gamma/ds + m\left[m_{xy}\cos 2\theta + \tfrac{1}{2}(m_{yy} - m_{xx})\sin 2\theta\right]. \tag{13}$$

This is the transformation law of the rate of change of curvature with respect to arc-length, under conformal mapping.

To apply (11) to the curves c_1 and c_2 at the point p, we merely give x, y, and θ the values which they have at p, and attach the subscript 1 to Γ and γ in the case of c_1, and the subscript 2 in the case of c_2 (by hypothesis, $\theta(p)$ is the same for c_1 and c_2). Taking the difference of the two resulting formulas, we have

$$\Gamma_2 - \Gamma_1 = m(\gamma_2 - \gamma_1). \tag{14}$$

Similarly from (13),

$$d\Gamma_2/dS_2 - d\Gamma_1/dS_1 = m^2(d\gamma_2/ds_2 - d\gamma_1/ds_1). \tag{15}$$

If we square (14) and divide by (15), we obtain (4). The sign of $\Gamma_2 - \Gamma_1$ is the same as that of $\gamma_2 - \gamma_1$, because the factor $m(p)$ in (14) is positive. Our proof is therefore complete.

3. Extension to surfaces. We wish to indicate briefly the extension of this theorem to the case of horn angles drawn on any (oriented) surface.

Consider a conformal mapping of a plane onto an arbitrary surface. Let x, y be rectangular cartesian coördinates in the plane, and let them be carried onto

* It can be looked on as practically a special case of a formula of Liouville for geodesic curvature, given in Monge's Application de l'Analyse à la Géométrie, 1850, p. 575.

A. Fialkow has observed that (11) can be written perhaps most compactly as $\Gamma = m\gamma + dm/d\sigma$, where σ is arc-length along a curve crossing c at an angle of 90°.

the surface as curvilinear coördinates by the mapping. The element of arc-length on the surface will be defined in these coördinates by

$$dS^2 = \frac{1}{m^2}(dx^2 + dy^2),$$

where $m(x, y)$ now merely stands for some positive point function on the surface. This is because the mapping is conformal.

The geodesic curvature Γ of a curve C on the surface, it may be shown from standard formulas, is given by (11), in which γ and θ are as before the (geodesic) curvature and the inclination of the image curve c in the plane. All the steps after (11) in the previous proof can now be repeated, and the same conclusions will be reached, but with the new meaning for Γ.

By an easy argument the result can be given the following general form:

*If a horn angle of first order contact is drawn on a curved surface, and if the surface is mapped conformally onto any other curved surface, the measure of the horn angle and the sign of its curvature difference remain invariant. The word curvature now has the significance of geodesic curvature.**

Kasner's new theory of *trihornometry* deals with a ξ, η plane in which the "distance" between two points ξ_1, η_1 and ξ_2, η_2 is defined by

$$(\xi_2 - \xi_1)^2/(\eta_2 - \eta_1).\dagger$$

Each theorem about distances between points in what we may call the Kasner plane can be interpreted as a theorem about measures of horn angles in an ordinary euclidean plane. For if we draw the set of curves in a euclidean plane which pass through a fixed point p in a fixed direction, and let ξ and η stand for the curvature and its arc derivative at p of these curves, then to each point in the Kasner plane there corresponds at least one curve in the euclidean plane, and the distance between two points in the Kasner plane (for which $\xi_2 - \xi_1 \neq 0$) becomes the measure of the horn angle of first order contact formed by the corresponding two curves in the euclidean plane.

In consequence of what was said above, we can now add that each theorem in the Kasner plane can in fact be interpreted as a theorem about horn angles on any curved surface.

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* See abstract in Bulletin of the American Mathematical Society, vol. 42, 1936, p. 806. This result was found independently by V. Hlavatý, who stated it in a review in the Zentralblatt, vol. 14, 1936, p. 178.

† Science, vol. 85, 1937, p. 480; Proceedings of the National Academy of Sciences, vol. 23, 1937, p. 337. Arc-length is defined by $\int dx^2/dy$. Thus the Kasner plane is a special Finsler space, and perhaps the simplest non-trivial one imaginable.

SOME ELEMENTARY ASPECTS OF TOPOLOGY*

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Much of my teaching is in the engineering college and I am frequently asked by my colleagues and students what my branch of mathematics is. On stating that it is topology, they promptly counter with, "What is topology?" A statement that it is the study of properties of sets and invariants under continuous transformations is meaningless to the non-mathematician. And mention of duality theorems, indecomposable continua, and the like proves equally meaningless.

I have tried to explain the Jordan Curve Theorem and that topology attempts to present proofs of certain facts which were simply assumed without proof before. This is not at all satisfactory to them, as they feel these things are obvious and need no proof. Incidentally I once heard a very prominent member of the American Mathematical Society argue that a paper which had just been presented was of no importance since it merely proved something that we had assumed true all along. Now if the professional mathematician does not see the necessity of proving the obvious, clearly it is useless to attempt to convince the layman.

I may define topology as "rubber sheet geometry." While in reality this gives a fairly accurate idea to my questioners, they suspect that I am pulling their leg and they leave with a somewhat irritated and dubious air. After many such painful experiences I have finally found that by connecting some well known puzzles and problems with simple topological theory I can give a fair idea of what we are studying and trying to do. This is the essence of what I wish to show in this paper.

First we may consider the famous problem of the Seven Bridges of Königsberg (Figure 1). According to this problem a Sunday afternoon stroller wishes to plan his walk so as to pass over each bridge once and only once. After some trial and error this is found to be impossible. But if we alter the situation by replacing the bridge from X to Y by a bridge from W to V, we may discover that the stroller can plan his walk as desired. He may start at any point, take his desired walk and finish at the starting point. Or if we again change the situation by leaving the seven bridges as pictured and adding an eighth bridge from W to V, we find that the stroller may take his desired walk if he chooses the proper starting point, but will find it impossible to complete the walk at the point of departure. The start must be made from the place marked A or B and the journey is finished at the other letter.

Naturally a simple test is desired to determine when a problem like this one has a solution. The problem is easily seen to be equivalent to the traversing of a connected graph. A graph is a configuration consisting of a finite number of points called vertices which are end points of arcs, no two arcs having any common point except perhaps a common vertex. We say that we can traverse a graph if we can find a path which passes through all the arcs exactly once. Of

* Presented to the Association on September 6, 1937, at State College, Pennsylvania.

course a vertex may be passed several times along different arcs. The graph whose traversing is equivalent to the Problem of the Seven Bridges of Königsberg is drawn in dotted lines in Figure 1.

Obtaining his start from the problem of the seven bridges, the general problem of the traversing of graphs was considered by Euler in 1736 [1]. It was found that *we may traverse any graph, starting at any point and returning to this point, if the graph contains even vertices only, i.e., vertices which are end points of an even number of arcs; and if the graph contains at most two odd vertices we may traverse it if we do not require that we return to the starting point. In general if the graph contains 2n odd vertices it will require n journeys to traverse it.*

The graph pictured by dotted lines in Figure 1 contains four odd vertices and thus may not be traversed in a single journey. The graph obtained in the first

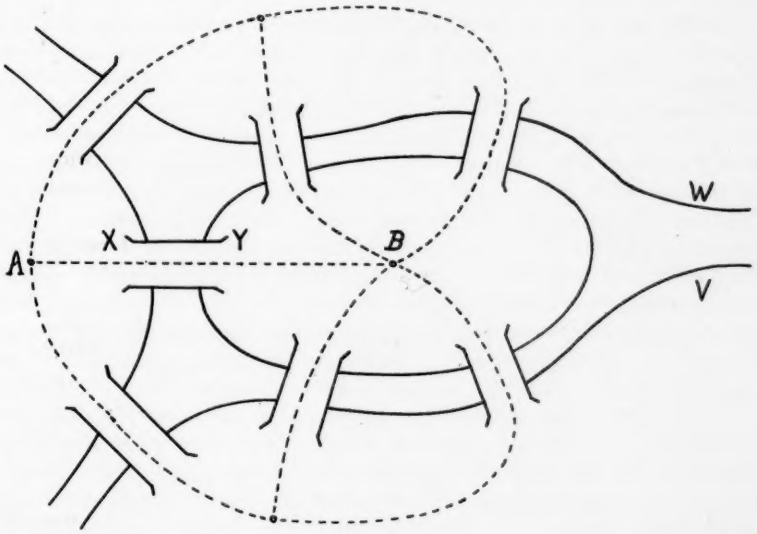

Fig. 1.

modification of the Figure contains three vertices of order 4 and one of order 2. As all are of even order we may traverse it returning to the starting point. The graph of the second modification has odd vertices at A and B only and hence may be traversed providing these points are used as start and finish.

As a somewhat similar problem we may ask how many journeys or tours are required by the Knight on the chessboard in order to move exactly once between every pair of squares between which it is possible for the Knight to move. This differs from the usual Knight's Tour Problem, of which I shall speak later, in that we demand all moves possible by the Knight and he must visit each

square several times to accomplish this. The answer is four, which I leave my readers to verify.

Another problem of the connected graph is that of finding a path which leads exactly once through all the vertices. We may or may not require that the path return to its starting point. In such a traversing there will in general be arcs of the graph which are not used in the path. We may call this *vertex-traversing* to distinguish it from the preceding *arc-traversing* problems. As an example of such a problem we have Sir William Hamilton's problem of finding a path along the edges of a regular dodecahedron which passes through all the vertices [2]. This problem was sold in England about 1860 under the name of the Icosian Game. The vertices were given the names of towns and the object was to make a journey through all the towns without visiting any twice. It may be seen that this problem may be solved in a number of ways and so as to return each time to the starting point.

Another example of the same type of problem is the classical Knight's Tour on the chessboard. We may consider the squares of the board as the vertices of our graph and two vertices joined by an arc if and only if they form a Knight's move. Then the problem of the Knight's Tour becomes simply the problem of vertex-traversing our graph. Other problems of tour on the chessboard may be similarly converted into vertex-traversing questions. The tour problems have been so thoroughly discussed in the literature (see for example Chapter 11 of W. Ahrens, *Mathematische Unterhaltungen und Spiele*), although not with this particular interpretation, that I shall say nothing more here conce ning them.

Yet another example of the same type of problem is the well known, but as yet unsolved, Four Color Problem. Here the question is whether any map of regions on a sphere, no matter how the regions are distributed, may be colored using but four colors so that no two regions of the same color shall meet along an arc of their frontier. This problem was mentioned by A. F. Möbius in his lectures in 1840 but apparently was known to practical map makers much earlier. A solution was offered as early as 1879 by A. B. Kempe [3] but unfortunately errors have been found in this proof and in all other proofs that have been proposed, and the problem remains unsolved to the present day. However, many partial solutions are known. It is known that any map may be colored if any cubic map can be colored, a cubic map being one in which at most three regions meet at a point. And Philip Franklin has recently shown that *any* map of less than 32 regions may be colored in four colors [4]. The frontiers of a map form a connected graph and it is known that if the vertices of this graph can be joined in a circuit along the frontiers, the map can be colored in four colors. Hence this famous problem reduces to the traversing of a graph; and any of my readers may make a name for himself in history by showing that any connected cubic graph on the sphere may be vertex-traversed with the end at the starting point. Strangely the problem of how many colors are required to color any map on more complicated surfaces such as the torus, sphere with several handles, and the simple one-sided surfaces have all been solved, while the problem for maps on a sphere, the simplest of all surfaces, remains unsolved [5].

As indicated above the problem of joining a finite number of points in a given set by a path *in the set* which does not cross itself, as well as its special case of vertex-traversing a graph, has been solved only in particular cases. Where the set is a locally connected continuum the problem has been solved only in the cases of two, three, and four points [6]. From some examples recently shown me by E. W. Miller, I do not believe that it is possible to find interesting necessary and sufficient conditions for the general finite problem. I gave perhaps the most general sufficient conditions in 1932 in showing that n given points of a locally connected continuum may be joined by a circuit (or simple closed curve) in the set if no two of them may be separated by $n-1$ points of the containing continuum, and may be joined by an arc if no two are separated by $n-2$ points [7]. It is observed that these conditions are by no means necessary, but the numbers $n-1$ and $n-2$ of the theorem may not be reduced and leave the results valid.

A problem of somewhat different nature is the Water, Gas, and Electricity Puzzle. Here we are given three plants supplying respectively water, gas, and electricity, and three houses; and the problem is to find an arrangement of houses so that water, gas, and electricity may be piped to each of the three houses so that no two pipes cross and no pipe passes underneath any house or plant. This puzzle dates far back into history and is said to appear under other names before the introduction of gas and electricity. The non-topologist will spend considerable time trying various arrangements, but a simple application of the Jordan Curve Theorem shows that the puzzle is impossible to solve. It is interesting to note that a generalization of this puzzle is found in the following theorem of G. T. Whyburn [8]: *Let A_1, A_2, A_3 be bounded continua in the plane such that the product* of any two is the set B and suppose each of the sets A_i-B is connected; then B is a continuum or the sum of two continua.* We get the puzzle by letting A_i consist of the three plants, one house and the supply pipes joining this house. Then B becomes the plants and Whyburn's theorem shows that there is no solution when there are more than two plants.

The above puzzle may be interpreted as a problem in graphs by considering the plants and houses as vertices and an arc joining each plant and house as supply pipe. The conclusion is that such a graph may not be drawn in the plane. This raises the general question of when may a graph be drawn or mapped in the plane. This problem was solved by C. Kuratowski in 1930 when he showed that a graph may be mapped in the plane if and only if it does not contain topologically one of two primitive skew curves [9]. The two primitive skew curves are (a) the edges of a tetrahedron plus a line joining the mid-points of two opposite edges, and (b) the edges of a tetrahedron plus lines joining the center of gravity to the four vertices. The first of these is observed to be the water, gas, and electricity graph of the problem above. A later proof of this Kuratowski theorem using combinatorial methods was given by H. Whitney in 1932 [10]. The problem of determining when a general locally connected continuum may be mapped on the surface of a sphere has been solved by S. Claytor

* The product of two sets consists of the elements common to the two sets.

[11]. He shows that a continuum which cannot be so mapped must contain one of four primitive skew curves—the two mentioned above and two more complicated ones also due to Kuratowski. The primitive skew curves for mapping graphs on the torus and the projective plane have been worked out by I. N. Kagno [12]. While two skew curves suffice for mapping a graph on a sphere, here we find a long list of irreducible skew curves. A simpler condition for the torus is due to Saunders MacLane [13].

The problem of finding conditions under which a graph may be topologically embedded in the 2-dimensional manifolds of higher connectivity is as yet unsolved. The question as to when a continuum is topologically embeddible in euclidean 3-space or n-space is unsolved and appears very difficult. The Menger-Nöbeling theorem [14] states a sufficient condition, *i.e.*, if the dimension of the continuum is $\leqq(n-1)/2$, then it may be embedded in euclidean n-space, but this is far from necessary.

These are but a few examples of a long list of problems of interest to the topologist of conditions under which a set of given properties may be embedded or mapped in another set of given properties.

We have discussed here a number of problems of topology which permit a simple formulation. These suffice to give an understanding of the nature of topology and its problems. Even in these simple problems we find several which are of fundamental importance and have not yet been solved. For a further discussion of unsolved problems in topology the reader is referred to the excellent article by R. L. Wilder in this MONTHLY for February, 1937.

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MATHEMATICAL ADVENTURES IN SOCIAL SCIENCE*

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1. Mathematics in economics. It is now just 99 years since there appeared on the scientific scene a small book entitled *Recherches sur les principes mathématiques de la théorie des richesses*, by A. A. Cournot (1801–1877). This book, written with an unsurpassed clarity of style and upon a subject that engages almost universal attention, was neglected for more than a third of a century after its publication. Not until 1872 when Stanley Jevons proclaimed its worth to English economists and a year later when Leon Walras called to the attention of the French Academy of Moral and Political Sciences the ideas of this notable work, did the influence of Cournot begin to rise. Today the *Theory of Riches* is recognized as the Principia of econometrics; and perhaps the quickest approach to much of the modern theory is through a careful study of its lucid pages. Mathematicians will be interested to note that Henri Poincaré, with his extraordinary curiosity about all approaches to knowledge through mathematics, devoted an extensive memoir in 1905 to *Cournot et les principes du calcul infinitésimal*. It is a credit to the perspicacity of George Sarton that he places Cournot among the 118 greatest mathematicians since the beginning of the nineteenth century.

It is my purpose today to indicate somewhat hastily and necessarily superficially some of the interesting mathematical problems that have arisen in the attempts of mathematical economists to found a theory of econometrics. To many, who casually survey the field, the vagaries of the economic state, the hourly rise and fall of prices in the grain market and in the stock exchange, the apparent randomness of the price and production indexes, and the perplexing difficulties of credit and money, the problems of econometrics may appear hopelessly unamenable to mathematical treatment. But first appearances are deceptive, for after a careful study much mathematical structure is discernible in economic situations, and where such structure fails to appear one may console himself by studying the curious properties of random series.

2. The isomorphism with mechanics. The subject of econometrics may be divided somewhat arbitrarily into two parts, one static, the other dynamic. The first is concerned with static equilibrium, curves of supply and demand, the concept of the potential of utility, surfaces of indifference, and the complex equations of the Walrasian theory of production. The second is for the most part a study of the behavior of time series, the detection of trends and cycles; the explanation of crises; and the description of the logistics of population increase and of the growth of productive enterprises.

One frequently hears the objection that by following the model of physical science we are dealing with invalid analogies, but this objection can be refuted

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on two grounds. The first is that all applications of interpretative methods to physical phenomena are themselves at heart logical isomorphisms. Thus the differential equation

$$\frac{d^2u}{dt^2} + k^2u = 0, \tag{1}$$

which describes the motion of a swinging pendulum, is a mathematical abstraction which is finally justified by the fact that the values of u are highly correlated with the measurements of the physical displacements of the bob over an interval of time.

On the empirical assumption that isomorphisms are given validity only by final reference to data, we are justified in following the model of mechanics, thermodynamics, or any other physical discipline, provided only that the isomorphism is established by the stubborn facts of statistical verification.

The second argument, which seems to refute the charge of dealing with invalid analogies, is found in the observation that all sciences, when they reach the stage which justifies an attempt at mathematical formulation, seek in some manner to find what we may call the *extremal* conditions which characterize their data. Something is either to be maximized or minimized in the complex of defined concepts which differentiate one science from another. Since the mathematical theory of extremals is a unique theory, whether it be applied to point functions or to such functionals as appear in the calculus of variations, it is but natural that the equations of one scientific discipline should bear a close resemblance to those of another. The maximizing of the profit function,

$$\int_{t_0}^{t_1} [y(t)p(t) - q(u)]dt,$$

where $y(t)$ is the amount of the commodity purchased at price $p(t)$, and $q(u)$ is the cost of manufacturing and marketing u units, employs the same concepts in economics as the minimizing of action employs in the theory of mechanics. This approach to a mathematical formulation of economic theory has been utilized by G. C. Evans, C. F. Roos, H. Hotelling, and others. It is not strange, therefore, that we should find similar formulas in the two disciplines, or that a student of mechanics will discover that the mathematical arguments of Pareto are easy to follow.*

A number of years ago Irving Fisher, under the stimulus of Josiah Willard Gibbs who was then at the height of his remarkable powers, wrote his now classical work *Mathematical Investigations in the Theory of Value and Prices*. Pro-

* Thus equation (1) is derived in its most significant way as a necessary condition in minimizing the action integral

$$\int_{t_0}^{t_1}(\text{Potential Energy} - \text{Kinetic Energy})dt$$

subject to the restriction that

$$\text{Potential Energy} + \text{Kinetic Energy} = \text{Constant}.$$

fessor Fisher in his Gibbs lecture delivered in Des Moines in 1929* on *The Application of Mathematics to the Social Sciences* comments thus: "Professor Gibbs showed a lively interest in this youthful work, and was especially interested in the fact that I had used geometric constructions and methods, including his own vector notation."

One of the remarkable things in this work of Fisher, which was published in 1892, was his table of mechanical analogies, a few of which we reproduce here as follows:

In mechanics	*In economics*
A particle	An individual
Space	Commodity space
Force	Marginal utility
Work	Disutility
Energy	Utility
Total work	Total disutility, or the integral of marginal disutility
Total energy	Total utility, or the integral of marginal utility
Net energy	Net utility
Equilibrium of forces	Equilibrium of utility

It is interesting to observe that the modern development of mathematical economics has followed closely the isomorphisms thus suggested. Some of these ideas we shall now point out.

3. The utility concept. The idea of utility seems to have been due independently to H. H. Gossen and W. S. Jevons.† Since its formulation it has exerted a great fascination upon mathematical economists and is today, perhaps, one of the most significant ideas in the theory of econometrics. *Utility*, or *ophelimity*, to employ the term invented by Pareto, is a measure of one's enjoyment. To make the notion more precise, let us designate by the symbol $U(x_1, x_2, \cdots, x_n)$ one's total pleasure in the possession of amounts $x_1, x_2, \cdots, x_n$ of n different commodities or services. Since measures of utility do not exist, and since, in fact, the very nature of U, whether it be a function or a functional, is as yet undetermined, we begin by considering the differential of U,

$$dU = U_1dx_1 + U_2dx_2 + U_3dx_3 + \cdots + U_ndx_n,$$

where $U_i = \partial U/\partial x_i$ is called the *marginal utility* of the commodity x_i.

Now in a modern economy, marginal utilities are acquired only through the medium of what is called price, and, as we all know from universal experience, price is related to a curious thing called money. The two great theories in modern science, whose mysteries are regarded in the popular mind as being shared by a very limited number of people, are the theory of relativity and the theory

* See The Bulletin of the American Mathematical Society, vol. 36, 1930, pp. 225–243.

† A selected bibliography on the theory of utility has been given by the author in the report of the third annual conference of the Cowles Commission for Research in Economics, 1937, pp. 76–79.

of money. Of these two, probably the theory of money is the more perplexing. How is it to be related to the concept of utility?

Curiously enough the first approach to this problem was due to Daniel Bernoulli, who, in his celebrated work *Specimen Theoriae Novae de Mensura Sortis* published in 1738, reduced the perplexities of the St. Petersburg paradox to the utility of money. In the theory of probability the mathematical expectation E of one who has the probabilities p_i of gaining amounts x_i is defined to be the sum

$$E = \sum p_i x_i = \int_a^b p(t)x(t)dt,$$

the integral, of course, being used in the case of continuous probabilities.

Now suppose that $p(t) = e^{-at}$, $x(t) = e^{at}$, while t ranges over the infinite interval $(0, \infty)$. These conditions are essentially those of the St. Petersburg paradox and we obtain the unrealistic answer $E = \infty$. Attempts to explain the paradox led to the assumption that for $t \geqq t_0$, $x(t) = e^{at_0}$, since any expectation of infinite gain was absurd, and hence one obtained

$$E = \int_0^{t_0} dt + \int_{t_0}^{\infty} e^{-a(t-t_0)}dt = t_0 + 1/a.$$

Bernoulli, however, placed the difficulty elsewhere by assuming that the pleasure of gaining a quantity of money x_i depended for each individual upon the amount x which he already possessed. That is to say, one's increment of *money utility* dU for an increment of gain, dx, depends upon one's initial wealth x. Bernoulli assumed that utility was directly proportional to the gain and inversely proportional to the initial wealth, that is, in symbols,

$$dU = \mu \frac{dx}{x}.$$

The total utility, therefore, would be equal to the integral of this amount, or

$$U = \mu \int_{x_0}^{x} \frac{dx}{x} = \mu \log (x/x_0),$$

where x_0 is a threshold value. Bernoulli then defined moral expectation, E_M, as a sum

$$E_M = \mu \sum p_i \log (1 + x_i/a) = \mu \int_a^b p(t) \log [1 + x(t)/a]dt,$$

where a is the total wealth of the individual. It is obvious that this integral will converge under the conditions imposed by the St. Petersburg problem, but it is equally obvious that values of $x(t)$ can be constructed for which the integral would not converge. This criticism of the Bernoulli solution has recently been

made by Karl Menger.* But however faulty Bernoulli's solution of the St. Petersburg paradox may be, his introduction of the concept of the utility of money is a contribution of real importance.

The form of the utility function for money, as given by Bernoulli, has been subjected to numerous objections. Lacking statistical evidence, it is best to express the marginal utility of money in the form

$$\frac{dU}{dx} = \mu F(x),$$

where $F(x)$ is subject to the restriction that

$$\lim_{x=\infty} F(x) = 0.$$

Presumably $F(x)$ is also monotonically decreasing, for we naturally assume that one can reach a point of satiation by gradual and continuous decrease of his pleasure in the accumulation of x. Charles Jordan† has suggested the value $F(x) = 1/x^2$ and Ragnar Frisch‡ has argued for the function $F(x) = 1 \log (x/a)$.

Returning now to the differential of utility, we define *utility* as the integral

$$U = \int \sum U_i dx_i,$$

and consider the equation

$$U(x_1, x_2, \cdots, x_n) = k,$$

where k is a variable parameter. If the differential form in U is integrable, then the equation just written down is an equation of a surface. This surface is called an *indifference surface*, and plays the rôle in econometrics that is played in mechanics by an equipotential surface.

Preference lines are then defined as lines in the direction of the maximum rate of increase of utility. The system of differential equations which defines them may be expressed in terms of the marginal utilities as follows:

$$\frac{dx_1}{U_1} = \frac{dx_2}{U_2} = \frac{dx_3}{U_3} = \cdots = \frac{dx_n}{U_n}.$$

Let us next consider the budget equation for an individual, namely

$$I = \sum p_i x_i,$$

where p_i are the prices, given by the market, and x_i are the amounts of the commodities which the individual will purchase from a fixed income I.

* Das Unsicherheitsmoment in der Wertlehre. Zeitschrift für Nationalökonomie, vol. 5 (1934), pp. 459–485.

† On Daniel Bernoulli's "Moral Expectation" and on a New Conception of Expectation. This MONTHLY, vol. 31 (1924), pp. 183–190.

‡ Sur un problème d'économie pure. Norsk Matematisk Forenings Skrifter, series 1, no. 16, 1926.

The simple problem in the calculus which is here involved is that of maximizing utility subject to the restraint of the budget surface. Employing Lagrange multipliers, we then have

$$\frac{\partial}{\partial x_i}(U + \lambda I) = 0, \qquad \text{that is,} \qquad U_i + \lambda p_i = 0,$$

or in customary form

$$\frac{U_1}{p_1} = \frac{U_2}{p_2} = \frac{U_3}{p_3} = \cdots = \frac{U_n}{p_n} = -\lambda,$$

where λ is the *marginal utility* of money.

Since we are dealing with the maximization of utility, the following supplementary condition is necessary

$$d^2U \equiv \sum \frac{\partial u_i}{\partial x_j} dx_i dx_j < 0,$$

that is, the quadratic form in commodity space must be negative definite.

The relationship between index numbers and the utility surface has been explored by R. G. D. Allen, H. Staehle, W. Leontief and others.* One of the most interesting of these investigations is due to Allen, who considered the index formulas of Laspeyres and Paasche, namely

$$i = I'/I, \qquad \text{and} \qquad j = J'/J,$$

where we write

$$I = \sum P_i Q_i, \qquad I' = \sum P_i' Q_i, \qquad J = \sum P_i Q_i', \qquad J' = \sum P_i' Q_i',$$

the primes referring to the comparison year.

Making use only of the assumption that the indifference surfaces are concave toward the origin, Allen was able to show that the true index of the cost of living, i', by which we mean that index which compares the respective utilities, lies between the Laspeyres and the Paasche indexes, that is, $i > i' > j$.

The problem of comparing the utilities which average individuals in different countries might have on a given money income has been studied by Ragnar Frisch through what he calls the *flexibility*, $\omega(r)$, of money, defined by the equation

$$\omega(r) = d[\log F(r)]/d[\log r],$$

where $F(r)$ is the marginal utility of money, and $r = \rho/P$, in which ρ is nominal income and P the price of living.†

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* See the bibliography previously referred to.

† For an extensive account of this situation see R. Frisch, Annual survey of general economic theory: The problem of index numbers. Econometrica, vol. 4, 1936, pp. 1–38.

4. The functional concept in the utility theory. Lacking at the present time any adequate data for testing utility concepts, we may raise the question as to the assumption that total utility is a potential function. We have seen that utility is defined as a line integral and, therefore, if it is a function, then the differential must be exact. This assumption, as is well known, imposes very stringent conditions on the marginal utilities.

It is a matter of historical interest to mathematicians that Vito Volterra in 1906 suggested to Vilfredo Pareto (1848–1923) that utility doubtless had a *fonctionnelle* aspect. Thus, defining utility by the integral

$$I = \int A dx + B dy + \cdots + N dt,$$

which is merely the same as our function U in different notation, Pareto in his *Manuel d'économie politique*, (Paris, 1909, p. 556) says:

"If one then considers a commodity path, which departing from a point $x, y, \cdots, t$, returns again to this point, one says that he has traversed a *closed cycle* provided he returns to this point with the same index of ophelimity with which he left it. This case corresponds to indifference in the order of consumption.

"One will say that he has traversed an *open cycle*, if he returns to the point of departure with an index of ophelimity different from that with which he departed. This case corresponds to that in which the order of consumption influences the pleasure which it occasions."

Although the terms open and closed cycle are not used here in that general sense which these words have acquired through the modern theory of functionals, they have given sufficient difficulty to econometrists. Thus J. R. Hicks in a recent article speaks of the "celebrated but mysterious" theory of open cycles.

It will be noticed from the foregoing discussion that the utility theory has followed an isomorphism with the classical potential theory of gravitation and mechanics. Suggestions, however, are not lacking that the potential theory which resides in the basic postulates of thermodynamics, may also furnish a heuristic guide to the solution of some of the problems of economics. Thus we find Pareto stating in his *Manuel* (p. 543) with reference to the utility integral:

"It would appear then, in order to abridge the discussion, that one should give some name to the quantity I; thus in mechanics it has been thought appropriate to give the name of *potential energy* to a certain integral, and in thermodynamics the name of *entropy* to another."

The author has been interested in pursuing this isomorphism with thermodynamic potentials and a rather surprising relationship is thus revealed which seems to be within the reach of statistical verification. The following set of postulates is suggested:

1. There exists a differential of expenditure

$$dE = P dQ,$$

where P is a price index and Q a measure of the quantity of goods purchased.

2. There is a differential of income, I,

$$dI = dS + dE,$$

where S denotes savings.

3. There is an equation of state

$$PQ = W.$$

where W is the total value of goods purchased.

4. There exists a money utility function, U, the differential of which is,

$$dU = \mu F(W)dI,$$

where $F(W)$ is a function which satisfies the condition

$$\lim_{W=\infty} F(W) = 0.$$

5. The function S plays the rôle of a potential function, that is,

$$S = \int_{C'} (dI - dE) = 0,$$

where C' is a closed path in the variables of state.

6. Money utility is a potential function, that is,

$$U = \int_{C'} \mu F(W)dI = 0,$$

where C' is a closed path in the variables of state.

Several results inherent in this system of postulates have been developed, of which only one will be mentioned. Thus if the coefficients A, B, C, and D are defined by the differentials

$$dI = AdP + BdW, \quad Q \text{ constant},$$
$$dI = CdQ + DdW, \quad P \text{ constant},$$

it may be shown that the following relationship holds between C and D:

$$C - D = -1/\omega(W),$$

where $\omega(W)$ is the marginal elasticity of money, similar, in form at least, to the money flexibility introduced into the theory of index numbers by Frisch. The importance of the relationship thus established is found in the fact that through it the validity of the assumptions are amenable to statistical check from budgetary data.

5. The dynamic problem. The problem of dynamics in the theory of econometrics is associated with the study of time series. This vast subject has been attacked by nearly all the powerful methods of analysis and the results have

been scrutinized by nearly all the methods known to modern statistical theory.

Perhaps the simplest way to study a time series is to discuss its variance, σ^2, which we may regard as consisting of four parts: (1) the variance due to secular trend, (2) the variance due to harmonic components, (3) the variance due to the erratic element, and (4) the variance due to crisis. If we designate these four variances respectively by the subscripts T, H, E, and C, then we may represent the total variance as

$$\sigma^2 = \sigma_T^2 + \sigma_H^2 + \sigma_E^2 + \sigma_C^2 .$$

A careful theoretical and statistical study indicates that the correlations between the four elements are sufficiently small to be neglected, so that, to a first approximation at least, we can neglect cross products which might be expected to appear in the total variance. That is to say, the four elements appear to be essentially independent ones, and each can be studied by itself. The independence of the trend and harmonic elements has been indicated by G. U. Yule.

In periods of comparatively stable economy, as the period from 1897 to 1914, the crisis variance is zero, but in the disruptive economy of the past 20 years this term has been one of great importance.

In the time series represented by the Dow-Jones averages of industrial stock prices between 1897–1914 we find that 50% of the variance can be attributed to trend, 33% to the harmonic element and 17% to the erratic element.

The erratic element in economic time series presents some intriguing problems. In the first place, it defines a kind of Heisenberg indeterminacy for economic series, that is to say, the rectangle within which complete statistical chaos exists. In economics this indeterminacy rectangle is that of price-time or production-time, and may be thought of as a quantity which depends upon the random character of human judgments. Some idea of its magnitude may be gathered from the fact that the naming of a committee for the arbitration of the steel strike earlier this summer was sufficient to turn a bear market, which had dropped 1.8 points on fair volume, to a bull market up 1.8 points. Since the naming of the committee was a random event, in so far as the market was concerned, it is obvious that a movement of 3.6 points in a single day must be regarded as lying within the indeterminacy rectangle of economics.

In the second place, the erratic element leads to the mathematical study of random series, that is to say, those series whose auto-correlation function is zero within statistical errors. More precisely, if $x(t)$ is a normalized random series, then the function

$$r(t) = \int_{-\infty}^{\infty} x(s)x(s+t)ds \tag{1}$$

has the property: $r(0) = 1$, $r(t) = 0$, $t \neq 0$.

Other definitions of random series have been given, but they depend upon a distribution function. Thus if the mean of $x(s)$ is zero, then Goutereau's constant,

$$(2) \qquad G = \int_{-\infty}^{\infty} |\Delta x(s)|\, ds \Big/ \int_{-\infty}^{\infty} |x(s)|\, ds,$$

is equal to $\sqrt{2}$, if x is normally distributed, but it is equal to 4/3 if the distribution is rectangular.

It is a fact, very astonishing when first encountered, that neither the difference nor the accumulation of a random series is random. Mr. Herbert Jones of the Cowles Commission, generalizing some of the conclusions found in G. U. Yule's classical paper: *Why Do We Sometimes Get Nonsense Correlations Between Time Series?* published in 1926,* derived the elegant formula

$$(3) \quad r_n(t) = \int_{-\infty}^{\infty} \Delta^n x(s) \Delta^n x(s+t)\, ds = \cos \pi t\, \Gamma^2(n+1) \Big/ [\Gamma(n+t+1)\Gamma(n-t+1)].$$

It is a matter of observation, also, that if we define

$$(4) \qquad S[x(t)] = \int_0^t x(s)\, ds - \frac{1}{L} \int_0^L (L-s) x(s)\, ds,$$

that is to say, if $S(x)$ is the first accumulation of a random series referred to its mean, its length being L, and if we define a sequence of functions by the iteration formulas

$$(5) \qquad S^2(x) = S[S(x)], \cdots, S^{(n)}(x) = S[S^{(n-1)}(x)],$$

then $S^{(n)}(x)$ tends toward a cosine function of period equal to the length of the series, that is,

$$(6) \qquad S^{(n)}(x) \sim A \cos \frac{2\pi}{L} (t + a),$$

where L is the length of the series and a is a phase constant.

These remarkable results on random numbers give a new and powerful approach to the statistical study of the actual nature of economic time series. We realize that a large random element, the indeterminacy range, must exist in these series. To what extent, therefore, are these series random and how far can we reasonably expect to go in their precise mathematical analysis?

The harmonic variance comes in for our next consideration. It is in the study of this element that we encounter the introduction of true dynamical principles, since we must account for the existence of fundamental oscillations in the economic system.

At first sight it might appear that the application of dynamical concepts to economic time series could be justified from the fact that the industrial system, its activities of production and consumption, rests essentially upon energy conversion. A quantity of money may be exchanged for a quantity of energy or

* Journal of the Royal Statistical Society, vol. 89, 1926, pp. 1–84.

that which has been produced by energy. But we also know that other important considerations enter the exchange of money for goods as is illustrated by some of the stories in O. Henry's *Gentle Grafters*. The grafters in these tales are probably called gentle, because their illegalities are always justified by the equating of marginal utilities. Thus a victim has entrusted to their care the sum of $5000, an accumulation of years of labor, which he is about to spend. The grafters determine the marginal utility of this sum by observing that their victim is about to wager the entire amount on a horse race. They decide that the utility of this expenditure is about equal to the utility which one might gain by the possession of 5 shares of stock in a corporation capitalized at $1,000,000, with a stock issue limited to 1000 shares. They incorporate such a company, print the 1000 stock certificates, and leave their victim in the possession of five of them. The equilibrium conditions are thus satisfied. It is obvious, however, that the energy equivalence of money has slipped.

How can dynamical principles, based upon the concept of energy exchange, be applied in the study of economic time series? The theory of utility throws some light on the matter. Thus, if $U(x_1, x_2, \cdots, x_n)$ is the utility associated with the possession of a group of objects, $x_1, x_2, \cdots, x_n$, let us say, industrial stocks, then the marginal utility of x_i is equal to the price p_i multiplied by the marginal utility of money, that is

$$\frac{\partial U}{\partial x_i} = \lambda p_i,$$

where λ is the marginal utility of money.

Clearly the total utility will be given by

$$U = \int \frac{\partial U}{\partial x_1} dx_1 + \frac{\partial U}{\partial x_2} dx_2 + \cdots + \frac{\partial U}{\partial x_n} dx_n$$

$$= \int \lambda (p_1 dx_1 + p_2 dx_2 + \cdots + p_n dx_n).$$

Now if the marginal utility of money varies little over an interval of time, then the total utility is proportional to total expenditure and the price index formed by the ratio of total expenditures for the base and comparison years is equal to the ratio of the corresponding utilities. Since utility in economics plays the rôle of potential energy in dynamics, and since in the problem of the pendulum the potential energy varies harmonically with the displacement, it is clear that the isomorphism which we contemplate here, where a price index is assumed to follow dynamical laws, is not altogether unrealistic.

A careful survey of the industrial stock price index from 1914 to the present time indicates that the movement of the averages was precisely what one would expect to find in a pendulum swinging under an impressed force of fixed period around 40 months, in which the phenomenon of *resonance* has suddenly appeared.

Let us consider the differential equation of motion of a pendulum subject to a periodic force $E \cos pt$; it has the form

$$A\frac{d^2y}{dt^2} + B\frac{dy}{dt} + Cy = E \cos pt;$$

the solution takes the form

$$y = Ke^{-\lambda t} \sin\left(\frac{2\pi}{T}t + \omega\right) + L \cos(pt - \alpha),$$

where

$$\tan \alpha = Bp/(C - Ap^2), \qquad L = E/\sqrt{(C - Ap^2)^2 + B^2p^2}.$$

The following computation tends to bear out the general thesis which we have set forth above:

Let us consider the industrial stock price averages, (percentage of trend minus the trend average) over the period from 1897 to 1914, and let us fit to them and their first and second derivatives the regression line

$$AX'' + BX' + CX + D \cos pt = 0, \qquad p = 2\pi/41.$$

By a simple application of the theory of multiple correlation we obtain the following differential equation:

$$X'' + .00004X' + .01744X = .07032 \cos pt.$$

The solution of this equation is

$$X = K \sin\left(\frac{2\pi}{47.6}t + \omega\right) + 11.6424 \cos pt,$$

where K is negligibly small.

The coefficient of $\cos pt$ may be regarded as statistically satisfactory since only 48% of the energy of the system is found in the 41-month cycle, and the standard error, σ, of X is approximately 15. But let us look more closely at the computation of this coefficient. Since apparently the normal period of the system is approximately 48 months (4 years) and the impressed period is only slightly less than this, the possibility of resonance is very great. Thus we find for the significance function $|C - Ap^8|$ the value .00604, and it is obvious that a very little change in any one of these parameters would lead to the phenomenon of resonance.

This resonance situation was spectacularly realized in the great bull market which began in 1926 and ended in the fall of 1929. Dynamical considerations again would lead to the expectation of a secondary high and a subsequent collapse, neither as severe as the primary ones, as the elements of the time series returned by damping to their normal oscillations about the line of trend.

CONCERNING A CONJECTURE IN THE PRECEDING PAPER

E. J. MOULTON, Northwestern University

It was stated in the preceding paper that, as a matter of observation, a sequence of functions $S^{(n)}(x)$ defined by equations (4) and (5) of the paper tends toward a cosine function given in (6). I understand that the statement is a conjecture based on computations with discrete series which were assumed to be random series. Some remarks concerning theoretical restrictions on the sequence of functions may tend to clarify the situation.

The S transform of a function $F(t)$ is given by the equivalent formulas

$$\text{(a)} \qquad \begin{aligned} S(F) &= \int_0^t F(s)ds - \frac{1}{L}\int_0^L (L-s)F(s)ds \\ &= \frac{1}{L}\int_0^L sF(s)ds - \int_t^L F(s)ds. \end{aligned}$$

We shall call the functions of the sequence $F_1(t)$, $F_2(t)$, $F_3(t)$, $\cdots$; they are defined by the formulas

$$F_1(t) = S(x), \qquad F_{k+1}(t) = S(F_k), \qquad k = 1, 2, 3, 4, \cdots.$$

If m is an upper bound of the absolute value of $x(t)$ on the interval $0 \leqq t \leqq L$, we have

$$\text{(b)} \qquad \begin{aligned} |F_1(t)| &\leqq \int_0^t mds + \frac{1}{L}\int_0^L (L-s)mds = m(\tfrac{1}{2}L + t), \\ |F_1(t)| &\leqq \frac{1}{L}\int_0^L smds + \int_t^L mds = m(\tfrac{3}{2}L - t). \end{aligned}$$

The former of these relations gives the smaller bound for $|F_1(t)|$ when $t < \frac{1}{2}L$, the latter when $t > \frac{1}{2}L$. If we define a function $T(t)$ by the relations

$$\text{(c)} \qquad \begin{aligned} T(t) &= \tfrac{1}{2}L + t, \qquad 0 \leqq t \leqq \tfrac{1}{2}L, \\ &= \tfrac{3}{2}L - t, \qquad \tfrac{1}{2}L \leqq t \leqq L, \end{aligned}$$

then we have, for $0 \leqq t \leqq L$,

$$\text{(d)} \qquad |F_1(t)| \leqq mT(t) \leqq mL.$$

By an argument similar to the preceding, taking mL as an upper bound of $|F_1(t)|$, we readily show that for $0 \leqq t \leqq L$, we have $|F_2(t)| \leqq mL^2$; and it is clear that

$$\text{(e)} \qquad |F_k(t)| \leqq mL^k, \qquad k = 1, 2, 3, \cdots.$$

By using $mT(t)$ as an upper bound of $|F_1(t)|$ a smaller upper bound could be obtained for $|F_2(t)|$, and consequently for $|F_k(t)|$.

From (e) it follows that when $L < 1$ the limit as n becomes infinite of $F_n(t)$, or of $S^n(x)$ in the notation of the preceding paper, is zero uniformly on the interval $0 \leqq t \leqq L$. Thus the conjecture, as far as it may be significant, must depend upon L, the length of the interval.

If for some value of k we have

$$F_k(t) = A \cos \frac{2\pi}{L}(t + a),$$

then we find that

$$F_{k+r}(t) = A_r \cos \frac{2\pi}{L}(t + a_r), \qquad (r = 1, 2, 3, \cdots),$$

where

$$A_r = A\left(\frac{L}{2\pi}\right)^r, \qquad a_r = a - \frac{rL}{4}.$$

Hence in this case as n becomes infinite the amplitude of $F_n(t)$, or of $S^n(x)$, tends to zero if $L < 2\pi$, remains constant if $L = 2\pi$, or becomes infinite if $L > 2\pi$.*

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems especially new problems, which are reserved for the department of Problems and Solutions.*

NOTE ON AN EXTENSION OF LAGRANGE'S FORMULA

J. J. CORLISS, De Paul University, Chicago

The polynomial $F(x)$ of degree less than or equal to m which takes on the preassigned values $f_0, f_1, f_2, \cdots, f_m$ for the distinct values $x = x_0, x_1, x_2, \cdots, x_m$ constitutes Lagrange's Formula.†

As usually given it reads:

$$F(x) = \sum_{i=0}^{m} F_i f_i, \tag{1}$$

where

$$F_i = \frac{(x - x_0)(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_m)}{(x_i - x_0)(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_m)}. \tag{1'}$$

The unique polynomial of minimal degree which takes on $m+1$ preassigned values at $m+1$ distinct points and whose 1st, 2nd, $\cdots$, sth derivatives take on preassigned values at these points may be written in the form:

$$F(x) = \sum_{i=0}^{m} F_i f_i + \frac{1}{\lfloor 1}\sum_{i=0}^{m} F_i^2 (x - x_i)^1 A_i^{(1)} + \frac{1}{\lfloor 2}\sum_{i=0}^{m} F_i^2 (x - x_i)^2 A_i^{(2)} + \cdots + \frac{1}{\lfloor s}\sum_{i=0}^{m} F_i^{s+1} (x - x_i)^s A_i^{(s)}. \tag{2}$$

The $A_i^{(s)}$ may be determined sequentially.

For $s = 1$ we have

* I expect to publish a complete solution of the problem in the near future. [Note added in page proof.]

† Lagrange's Formula may be very conveniently written as the ratio of two determinants—see Nörlund's Leçons sur les séries d'interpolation, p. 4. The author suggests the use of this form when computing the $A_i^{(s)}$ in equation (2).

$$\left.\frac{dF(x)}{dx}\right|_{x=x_r} = \left.\frac{d}{dx}\sum_{i=0}^{m} F_i f_i\right|_{x=x_r} + A_r^{(1)}, \qquad (r = 0, 1, 2, \cdots, m). \tag{3}$$

Knowing $A_r^{(1)}$ $(r=0, 1, 2, \cdots, m)$ the $A_r^{(2)}$ are given by:

$$\left.\frac{d^2F(x)}{dx^2}\right|_{x=x_r} = \left.\frac{d^2}{dx^2}\sum_{i=0}^{m} F_i f_i\right|_{x=x_r} + \left.\frac{d^2}{dx^2}\sum_{i=0}^{m} F_i^2 (x - x_i) A_i^{(1)}\right|_{x=x_r} + A_r^{(2)}, \tag{4}$$

$$(r = 0, 1, 2, \cdots, m).$$

Continuing the process gives all the $A_i^{(s)}$.

If $f(x)$ represent the true function and $R(x)$ the remainder or error term then

$$f(x) = F(x) + R(x), \tag{5}$$

and if $f(x)$ has a continuous derivative of order $(m+1)(s+1)$, we may write

$$R(x) = [(x - x_0)(x - x_1)(x - x_2) \cdots (x - x_m)]^{s+1} \frac{f(x^1)^{(s+1)(m+1)}}{\lfloor (m+1)(s+1)} \tag{6}$$

where x^1 lies in the interval within which $x_0, x_1, \cdots, x_m$ lie.*

Formula (2) is really an extension of Taylor's Series also, for here the function is expanded in a series, given $m+1$ points and the values of the function and its successive derivatives at each of these points. Further the form of formula (2) clearly brings out its relationship to the Lagrange Formula and to Taylor's Series.

The trigonometric approximation formula for periodic functions which corresponds to Lagrange's Formula is due to Gauss and reads:

$$F(x) = \sum_{i=0}^{m} F_i f_i, \tag{7}$$

where

$$F_i = \frac{\sin(x-x_0)\sin(x-x_1)\cdots\sin(x-x_{i-1})\sin(x-x_{i+1})\cdots\sin(x-x_m)}{\sin(x_i-x_0)\sin(x_i-x_1)\cdots\sin(x_i-x_{i-1})\sin(x_i-x_{i+1})\cdots\sin(x_i-x_m)}. \tag{7'}$$

The periodic function corresponding to the formula (2) is:

$$F(x) = \sum_{i=0}^{m} F_i f_i + \frac{1}{\lfloor 1}\sum_{i=0}^{m} F_i^2 \sin(x - x_i) A_i^{(1)} + \frac{1}{\lfloor 2}\sum_{i=0}^{m} F_i^3 \sin^2(x - x_i) A_i^{(2)} + \cdots + \frac{1}{\lfloor s}\sum_{i=0}^{m} F_i^{s+1} \sin^s(x - x_i) A_i^{(s)}. \tag{8}$$

The $A_i^{(s)}$ may be determined sequentially and the error term computed as before.

* The remainder term is computed in the usual manner—see: L. R. Ford, Differential Equations, p. 121; L. M. Milne-Thompson, The Calculus of Finite Differences, p. 5; Steffensen, Interpolation, p. 24.

ON THE DERIVATION OF ARCTANGENT EQUALITIES

J. W. WRENCH, JR., Yale University

For more than two centuries the subject of arctangent equalities has been one of recurring interest. Stated explicitly, the problem has been to express π, or an integral multiple of π, as a linear combination with integral coefficients of angles whose tangents are rational fractions, particularly reciprocals of integers.

The method employed in this paper to derive such relations is due apparently to Gauss,* and seems to have been neglected by the majority of writers on this subject.

The fundamental relation in this discussion is

$$\tan^{-1} 1/x = \frac{i}{2} \log (x - i)/(x + i), \tag{1}$$

where the principal value of the arctangent corresponds to the principal value of the logarithm.†

Consequently the expression

$$a_1 \tan^{-1} 1/x_1 + a_2 \tan^{-1} 1/x_2 + \cdots + a_n \tan^{-1} 1/x_n \tag{2}$$

may be replaced by

$$\frac{i}{2} \log (x_1 - i)^{a_1}(x_2 - i)^{a_2} \cdots (x_n - i)^{a_n}/(x_1 + i)^{a_1}(x_2 + i)^{a_2} \cdots (x_n + i)^{a_n}, \tag{3}$$

where the inverse functions are to be restricted to their principal values in this discussion.

Thus the problem of obtaining arctangent relations involving $k\pi/4$, k being a positive integer or zero, reduces to the determination‡ of n positive integers $x_1, \cdots, x_n$ corresponding to which there exist n integers $a_1, \cdots, a_n$, not all zero, such that the fraction in (3) reduces to $(1-i)^k/(1+i)^k$. The trivial case of linear dependence occurs when k is zero; the most interesting cases from our point of view arise when k is positive.

If the integers $x_1, \cdots, x_n$ are selected in advance, the determination of the coefficients $a_1, \cdots, a_n$ depends upon the solution in integers of a homogeneous system of m linear equations with integral coefficients in n variables, where m denotes the number of distinct odd primes dividing the product of the norms of the algebraic integers $x_j \pm i$, $j = 1, \cdots, n$, in the field $k(i)$. If r is the rank of

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* Gauss, Werke, Bd. 2, p. 478 et seq.

† Weber-Wellstein, Enzyklopädie der Elementarmathematik, Bd. 1, p. 526.

‡ Accomplished with the aid of a table of factorizations of integers of the form x^2+1, or a table of solutions of $x^2 = -1$ for various prime moduli. See Gauss, loc. cit., and Cunningham, Binomial Factorizations, vol. I, pp. 1–22.

the matrix of the coefficients, the system has a set of $n-r$ linearly independent solutions, each of which is a set of relatively prime integers. Moreover, all solutions are expressible with integral coefficients in terms of these independent solutions.* In the majority of applications of this theory it has been found necessary to take $n=m+1$; however, there exist exceptions where $m=n$, such as (5) below.

The necessary and sufficient conditions for the solution of the Diophantine equation obtained by setting (2) equal to $k\pi/4$, for the cases when k is either 0 or 1, have been given in a theorem of Störmer.†

The theory outlined above readily yields all the known arctangent equalities involving π, as well as many new equalities, of which the most important for purposes of calculation are:

$$44\tan^{-1}1/57+7\tan^{-1}1/239-12\tan^{-1}1/682+24\tan^{-1}1/12943=\pi/4, \tag{4}$$

$$7\tan^{-1}1/10+2\tan^{-1}1/50+4\tan^{-1}1/100+\tan^{-1}1/682+4\tan^{-1}1/1000+3\tan^{-1}1/1303-4\tan^{-1}1/90109=\pi/4. \tag{5}$$

The following variation of (5) might be preferred if an extensive machine calculation were undertaken.

$$7\tan^{-1}1/10+8\tan^{-1}1/100+\tan^{-1}1/682+4\tan^{-1}1/1000+3\tan^{-1}1/1303-4\tan^{-1}1/90109-2\tan^{-1}1/500150=\pi/4. \tag{6}$$

In the preceding relations the constant k is unity; the following equality is interesting as a solution of the case where k is 3.

$$56\tan^{-1}1/23-\tan^{-1}1/182-25\tan^{-1}1/500-20\tan^{-1}1/924+5\tan^{-1}1/99557+10\tan^{-1}1/298307=3\pi/4. \tag{7}$$

As far as the writer is aware, no primitive solutions of the Diophantine equation obtained by setting (2) equal to $k\pi/4$ have been found for the cases in which k exceeds 3.

ON VOLUMES BOUNDED BY CYLINDRICAL SURFACES

J. D. MANCILL, University of Alabama

The problem with which this note is concerned is that of determining certain volumes common to two given cylindrical surfaces. The usual procedure of integral calculus leads to extremely difficult integration even in the simplest problems unless the cylindrical surfaces have their elements perpendicular. We shall show how these procedures may be slightly altered to greatly simplify the integration in the general case. This note is prompted by repeated requests for solutions of problems of this nature.

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* Veblen, Analysis Situs, p. 183.

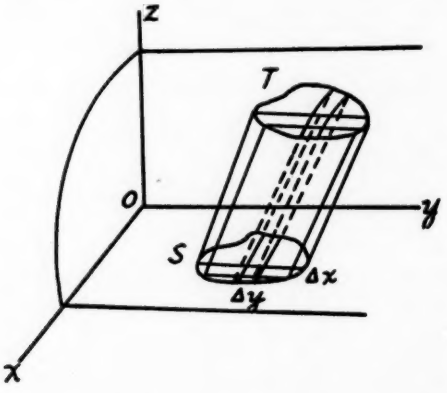
† Comptes Rendus, vol. 122, 1896, p. 175.

Consider the solid bounded by a portion T of the cylindrical surface C_1 with equation of the form

$$C_1: \qquad z = f(x),$$

a cylindrical surface C_2, and the area S whose boundary is the trace of C_2 in the xy-plane as shown in the figure. We shall assume that the x and z axes are chosen so that the elements of the surface C_2 are parallel to the yz-plane.

We can get an expression for an approximation to this volume as follows. Draw in S a set of n lines parallel to the y-axis and a set of m lines parallel to the x-axis as indicated in the figure, thus dividing the region S into rectangles of area $\Delta x \Delta y$, together with a number of irregular portions around the boundary. By passing through each line of the two sets of lines a plane parallel to the ele-



ments of the surface C_2, we divide the volume V into regular oblique columns, together with irregular ones. The upper boundary of each column is a portion of the cylindrical surface T. Through that point of the upper boundary of each column which is nearest the xy-plane pass a horizontal plane thus forming a set of prisms lying wholly within V, each of altitude z. The sum of the volumes of these prisms is evidently an approximation to the required volume, the error committed being the sum of the volumes of the irregular columns around the outside together with the portions lying above the upper bases of the prisms. It is obvious that the error in this approximation may in general be made arbitrarily small by taking both Δx and Δy sufficiently small. Hence, the required volume is

$$V = \int_S \int z dy dx.$$

We see from this formula that the volume V depends solely upon z and the region S. Consequently, all cylindrical surfaces through the boundary of the re-

gion S and whose elements are parallel to the ys-plane have equivalent volumes in common with the surface C_1.

Let V_1 denote the volume common to the two cylindrical surfaces C_1 and C_2 when their elements are perpendicular. Now let the surface C_2 be rotated by rotating each of its elements through an angle $\pi/2-\theta$ in a plane parallel to the ys-plane. It is easily seen that the volume V common to C_1 and C_2 in this position is given by the relation

$$V = \csc \theta V_1.$$

As an illustration let us find the volume common to two cylinders of revolution with radii a and r $(a \geqq r)$ whose axes intersect at an angle θ. The area S in this case is bounded by an ellipse whose equation may be taken in the form

$$x^2/r^2 + y^2/r^2 \csc^2 \theta = 1.$$

Therefore, we have

$$\begin{aligned} V &= 8\int_0^r \int_0^y z\,dy\,dx \\ &= 8\int_0^r \int_0^y [a^2 - x^2]^{1/2} dy\,dx \\ &= 8 \csc \theta \int_0^r [(a^2 - x^2)(r^2 - x^2)]^{1/2} dx \\ &= (16/3)r^3 \csc \theta, \quad \text{if} \quad a = r, \\ &= (8a/3) \csc \theta [(a^2 + r^2)E(k, \pi/2) - (a^2 - r^2)K(\text{mod } k)], \text{ if } a > r, \end{aligned}$$

where $k = r/a$ and E and K denote complete elliptic integrals of the second and first classes respectively.

RECENT PUBLICATIONS

EDITED BY TOMLINSON FORT, Lehigh University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

Leçons d'algèbre et de géométrie à l'usage des étudiants des facultés des sciences. Tome III. By R. Garnier. Paris, Gauthier-Villars, 1937. 6+280 pages. 80 fr.

The third volume of M. Garnier's excellent course is devoted to elimination, line geometry, the Lie line-sphere transformation, and to contact transformations which preserve lines of curvature.

The algebraic theories of symmetric functions and elimination are introduced for themselves, and for application in the later chapters. The four chapters on line geometry constitute a more readable and illuminating introduction than can be found in any of the standard treatises. The treatment of the last two topics is a little sophisticated, but it is not difficult reading for the graduate student.

It is good to see these important and striking contributions of 19th century geometry presented in such an attractive manner. Whatever the trends in geometry may be today and tomorrow, the work of Pluecker and Laguerre, of Klein and Lie and Darboux will not shrink to insignificance. Graduate students must continue to master the content and the import of their researches. It should be a pleasure, not a chore, to do this from the beautifully planned course of M. Garnier.

B. H. BROWN

Mathematics of Finance. By D. H. Mackenzie. New York and London, McGraw-Hill Book Company, 1937. 313 pages. Tables, 214 pages. $3.75.

There are two distinct kinds of courses in mathematics of finance in this country. There is the kind where the instructor can expect a certain amount of facility on the part of the student in algebra and even some calculus. In this kind the instructor has mainly to emphasize certain fundamental principles and then thoroughly drill his students in analyses of the different applications. It is the other kind for which this text is adapted and that is the kind where the student must be taught and drilled in algebra at the same time and, hence, must be fed the various topics in well diluted doses. While the reviewer feels that this is the longest, most difficult, and least satisfactory way to study the mathematical part of finance, he sees no other way out. When one considers that to each student of the former type there are at least twenty-five of the latter, who are primarily interested in some form of business, accounting, and so on, and have had little mathematical training, it is obviously a situation and not a theory with which we have to deal. So far we have had too many texts prepared by authors who are also still weak in their mathematics. Let us say at once that this text is far and away the best text of its kind that we have seen. Above all

it is sound and reliable throughout. No novel types of notation have been invented and introduced.

It is true that the most important formulas (strangely called "equations" throughout the text) are presented in different forms—for the different types of applications—which are apt to prove rather formidable and it is doubtful whether the average student of such a course will appreciate fully their true relations but this is rather a criticism of the situation and not of the text. The author seems to be quite familiar with all the difficulties of his students and in the more difficult sections separates the various applications into specific and numbered "cases." He takes five chapters and over one hundred pages to cover the very fundamental work on annuities.

The text includes the usual topics but leads off with a good chapter devoted entirely to all the abstract work in arithmetic and algebra that the student must know, from the fundamental processes of arithmetic to progressions and the binomial expansion, and finishes with an elementary treatment of life insurance. One of the outstanding features is the tables—to ten and fifteen places! The reviewer feels that the subject of interpolation should have been omitted even from the introduction to the tables since the differences of interest functions are notoriously slow in convergence. Also, few readers will think to look for the index in the middle of the text—just before the tables. On the other hand, no important errors of any kind were found. Much care must have been devoted both to the original preparation and also to the final proofreading.

C. H. FORSYTH

Calculus. By J. V. McKelvey. New York, The Macmillan Company, 1937. 9+420 pages. $3.00.

Too often a student's comment on a first course in calculus is, "I never knew what it was all about." Professor McKelvey in the introduction to his calculus attempts to forestall this criticism by exhibiting in three or four pages the core of differential and integral calculus with a few important illustrations of its applications.

As is customary, the book is divided into four sections: differential calculus, integral calculus, infinite series, and a chapter of twenty-six pages on differential equations. Among the topics not always included in texts of this type we note the following: envelopes, evolutes, involutes, hyperbolic functions, singular points, the order of partial differention, Rolle's theorem, the integral test for convergence, and Taylor's series with a remainder. On the other hand, no mention is made of asymptotes, Duhamel's theorem, comparison of infinitesimals, the second form of Taylor's series, or its extension for functions of two or more variables.

Certain details in the treatment of the material may be of interest. The subject of limits is dealt with by stating carefully the essential definitions and theorems, but instead of including the proofs of these theorems simple examples are given to make them clear, indeed almost self evident. Early in the book the

subject of differentials is introduced and receives more than the usual attention. Differentials of length of arc, plane area, surface and volume of revolution are determined and a few examples given to illustrate the importance of finding a function when its derivative is known. Parametric equations are also fully discussed in a chapter by themselves. To the reviewer it seems that it will be impossible for any student who comes in contact with this text to forget that the definite integral is the limit of a sum, since the author devotes seventeen pages of explanation, diagrams, and examples to summation problems. The definite integral is then introduced as another and simpler notation. This is followed by careful and detailed suggestions concerning the methods of setting up the definite integral, single and iterated. In general it may be said that the text is a scholarly one, including more theory than a calculus designed merely for practical purposes, thus preparing the student for more advanced theoretical work. One finds an abundance of examples, many of them done in great detail. In fact much of the instruction in this treatise is done by the solution of model examples. Frequent summaries are given. This calculus, to an unusual degree, does its own teaching leaving few explanations for the instructor.

The diagrams in the book are excellent. References to articles instead of pages make reading more time consuming than need be. The pages are small, four inches by six of reading matter, and the type is rather large; hence it is a book of considerable bulk.

LENNIE P. COPELAND

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 462 East Foster Ave., State College, Pa.

NOTEWORTHY PAPERS BY STUDENTS

Because of limited space, it is impossible to print the interesting papers by undergraduates submitted to this department. In order to give deserved recognition to such papers, a list of titles, names of authors, and clubs where they were presented is being compiled for early publication. For this first listing, papers presented in the academic year 1936–37 and in the first semester of the year 1937–38 will be considered. Not more than three titles for this period should be sent. These should be mailed before May 1, 1938, accompanied by a statement of worth from the faculty member in charge of the club before which the papers were presented.

A NEW NOTICE OF A CLUB MEETING

A mimeographed page, appearing recently at the University of Toledo, created interest among non-mathematical students and members of Delta x. Headed by three outline figures was the following announcement: "There are three holes in a board: one circular, one square, one in shape of a Greek Cross, *i.e.*, a cross made from a square by removing smaller squares from the four corners of the original one. Diameter of circle, edge of square, and altitude of cross are equal. Design a rigid solid which will pass through each of the three holes, fitting tightly in each hole and completely filling it. Members are urged to construct models (of soap, clay, wood, etc.) of a solid, meeting specifications and bring them to the exhibition to be held at the next meeting."

CLUB REPORTS
1936–37

Pi Mu Epsilon, Hunter College

Director, Professor R. Lucile Anderson; Vice-Director, Catherine Shanahan; Secretaries, Claudia Cohen, Ellen Sharfstein; Treasurer, Ida White; Permanent Secretary, Professor Laura Guggenbühl. The chapter held eight program meetings, an initiation banquet, a tea, and an open house with buffet supper. Topics for first term were selected from the theory of functions of a complex variable, including: Mapping by means of the general linear fractional function; Exponential and trigonometric functions of a complex variable; Cauchy's integral theory; Infinite series of complex terms; Poles and residues. The second term topics on infinite series included: Infinite sequences; Limits; Infinite series of constant real terms; Infinite series of functions of a real variable; Uniform convergence. Prizes were awarded to the best speakers of each term. First term honors were awarded to Ruth Ashkenazy, second term honors to Natalie Krauss. Chapter members received other honors as follows: Joseph A. Gillet Scholarship, Elsie Peterson, Honorable mention, Ruth Ashkenazy; Dr. Thomas Hunter Prize, Claudia Cohen, Honorable mention, Jeanette Demoff; Fellowship in Mathematics at New York University, Bella Manel; Hebrew Technical School for Girls Scholarships, Anna Shames at Columbia University, Ida White at New York University, Claudia Cohen and Esther Lippenholtz at University of Michigan.

Mathematics Club of Hunter College, Lexington Avenue Branch

President, Mildred Wurman; Vice-President, Lillian Leight; Publicity Manager, Mary Eigerman; Secretary, Helene Parnass; Treasurer, Betty Pead; Faculty Adviser, Professor L. S. Hill. Program meetings, two prize problems, cake sales, hikes, and a bingo party filled an active year. The annual spring party at which the Pi Mu Epsilon chapter and the three Branches of the Mathematics Club unite was celebrated with great festivity. A play by Professor A. Marie Whelan entitled "Grabit all," a take-off on the Townsend plan, was presented at this time. The year's program began with a series on algebra in finite fields and the more elementary associated geometry; the secretary's report says: "At first, only finite fields in which the number of elements is a prime were mentioned. Then a prize problem was proposed which really demanded the construction of addition and multiplication tables for two fields in each of which the number of elements was a power of a prime. Since such fields had received no consideration at all, even their existence being unknown to the club members, the problem constituted a puzzle of no mean difficulty. Subsequently, these fields were studied briefly, and the work of the first term came to an end in the elucidation of all the curious relations brought out." Other topics included: The Euler-Maclaurin asymptotic summation series; Transfinite cardinal numbers; The work of a statistician in the textile industry. In two meetings the speakers prepared the way for the second prize problem. They defined, for the "point-pairs"—pairs of diametrically opposite points—on a spherical surface, a system of homogeneous coördinates (x, y, z), and developed the most important and useful elementary formulas of the resultant algebraic geometry on the surface. The problem proposed was that of employing this algebraic analysis to establish the existence of the regular convex icosahedron and dodecahedron, and to compute their volumes (in terms of the lengths of edges) and their dihedral angles.

Rutgers University Mathematics Club

President, L. W. Wilchinsky; Vice-President, O. Gartner; Secretary-Treasurer, N. A. Brigham; Faculty Adviser, Professor E. P. Starke. The club held monthly meetings and cooperated with honor fraternities in presenting to the University a lecture by Professor Kasner, Columbia University, on "Imagination and Conception in Mathematics." The spring banquet was held in conjunction with the Mathematics Club of New Jersey College for Women. Professor E. T. Bell was guest of honor. His lecture, "The Economic Interpretation of the History of Mathematics"

was open to the public. The two mathematical prizes of the university, one open to any senior, the other to seniors with mathematics majors, were both won in competitive examinations by N. A. Brigham. Program topics included: Definite integrals; The power of observation, illustrated by Farey's series; Fallacies and puzzles; History of the squaring the circle problem. A. Gelford received the award of best paper of the year for his original translation from the French of *Construction of the regular seventeen-sided polygon with compasses only*, the French author being M. L. Gérard at Lyon.

Mathematics Club, University of Rochester

President, H. R. Ellis; Secretary-Treasurer, H. V. Strong; Faculty Advisers, Dr. W. Seidel, Dr. G. B. Van Schoack. Two supper meetings and a spring outing furnished outside interest to club members. Prizes for the best student papers at club meetings and the subjects were: First, K. Cliquennoi, Multiplication axioms; Second, P. Gilbert, Integral equations; Third, H. Ellis, Mathematical fallacies. Other program topics included: Circular inversions; Cauchy's condensation test; Calculus of variations; Sturm's theorem; Geometry's application to the Supreme Court; Mathematical philosophy.

Students' Mathematical Round Table, University of Illinois

President, J. J. Eachus; Secretary, Margaret Hasman. Meetings are held on alternate Thursdays. All papers are presented by student members. Program topics for the second semester included: Discontinuous functions; A particular involution in space; Generalized Lambert series; A problem in Abelian groups; Aspects of the theory of partitions; Configuration and double points of the cubics of a pencil.

Mathematics Club, University of Buffalo

President, Evelyn Jaeckle; Secretary, R. Cord; Faculty Adviser, Dr. Harriet F. Montague. The club held monthly meetings and a spring party at which the annual Wilfred E. Sheck Memorial Prize was awarded to Evelyn Jaeckle for her paper, "A rectangle both inscribed and circumscribed to a pair of conjugate hyperbolas." This paper was a solution of a problem in this MONTHLY. At one meeting seniors from East High School, Buffalo, were guests. Program topics included: Survey of the branches of mathematics; Oblique coördinates; Loci of inequalities; Mechanical integrators; Determination of complex roots by graphical methods; Demonstration and explanation of the stroboscope; The calendar; An introduction to analysis situs.

Mathematics Club, Milwaukee-Downer College

President, Kathryn Wambach; Secretary-Treasurer, Erngarde Kaddatz; Representative to Intercollegiate Mathematics Association, Norma Fedders. Topics at meetings included: Mathematical bibliography; History of mathematical notation; Symmetry; Mathematical paradoxes and fallacies; The calendar, old and new. The report includes this interesting note:

"One of our members has recently made a set of ten Napier's Rods, which we will gladly loan to anyone wishing to use them for a meeting. They illustrate very well the primitive state of mathematical learning in the seventeenth century."

Requests for these Rods should be made to the Department of Mathematics, Milwaukee-Downer College, Milwaukee, Wisconsin.

Sigma Phi Mu, New Jersey State Teachers College, Montclair

President, E. Devereux; Vice-President, Marjorie Arkenberg; Secretary, Helen Robertson; Treasurer, Lorraine Grauss; Adviser, Howard Fehr. Three social meetings were held in addition to the regular monthly meetings. Topics discussed included: Dynamic teaching of plane geometry; Quaternions; Two schools of thought in foundations of mathematics; The astralobe; The sextant; Visual aids for teaching geometry.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 315. *Proposed by F. A. Alfieri, New York City.*

In any plane triangle, prove that the ratio of the sum of the squares of the sides to four times the sum of the cotangents of the angles is equal to the area.

E 316. *Proposed by J. E. Trevor, Cornell University.*

The profile of a certain straight road is the curve

$$125x^2y + 5y - 1 = 0,$$

where x and y are measured in miles. In miles per hour and with reference to the road, the speed at which Mr. Smith walks is a linear function of the slope of the road, is four miles per hour on the level, and becomes zero when the slope reaches the value 0.8. Smith lives on the road at $x_0 = -0.8$ miles. He left home on foot at 9:35 A.M., saying to his son, "Telephone to your mother that I will meet her at the crest of the hill at ten o'clock." Did he arrive in time for this appointment?

E 317. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

PRYP, *LARAP*, and *LLUMP* are three numbers in harmonic progression, where each letter is the code representative of some digit in the decimal system. Solve the code.

E 318. *Proposed by N. A. Court, University of Oklahoma.*

A tetrahedron $KLMN$ is inscribed in the tetrahedron $ABCD$, and O is a fixed point in space. If the parallels AP, BQ, CR, and DS to the lines OK, OL, OM, and ON meet the respective faces of $ABCD$ in P, Q, R, and S, show that

$$\frac{OK}{AP} + \frac{OL}{BQ} + \frac{OM}{CR} + \frac{ON}{DS} = 1.$$

E 319. *Proposed by Joseph Rosenbaum, Bloomfield, Connecticut.*

An electric light bulb is connected to n switches in such a way that the light is lit only when each switch is closed. Each switch is controlled by a push button, successive depressions of which will alternately open and close that switch. The push buttons are not provided with the usual marks, "on" and "off."

It is required to find in what order the push buttons should be pressed so that the greatest possible number of pushes which may be required to turn on the light, will be as small as possible. Generalize the solution to cover the case in which each switch closes only at every pth push.

E 320. *Proposed by D. H. Lehmer, Lehigh University.*

A table of the binomial coefficients of the 48593rd power would begin as follows:

1

48593

1180615528

19122429707016

232289714865976860

2257344991124589930108

182799797381269292540145 84

. . . . . .

Show that if this table were completed, the first digits of these numbers would lie very nearly on a parabola assuming equal spaces for all digits.

SOLUTIONS

E 282 [1937, 384]. *Proposed by A. Gloden, l'Athénée de Luxembourg.*

Find a seven-digit cube of the form, $1{,}000{,}000x+1{,}000y+z$, with $y=x+z+1$. Is the solution unique?

Solution by D. C. Duncan, Compton, California.

Eliminating y, the desired number must be of the form, $1{,}000{,}000x+1000(x+z+1)+z$, so that $N^3=1001(1000x+z)+1000$. Thence, $N^3\equiv 1000$ (modulo 1001). The complete solution of this congruence for $N<1001$ gives 10, 153, 285, 472, 615, 857, 901, and 1000. Of these positive solutions, the only one whose cube consists of seven digits is 153. Hence the unique positive solution is 3,581,577.

If negative solutions are admissible, there are two additional solutions, namely $(-100)^3$ and $(-144)^3$, or $-1{,}000{,}000$ and $-2{,}985{,}984$.

Also solved by W. E. Buker, J. E. Burnam, Daniel Finkel, Jack Lorell, Daniel Finkel, E. P. Starke, C. W. Trigg, and the proposer.

E 283 [1937, 384]. *Proposed by W. B. Clarke, San Jose, California.*

Find the smallest pair of differently shaped right triangles with integer sides and the same area.

Solution by William Douglas, Courtenay, British Columbia.

Let the legs of a right triangle be $2mn$ and m^2-n^2, with $n<m$. Then the

area is $mn(m^2-n^2)$. By assigning to m successively the values 2, 3, 4, 5, and 6, with n taking all possible smaller values for each m, we find that the area of 210 appears both for $m=5$, $n=2$ and for $m=6$, $n=1$. The sides of these two triangles are 20, 21, 29 and 12, 35, 37.

Editorial Note. Three readers in widely separated states sent in "proofs" that two differently shaped right triangles could not have the same area.

Also solved by H. T. R. Aude, W. E. Buker, E. P. Starke, C. W. Trigg, and the proposer.

E 284 [1937, 384]. *Proposed by V. Thébault, Le Mans, France.*

Find the set of numbers, each of which equals thrice the square of the sum of its digits.

Solution by Mary L. Constable, Philadelphia.

Let the number be $3s^2$, where s is the sum of the digits. Since the number is divisible by 3, s is divisible by 3, and therefore the number is divisible by 9. Therefore s is divisible by 9 and the number may be represented by $3(9n)^2$, where n is an integer.

If $n=0$, 1, or 2, the number equals 0, 243, or 972. Substitution shows that n may not be 3, 4, or 5.

Suppose $n=6$. Then $3(9n)^2$ may not contain more than five digits. But the smallest number whose digit sum is 54 is 999999, a number of six digits. Therefore n may not be 6, and in like manner may not exceed 6. Hence the set required consists of 0, 243, and 972.

Also solved by H. T. R. Aude, W. E. Buker, J. E. Burnam, D. C. Duncan, Daniel Finkel, E. P. Starke, C. W. Trigg, and the proposer.

E 285 [1937, 384]. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

If in triangle ABC, $\sin^2A+\sin^2B+\sin^2C=1$, prove that the circumcircle cuts the nine point circle orthogonally.

Solution by C. E. Springer, University of Oklahoma.

If we employ areal coördinates with the triangle ABC as the triangle of reference, the circle $Lx^2+My^2+Nz^2+2Pyz+2Qzx+2Rxy=0$ cuts the circumcircle $a^2yz+b^2zx+c^2xy=0$ orthogonally if $aL\cos A+bM\cos B+cN\cos C=abcK$, where a, b, and c are the sides of the triangle of reference, and where

$$K=\frac{M+N-2P}{a^2}=\frac{N+L-2Q}{b^2}=\frac{L+M-2R}{c^2}.$$

(Askwith, *The Analytical Geometry of the Conic Sections*, Black, 1918, p. 308, art. 303.)

For the nine point circle, $L=b^2+c^2-a^2$, $M=c^2+a^2-b^2$, $N=a^2+b^2-c^2$, $P=-a^2$, $Q=-b^2$, $R=-c^2$. If we use these values in the condition above, it becomes $\cos^2A+\cos^2B+\cos^2C=2$, or $\sin^2A+\sin^2B+\sin^2C=1$.

Also solved by F. A. Alfieri, E. F. Allen, J. H. Butchart, D. C. Duncan, V. W. Graham, L. M. Kelly, C. W. Trigg, and the proposer.

E 286 [1937, 384]. *Proposed by Mannis Charosh, New Utrecht High School, Brooklyn, N. Y.*

Prove that

$$\sum_{r=0}^{n-1} 2^r \tan(2^r x) = \cot x - 2^n \cot(2^n x).$$

Solution by L. S. Johnston, University of Detroit.

The identity, $\tan\theta = \cot\theta - 2\cot 2\theta$, is well known, and quite easily proved. We have therefore:

$$\begin{aligned}
\tan\ x &= \cot\ x - 2\cot 2x,\\
\tan 2x &= \cot 2x - 2\cot 4x,\\
\tan 4x &= \cot 4x - 2\cot 8x,\\
&\cdots\cdots\cdots\cdots\cdots\\
\tan 2^{n-1}x &= \cot 2^{n-1}x - 2\cot 2^n x.
\end{aligned}$$

Multiplying both members of the first equation by 2^0, the second by 2^1, the third by 2^2, $\cdots$, the nth by 2^{n-1}, and adding the respective members of the resulting equations, we have the solution at once.

The problem is also easily solved by mathematical induction, as the reader will find upon trial. This problem is set as an exercise in some of the older, more classical texts in trigonometry, *e.g.*, Hall and Knight, *Elementary Trigonometry*, 1919, page 294, exercise 7. On the same page is proved the somewhat similar identity

$$\sum_{r=0}^{n} 2^{-r} \tan(2^{-r} x) = 2^{-n} \cot(2^{-n} x) - 2\cot 2x.$$

It is easy to prove, by the use of the identity by which the proposed problem has been proved, that

$$\sum_{r=a}^{b} 2^r \tan(2^r x) = 2^a \cot(2^a x) - 2^{b+1} \cot(2^{b+1} x),$$

where a and b may have integral values, positive or negative, or may have the value zero, provided only that b is greater than a.

Also solved by E. F. Allen, H. T. R. Aude, J. E. Burnam, J. H. Butchart, R. W. Cowan, D. C. Duncan, Abe Gelbart, J. F. Heyda, C. E. Springer, E. P. Starke, and the proposer.

E 287 [1937, 384]. *Proposed by W. B. Campbell, Drexel Institute.*

Find the maximum and minimum values of the function Z, defined as follows:

when $0 \leqq x \leqq \frac{1}{2}$ and $0 \leqq y \leqq x$, or when $\frac{1}{2} \leqq x \leqq 1$ and $0 \leqq y \leqq 1-x$,

$$Z = y - y^2/3x;$$

when $\frac{1}{2} \leqq x \leqq 1$ and $1-x \leqq y \leqq x$,

$$Z = y - y^2/3x - (x + y - 1)^3/(6xy).$$

Solution by E. P. Starke, Rutgers University.

The familiar necessary condition for maximum and minimum values of a function of two variables, viz. $\partial Z/\partial x = 0 = \partial Z/\partial y$, is not satisfied at any *interior* point of the three regions cited. We may then find the desired values by studying the behavior of Z along the boundaries.

If $y=0$ and $x \neq 0$, then $Z=0$.

$\text{Lim}_{x,y \to 0} Z = 0$ along any path in the given region.

If $x=y$ and $0 \leqq y \leqq 1/2$, then $Z=2y/3$ has the maximum value $1/3$ at $(1/2, 1/2)$.

If $x=1/2$ and $0 \leqq y \leqq 1/2$, then $Z=y-2y^2/3$ hast he maximum value $1/3$ at $(1/2, 1/2)$. $(dZ/dy \neq 0$ within the interval.)

If $x=1-y$ and $0 \leqq y \leqq 1/2$, then $Z=y-y^2/(3-3y)$ has the maximum value $1/3$ at $(1/2, 1/2)$. $(dZ/dy=0$ at $y=1/2.)$

If $x=1$ and $0<y \leqq 1$, then $Z=y-y^2/2$ has the maximum value $1/2$ at $(1, 1)$.

$\text{Lim}_{x \to 1, y \to 0} Z = 0$ along any path in the given region.

If $x=y$ and $1/2 \leqq y \leqq 1$, then $Z=2y/3-(2y-1)^3/6y^2$ has the maximum value $1/2$ at$(1, 1)$ and the minimum value $1/3$ at $(1/2, 1/2)$.

Furthermore Z is never negative. It is then evident that Z takes its minimum value, 0, for all points $y=0$ and $0 \leqq x \leqq 1$ and its maximum value $1/2$ for $(1, 1)$.

Also solved by V. W. Graham and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3858. *Proposed by A. W. Tucker, Princeton University.*

Prove constructively that any collineation in the projective plane is a product of a finite number of involutoric collineations.

3859. *Proposed by J. R. Musselman, Western Reserve University.*

For any five tangent lines L_i of a parabola there is a centric circle A passing through the five points c_i, where c_i are the centric centers of the five four-lines which can be formed from the five tangents. Thus c_5 is the center of the circle

passing through the four circumcenters C_{123}, C_{124}, C_{134} and C_{234} of the lines L_1, L_2, L_3 and L_4. See Musselman, this MONTHLY, 1937, p. 513. Join c_1 to C_{234}, c_2 to C_{134}, c_3 to C_{124}, c_4 to C_{123}. The four lines meet at a point P_5 on the circle A. Cyclic permutation of the subscripts will produce five sets of four lines meeting in points on the centric circle A. This theorem will generalize for n tangent lines of a parabola.

3860. *Proposed by Joseph Rosenbaum, Bloomfield, Connecticut.*

Given a tetrahedron, find the point such that the sum of its distances from the vertices is a minimum.

3861. *Proposed by V. Thébault, Le Mans, France.*

A convex polygon $A_1A_2 \cdots A_{2n}$, with its opposite sides equal and parallel, is inscribed in a conic. If from a point of the conic parallels are drawn to the directions conjugate to those of the sides, the intersections of these parallels with the conjugate sides form a polygon of constant area, and its opposite sides have conjugate directions.

3862. *Proposed by V. Thébault, Le Mans, France.*

Through the vertices of a triangle ABC perpendiculars to its plane are drawn

$$AA_1 = -AA_2 = BC, \qquad BB_1 = -BB_2 = CA, \qquad CC_1 = -CC_2 = AB.$$

Show that the points A_1, B_1, C_1, A_2, B_2, C_2 and the vertices of the triangle anti-complementary to ABC lie on the same sphere.

SOLUTIONS

3771 [1936, 110]. *Proposed by Hansraj Gupta, Hoshiarpur, India.*

Prove that

$$61! + 1 \equiv 0 \bmod 71,$$
$$63! + 1 \equiv 0 \bmod 71.$$

Prove the general result of which these are particular cases.

Solution by E. L. Harp, Jr., Roswell High School, Roswell, New Mexico.

If p is a prime it is well known that (1) $(p-1)!+1 \equiv 0 \bmod p$; and it is easily seen that

$$(2) \qquad (p-1)(p-2)\cdots(p-r) \equiv (-1)^r r! \bmod p.$$

Let r be any positive integer less than p such that

$$(3) \qquad (-1)^r r! \equiv 1 \bmod p.$$

Then from (1), (2), (3) we have

$$(4) \qquad (p-r-1)! + 1 \equiv 0 \bmod p;$$

and conversely if (4) is true, (3) is true. For the numerical case of the problem

we easily verify that $(-1)^{7}7!\equiv 1 \bmod 71$; and since $8\cdot 9\equiv 1 \bmod 71$, we have also $(-1)^{9}9!\equiv 1 \bmod 71$. Hence the congruences of the problem are true.

Solved also by G. E. Raynor, E. P. Starke, and F. L. Wilmer.

Editorial Note. All of the solvers gave the same general result. Starke stated that the same kind of reasoning gives $\{[(p-1)/2]!\}^2\equiv(-1)^{(p+1)/2} \bmod p$. If p is of the form $4k+3$, then

$$[(p-1)/2]!\equiv(-1)^m \bmod p,$$

where m is the number of quadratic non-residues less than $p/2$. See the solution of 3682 [1935, 630]. Thus $9!\equiv 1 \bmod 19$; $11!\equiv 1 \bmod 23$, and so on.

3772 [1936, 110]. *Proposed by H. D. Ruderman, Brooklyn, N. Y.*

Given two perspective triangles $A_1A_2A_3$ and $A_1'A_2'A_3'$, let d_{ij} denote the perpendicular distance from A_i' to A_iA_k, where d_{ij} is positive or negative according as A_i' and A_j are on the same or opposite sides of A_iA_k. Prove that $d_{12}d_{23}d_{31}=d_{13}d_{32}d_{21}$.

Solution by J. M. Feld, New York City.

A more general theorem is obtained if the two perspective triangles are replaced by two perspective polygons, $A_1A_2\cdots A_n$ and $A_1'A_2'\cdots A_n'$. We then have to prove that

$$d_{12}d_{23}\cdots d_{n1}=d_{1n}d_{n,n-1}d_{n-1,n-2}\cdots d_{21}. \tag{1}$$

We first prove a preliminary formula.

Let the vertices $A_1, A_2, \cdots$, follow one another in clockwise order. Join each vertex to a point P distinct from the A_i. Let α_i and β_i be the angles that PA_i forms with A_iA_{i+1} and A_iA_{i-1} respectively. We take $A_0\equiv A_n$ and $A_{n+1}\equiv A_1$. Let $PA_i=c_i$. Then $\sin\alpha_i/c_{i+1}=\sin\beta_{i+1}/c_i$ for $i=1, 2, \cdots, n$. Multiplying these n equations together and removing a common factor we obtain

$$\sin\alpha_1\sin\alpha_2\cdots\sin\alpha_n=\sin\beta_1\sin\beta_2\cdots\sin\beta_n. \tag{2}$$

As P recedes to infinity along any direction this equation remains valid and, in consequence of the continuity of each member, the equation obtains when P is on the ideal line at infinity. Since the polygons are perspective the lines A_iA_i' are concurrent at a point P, which may lie at infinity. Let the segment $A_iA_i'=p_i$. Then

$$d_{i,i+1}=p_i\sin\beta_i \quad \text{and} \quad d_{i,i-1}=p_i\sin\alpha_i.$$

Therefore by virtue of (2) the two members of (1) are equal in absolute value.

To show that they have the same sign, the given convention regarding signs must first be restated to apply to polygons. On the assumption that the polygons are convex, the convention may be stated as follows: if A_i' lies on the same side of A_iA_{i-1} as polygon $A_1A_2\cdots A_n$ (to be called hereafter A), let $d_{i,i+1}$ be positive; otherwise negative. We shall use the following observation, which is an obvious consequence of the adopted convention. If a line joining a point, Q,

not on a side of A, to the vertex A_i has points inside the polygon, the normals from Q to A_iA_{i+1} and A_iA_{i-1} have like signs; otherwise the normals have unlike signs.

Polygons A and A' being perspective, let P be their center of perspectivity. If P falls within polygon A, $A_i'A_i$ has points within A and therefore, for every i, the normals from A_i' to A_iA_{i+1} and A_iA_{i-1} have like signs. But of each pair of such normals one appears in the left and one in the right member of (1). Hence the two products have the same sign. If P falls outside polygon A, two and only two of the lines joining P to the vertices of A have no points inside A. Let A_j' and A_k' be the vertices of A' that lie on these two lines. It follows that the pair of normals from A_j', $d_{j,j+1}$ and $d_{j,j-1}$ have unlike signs; likewise $d_{k,k+1}$ and $d_{k,k-1}$ have unlike signs. However, the pairs of normals from any other vertex of A_i' have like signs. Because the normals of each of these pairs occur on opposite sides of (1), the products of the two members must have the same sign.

Solved also by L. M. Kelly and C. E. Springer.

Editorial Note. The solution by Springer used areal coördinates with $A_1A_2A_3$ as the triangle of reference. It is simpler to use absolute normal coördinates. Let O be the center of perspectivity with the normal coördinates x_1, x_2, x_3, which are the actual lengths of the perpendiculars from O to the sides of the reference triangle provided with signs by the rule in the problem. With this slight change his solution is as follows: From the equation of $A_1A_1'O$ we have at once $d_{12}/x_2 = d_{13}/x_3$; there are two more equations resulting from a cyclic permutation, and their product gives the theorem. In Kelly's solution A_2A_3 and $A_2'A_3'$ meet in P_1, giving $d_{31}/d_{21} = P_1A_3'/P_1A_2'$. Considering $P_1A_2A_3$ as a transversal of triangle $OA_2A_2'A_3'A_3$, we have by the theorem of Menelaus

$$\frac{P_1A_3'}{P_1A_2'}\,\frac{OA_3}{OA_2}\,\frac{A_2A_2'}{A_3A_3'} = 1.$$

Two more pairs of equations result from the cyclic interchange, and the theorem easily results from these after an examination of the sign of the d's. The proposer stated that Ceva's theorem is the result of a special case.

3773. [1936, 111]. *Proposed by William Hoover, Columbus, Ohio.*

Two planes are tangent to a central quadric surface; they are also at right angles to each other and their line of intersection has a given direction. Find the form of the cylinder generated by their line of intersection.

Solution by C. E. Springer, University of Oklahoma.

If $\sum ax^2 = 1$ is the equation of the surface, the two tangent planes are given by

$$\sum l_i x = \left[\sum \frac{l_i^2}{a}\right]^{1/2}, \qquad \text{where } i = 1 \text{ and } 2. \tag{1}$$

Since these planes are perpendicular

$$\sum l_1 l_2 = 0 \tag{2}$$

and, if the given direction has direction numbers $(\alpha:\beta:\gamma)$, then we may write

$$m_1 n_2 - m_2 n_1 = \alpha, \qquad n_1 l_2 - n_2 l_1 = \beta, \qquad l_1 m_2 - l_2 m_1 = \gamma. \tag{3}$$

To eliminate the six parameters l_i, m_i, and n_i from (1), (2), and (3), multiply equations (3) by l_i, m_i, and n_i respectively and add, to find

$$\alpha l_i + \beta m_i + \gamma n_i = 0, \qquad i = 1 \text{ and } 2. \tag{4}$$

Eliminate n_i between (1) and (4) to find a quadratic expression which is satisfied by the ratios l_1/m_1 and l_2/m_2, namely,

$$\begin{aligned} bl^2[ac(\gamma x - \alpha z)^2 - (a\alpha^2 + c\gamma^2)] + 2ablm[c(\gamma x - \alpha z)(\gamma y - \beta z) - \alpha\beta] \\ + am^2[bc(\gamma y - \beta z)^2 - (b\beta^2 + c\gamma^2)] = 0. \end{aligned} \tag{5}$$

Now, eliminate n_1 and n_2 between (2) and (4) to obtain

$$\alpha\beta\left[\frac{l_1}{m_1} + \frac{l_2}{m_2}\right] + (\alpha^2 + \gamma^2)\frac{l_1}{m_1}\cdot\frac{l_2}{m_2} + \beta^2 + \gamma^2 = 0. \tag{6}$$

Substitute the expressions for the sum and product of the roots from (5) into (6) and simplify to obtain the cylinder

$$\sum (\gamma y - \beta z)^2 = \sum \alpha^2\left(\frac{1}{b} + \frac{1}{c}\right). \tag{7}$$

We may suppose $\sum l_i^2 = 1$ in (1) and write the two tangent planes in the form $\sum l_i x = p_i$. The perpendiculars (p_i) from the center O of the quadric to the planes (i) determine a rectangle, one diagonal of which is the chord of contact of the planes (i) and the other is the perpendicular (of length R) from O to the intersection. Now, from relations (3) we have

$$\begin{aligned} \gamma y - \beta z &= l_1 p_2 - l_2 p_1, \\ \alpha z - \gamma x &= m_1 p_2 - m_2 p_1, \\ \beta x - \alpha y &= n_1 p_2 - n_2 p_1. \end{aligned}$$

Squaring and adding we have

$$\sum (\gamma y - \beta z)^2 = p_1^2 + p_2^2. \tag{8}$$

Thus,

$$R^2 = p_1^2 + p_2^2 = \sum \alpha^2\left(\frac{1}{b} + \frac{1}{c}\right) = \text{constant},$$

so that (7) represents a right circular cylinder which is real, null, or imaginary according as

$$\sum \alpha^2\left(\frac{1}{b}+\frac{1}{c}\right) \gtreqless 0.$$

Editorial Note. This problem is closely related to the following well known theorem:

The locus of the intersection of orthogonal tangents to an ellipse is the director circle, its center at the center of the ellipse and its radius equal to the length of the chords joining the extremities of the major and minor axes. For a hyperbola it is again a circle with its center at the center of the hyperbola. The circle is real if the angle between the asymptotes is less than a right angle; a null circle if the angle is a right angle; an imaginary circle if the angle is greater than a right angle. If the conic is a parabola the locus is the directrix.

In the problem the generators of the cylinder S' meet in P at infinity. The polar plane of P with respect to the given central quadric S is a diametral plane π conjugate to the axis OP, where O is the center of S. The plane π cuts from S a conic C and from the cylinder S' a directrix curve C'. Now consider a plane α orthogonal to the elements of S'. The section of S' by α is K', the orthogonal projection of C'; the projection of C is a conic K. It is now clear that K' is the locus of the intersection of orthogonal tangents to the conic K. Hence, if S is an ellipsoid, the cylinder S' is real and right circular. If S is an hyperboloid of one sheet and if π does not contain an element of the asymptotic cone, S' is again real and right circular. If π is tangent to the cone, C consists of two parallel straight lines and the conjugate axis OP lies in π equally distant from these two parallels, and S' is real and right circular. If π contains two distinct cone elements, S' is right circular, real or imaginary, and it may be a straight line. If S is an hyperboloid of two sheets and if π cuts the cone in two distinct elements, we shall have the same cases possible as before for the hyperboloid of one sheet. If π does not contain a cone element, there is no real S'. If π is tangent to the cone, it is also tangent to S at P, where OP is the cone element of tangency. Thus π is the only tangent plane to S passing through parallels to OP, and there is no real cylinder in this case.

In the above proof we may take the three sets of direction numbers as actual direction cosines so that their determinant in the proper order is unity. Then we have three equations such as $l_1^2+l_2^2+\alpha^2=1$; and then from (1) we have

$$p_1^2+p_2^2=\sum\frac{l_1^2+l_2^2}{a}=\sum\frac{1-\alpha^2}{a}.$$

This with the proof of (8) would replace part of the analysis.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The examination in this competition will be held Saturday, April 16, 1938.

Much interest has been expressed in this competition over the country. A detailed announcement was made in the MONTHLY for January and has been sent to the heads of departments of mathematics in the colleges and universities of the United States and Canada.

A postcard request to the Secretary of the Association, Oberlin, Ohio, from the head of a department will bring an application blank to be used by him in entering a team of three or in entering individual contestants before April first. It will be noted that the largest prizes, except for the thousand dollar scholarship, are associated with the competition of mathematical teams; however, both these and individual contestants are welcomed for the competition.

W. D. CAIRNS, *Secretary*

The University of Notre Dame announces a symposium on "The physics of the universe and the nature of primordial particles" to be held at the University of Notre Dame on May 2 and 3, 1938. Among those who will participate are C. D. Anderson, Gregory Breit, A. H. Compton, J. F. Carlson, Eugene Guth, A. E. Haas, W. D. Harkins, Canon Georges Lemaître, Harlow Shapley, and M. S. Vallarta.

Professor M. H. Ingraham of the University of Wisconsin was elected President of the American Association of University Professors at the annual meeting of that association recently held in Indianapolis. That association is considered to be the greatest factor in preserving academic freedom in America at the present time, and selection as its president carries with it important responsibility. Professor Ingraham has been active for a number of years in the affairs of that association, and his election to the presidency brings high honor in the academic world to a member of our Association.

D. W. Hall of the University of Virginia has been appointed lecturer in mathematics at the University of California at Los Angeles for the second semester of 1937–38.

W. R. Hanson has been appointed an instructor in mathematics for the second semester of the year 1937–38 at North Dakota Agricultural College.

At the Massachusetts Institute of Technology, Dr. C. W. MacGregor has been promoted to an assistant professorship.

A. B. Mewborn of the University of Arizona has been promoted to an assistant professorship.

Dr. Ruth B. Rasmusen has been appointed an instructor at the State Teachers College, Platteville, Wisconsin.

Dr. W. E. Sewell of the Georgia School of Technology has been promoted to an assistant professorship.

The following appointments to instructorships are announced:
Agricultural and Mechanical College of Texas: R. R. Lyle
Michigan College of Mining and Technology: V. O. York
University of Nevada: Mr. Ingo Maddaus
Rose Polytechnic Institute: T. P. Palmer
Virginia Military Institute: I. G. Foster
Washburn College: Paul Eberhart.
Wayne University: Dr. T. H. Southard; Dr. Margarete C. Wolfe.

Professor C. A. Isaacs of the State College of Washington died August 12, 1937. He was a charter member of the Mathematical Association and had taught at State College since 1909.

Professor H. W. Tyler, professor emeritus of mathematics at the Massachusetts Institute of Technology and consultant in science at the Library of Congress, died on February 2, 1938 at the age of seventy-four. He was a charter member of the Mathematical Association.

Miss Orpha E. Worden, associate professor of mathematics at Wayne University, died at Miami Beach, Florida, February 7, 1938. On February 1, she had retired from service after forty years of teaching, the last twenty of which were in Detroit Teachers College and Wayne University.

THE TWENTY-SECOND ANNUAL MEETING OF THE ASSOCIATION

The twenty-second annual meeting of the Mathematical Association of America was held at Indianapolis, Indiana, on Thursday and Friday, December 30–31, 1937, in conjunction with the meetings of the American Association for the Advancement of Science, the American Mathematical Society, The National Council of Teachers of Mathematics, the Association for Symbolic Logic, and the Institute of Mathematical Statistics. Four hundred twelve were in attendance at the meetings, including the following two hundred fifty-eight members of the Association:

V. W. Adkisson, University of Arkansas
R. P. Agnew, Cornell University
A. A. Albert, University of Chicago
Nola L. Anderson, Sophie Newcomb College
R. C. Archibald, Brown University
W. C. Arnold, DePauw University
Max Astrachan, Antioch College
C. S. Atchison, Washington and Jefferson College
H. G. Ayre, Western Illinois State Teachers College
W. L. Ayres, University of Michigan

R. W. Babcock, Kansas State College
H. W. Bailey, University of Illinois
D. F. Barrow, University of Georgia
Juna Lutz Beal, Butler University
E. F. Beckenbach, Rice Institute
F. A. Beeler, Hillsdale College
May M. Beenken, State Teachers College, Oshkosh, Wis.
A. A. Bennett, Brown University
William Betz, University of Rochester
G. D. Birkhoff, Harvard University
G. A. Bliss, University of Chicago
Henry Blumberg, Ohio State University
L. M. Blumenthal, University of Missouri
O. K. Bower, University of Illinois
M. G. Boyce, Western Reserve University
J. W. Bradshaw, University of Michigan
J. W. Branson, New Mexico State College
R. W. Brink, University of Minnesota
H. W. Brinkmann, Swarthmore College
Foster Brooks, Kent State University
H. E. Buchanan, Tulane University
C. T. Bumer, Kenyon College
R. S. Burington, Case School of Applied Science
I. W. Burr, Antioch College
L. E. Bush, College of St. Thomas
Jewell Hughes Bushey, Hunter College
J. H. Butchart, Phillips University
W. E. Byrne, Virginia Military Academy

W. D. Cairns, Oberlin College
M. Patricia Callaghan, Fontbonne College
B. H. Camp, Wesleyan University
E. J. Camp, Macalester College
Mildred E. Carlen, Brown University
G. E. Carscallen, Wabash College
C. C. Carter, Bluffs, Ill.
W. E. Cederberg, Augustana College
Laura E. Christman, Senn High School, Chicago
H. C. Christofferson, Miami University
E. H. Clarke, Hiram College
L. M. Coffin, Coe College
L. W. Cohen, University of Kentucky
J. B. Coleman, University of South Carolina
E. G. H. Comfort, University of Arkansas
N. B. Conkwright, University of Iowa
A. H. Copeland, University of Michigan
W. A. Cordrey, Lambuth College
J. J. Corliss, DePaul University, Chicago
A. T. Craig, University of Iowa
Forrest Cumming, University of Georgia
H. B. Curry, Pennsylvania State College
D. R. Curtiss, Northwestern University
J. H. Curtiss, Cornell University
E. H. Cutler, Lehigh University

D. R. Davis, State Teachers College, Montclair, N. J.
H. T. Davis, Northwestern University
W. M. Davis, Armour Institute of Technology
L. A. V. DeCleene, St. Norbert College
Brother Louis De LaSalle, St. Mary's College, Winona, Minn.
L. L. Dines, Carnegie Institute of Technology
E. L. Dodd, University of Texas

J. E. DOTTERER, Manchester College
OLIVE M. DRAPER, Taylor University
D. M. DRIBIN, Yale University

W. E. EDINGTON, DePauw University
P. D. EDWARDS, Ball State Teachers College
Sister ANN ELIZABETH, St. Mary College, Leavenworth, Kans.
W. J. ETTINGER, Edison General Electric Appliance Co., Chicago
G. C. EVANS, University of California
G. W. EVANS, Swampscott, Mass.
H. P. EVANS, University of Wisconsin

B. F. FINKEL, Drury College
C. H. FISCHER, Wayne University
W. W. FLEXNER, Cornell University
M. M. FLOOD, Princeton University
K. W. FOLLEY, Wayne University
L. R. FORD, Armour Institute of Technology

F. J. GERST, Loyola University, Chicago
H. H. GOLDSTINE, University of Chicago
G. D. GORE, Y.M.C.A. College, Chicago
CORNELIUS GOUWENS, Iowa State College
G. H. GRAVES, Purdue University
F. L. GRIFFIN, Reed College

D. C. HARKIN, Alabama Polytechnic Institute
E. G. HARRELL, Ouachita College
M. C. HARTLEY, University of Illinois High School
M. L. HARTUNG, Ohio State University
G. G. HARVEY, College of the City of New York
J. O. HASSLER, University of Oklahoma
C. T. HAZARD, Purdue University
L. A. HAZELTINE, Stevens Institute of Technology
E. R. HEDRICK, University of California at Los Angeles
GERTRUDE HENDRIX, Eastern Illinois State Teachers College
CORA B. HENNEL, Indiana University
M. R. HESTENES, University of Chicago
DEBORAH M. HICKEY, Delta State Teachers College
RUBY U. HIGHTOWER, Shorter College
MARTHA HILDEBRANDT, Proviso Township High School
T. H. HILDEBRANDT, University of Michigan
R. C. HILDNER, Mount Union College
T. R. HOLLCROFT, Wells College
E. MARIE HOVE, State Teachers College, Wayne, Nebr.
R. C. HUFFER, Beloit College
M. GWENETH HUMPHREYS, Sophie Newcomb College
E. V. HUNTINGTON, Harvard University
W. R. HUTCHERSON, Berea College
C. A. HUTCHINSON, University of Colorado

M. H. INGRAHAM, University of Wisconsin

DUNHAM JACKSON, University of Minnesota
G. H. JAMISON, State Teachers College, Kirksville, Mo.
E. D. JENKINS, Eastern Kentucky State Teachers College
FRITZ JOHN, University of Kentucky

D. K. KAZARINOFF, University of Michigan.
M. W. KELLER, Purdue University
A. J. KEMPNER, University of Colorado
EVELYN M. KENNEDY, High School, Cincinnati, Ohio
J. M. KINNEY, Woodrow Wilson Junior College, Chicago
W. J. KIRKHAM, Oregon State Agricultural College
ELIZABETH E. KNIGHT, State Teachers College, Milwaukee, Wis.
L. A. KNOWLER, Hunter College
J. H. KUSNER, University of Florida

RUTH O. LANE, University of Iowa High School
G. B. LANG, West Georgia College
R. E. LANGER, University of Wisconsin
GILLIE A. LAREW, Randolph-Macon Woman's College
H. D. LARSEN, University of New Mexico
C. G. LATIMER, University of Kentucky
CAROLINE A. LESTER, New York State College for Teachers
C. F. LEWIS, Kansas State College
G. H. LING, University of Saskatchewan
MAYME I. LOGSDON, University of Chicago
W. R. LONGLEY, Yale University
R. G. LUBBEN, University of Texas

C. C. MACDUFFEE, University of Wisconsin
H. M. MACNEILLE, Harvard University
GLADYS B. MCCOLGIN, Indianapolis, Ind.
E. J. MCSHANE, University of Virginia
ANNA MARM, Bethany College, Lindsborg, Kans.
R. H. MARQUIS, Ohio University
WILLIAM MARSHALL, Purdue University
Sister MARY FELICE, Mount Mary College
RUTH G. MASON, University of Illinois

W. L. MASSEY, University of Chattanooga
H. R. MATHIAS, Bowling Green State University
A. E. MEDER, Jr., New Jersey College for Women
KARL MENGER, University of Notre Dame
H. J. MILES, University of Illinois
G. T. MILLER, Purdue University
W. E. MILNE, Oregon State Agricultural College
W. L. MISER, Vanderbilt University
VIRGINIA MODESITT, Randolph-Macon Woman's College
C. N. MOORE, University of Cincinnati
G. E. MOORE, University of Illinois
R. L. MOORE, University of Texas
T. W. MOORE, U. S. Naval Academy
W. L. MOORE, University of Louisville
Sister CHARLES MARY MORRISON, Nazareth College
MARSTON MORSE, Institute for Advanced Study
E. J. MOULTON, Northwestern University

A. L. NELSON, Wayne University
C. A. NELSON, New Jersey College for Women
C. V. NEWSOM, University of New Mexico

E. G. OLDS, Carnegie Institute of Technology

SALLIE E. PENCE, University of Kentucky
H. A. PERKINS, Hampton Institute
I. E. PERLIN, Armour Institute of Technology
O. J. PETERSON, State Teachers College, Emporia, Kans.
H. P. PETTIT, Marquette University
D. T. PETTY, F. W. Parker School, Chicago
A. E. PITCHER, Harvard University
E. W. PLOENGES, James Millikin University
H. S. POLLARD, Miami University
G. B. PRICE, University of Kansas
IRENE PRICE, State Teachers College, Oshkosh, Wis.
H. R. PYLE, Earlham College

TIBOR RADÓ, Ohio State University
G. Y. RAINICH, University of Michigan
E. D. RAINVILLE, University of Michigan
W. C. RANDELS, Northwestern University
J. F. RANDOLPH, Institute for Advanced Study
E. A. RASOR, Social Security Board, Washington, D. C.
S. E. RASOR, Ohio State University
F. W. REED, Ohio University
HARRIET REES, University of Utah
W. T. REID, University of Chicago
R. G. D. RICHARDSON, Brown University
P. R. RIDER, Washington University
H. L. RIETZ, University of Iowa
R. F. RINEHART, Case School of Applied Science
C. K. ROBBINS, Purdue University
FRED ROBERTSON, Iowa State College
ROBIN ROBINSON, Dartmouth College
J. B. ROSENBACH, Carnegie Institute of Technology
M. F. ROSSKOPF, John Burroughs School, Clayton, Mo.
D. A. ROTHROCK, Indiana University
LULU L. RUNGE, University of Nebraska

S. T. SANDERS, Louisiana State University
R. G. SANGER, University of Chicago
I. J. SCHOENBERG, Colby College
E. W. SCHREIBER, Western Illinois State Teachers College
JOSEPH SEIDLIN, Alfred University
W. E. SEWELL, Georgia School of Technology
M. E. SHANKS, University of Michigan
R. S. SHAW, College of the City of New York
W. A. SHEWHART, Bell Telephone Laboratories
J. A. G. SHIRK, State Teachers College, Pittsburg, Kans.
L. S. SHIVELY, Ball State Teachers College
C. GRACE SHOVER, Carleton College
D. T. SIGLEY, University of Kansas City
J. H. SIMESTER, University of Louisville
MARY EMILY SINCLAIR, Oberlin College
C. H. SMILEY, Brown University
F. C. SMITH, College of St. Francis
G. W. SMITH, University of Kansas
J. P. SMITH, St. Peter's College
ZENS L. SMITH, University of Chicago
D. E. SOUTH, University of Kentucky
G. W. STARCHER, Ohio University
H. E. STELSON, Kent State University
GUY STEVENSON, University of Louisville
ELLEN C. STOKES, New York State College for Teachers
RUTH W. STOKES, Winthrop College
E. B. STOUFFER, University of Kansas
ANNA K. SUTER, Butler University

E. H. TAYLOR, Eastern Illinois State Teachers College
J. S. TAYLOR, University of Pittsburgh
MILDRED E. TAYLOR, Mary Baldwin College
H. P. THIELMAN, College of St. Thomas

LOUISE THOMPSON, Shorter College
C. B. TOMKINS II, Institute for Advanced Study
C. C. TORRANCE, Case School of Applied Science
P. L. TRUMP, University of Wisconsin High School

HENRY VAN ENGEN, Iowa State Teachers College
H. E. VAUGHAN, University of Illinois
J. H. WEAVER, Ohio State University
D. L. WEBB, Georgia School of Technology
MARIE J. WEISS, Vassar College
J. J. WHEELER, University of Kansas
G. T. WHYBURN, University of Virginia
W. M. WHYBURN, University of California at Los Angeles
D. V. WIDDER, Harvard University
K. P. WILLIAMS, Indiana University
A. H. WILSON, Haverford College
MARGARETE C. WOLF, University of Wisconsin
H. E. WOLFE, Indiana University
F. E. WOOD, Northwestern University
H. A. WRIGHT, Transylvania College
E. KATHRYN WYANT, Athens College

MABEL M. YOUNG, Wellesley College

The A.A.A.S. held its opening session on Monday evening under the presidency of Professor G. D. Birkhoff of Harvard University. Doctor E. G. Conklin of Princeton University gave his retiring presidential address on the subject "Science and ethics." The Mathematical Association was represented on the Council of the A.A.A.S. each morning by Professor C. S. Atchison and Secretary Cairns. The Council elected Professor J. R. Kline vice-president and chairman of Section A for the year 1938 and Professor J. M. Thomas member of the Section Committee.

About two hundred of the mathematicians were housed in fraternity and sorority houses and private homes near Butler University, nearly a hundred had rooms at the Marott Hotel and others at downtown hotels. Convenient social rooms were provided both at Butler University and at the Marott, and constant use was made of these facilities. Tea was served by the ladies of the University on Thursday afternoon. The many courtesies extended to the visitors were recognized in a resolution offered by Professor G. T. Whyburn and adopted unanimously at the joint dinner.

Three hundred twenty-one attended this dinner at the Marott Wednesday evening. Following the dinner the toastmaster, Professor R. E. Langer, introduced Doctor J. McKeen Cattell, and Miss Martha Hildebrandt, president of the National Council of Teachers of Mathematics. In brief speeches which followed, President Birkhoff spoke of the importance to mathematicians of giving support to the American Association, Professor G. C. Evans spoke as the retiring chairman of Section A, Doctor F. R. Moulton brought greetings from the American Association officers, and vice-president T. H. Hildebrandt represented the Mathematical Association.

The American Mathematical Society held sessions for the reading of short papers on Tuesday morning and afternoon, Wednesday morning and afternoon, and Thursday morning. The Josiah Willard Gibbs Lecture was delivered by Professor Charles A. Kraus late Thursday afternoon in the Ballroom of the Antlers Hotel, his subject being "The present status of the theory of electrolytes." Wednesday afternoon Professor Alonzo Church of Princeton University gave an address entitled "The constructive second number class"; following this address

one section of the Society met jointly with the Association for Symbolic Logic. On Thursday morning a section of the Society met jointly with the Institute of Mathematical Statistics.

The National Council of Teachers of Mathematics held two sessions on Wednesday in the Memorial Shrine Auditorium. The program consisted of addresses by Professor A. J. Kempner on "The need of cooperation between high school teachers and college teachers," Professor J. O. Hassler on "High school teaching values derived from the study of higher mathematics," Professor Joseph Seidlin on "What shall we do with these, our unfit?," Miss Laura E. Christman on "Mathematics exhibits and the opportunities which they present," Miss Edith L. Mossman on "Omissions, enrichment, and improved machinery needed in all three types of courses found in ninth grade mathematics," Professor Harold Fawcett on "The role of demonstrative geometry in the cultivation of reflective thought," and Professor H. C. Christofferson on "Geometry, a way of thinking." At the luncheon at the Spink-Arms Hotel a report of progress was made by Professor W. D. Reeve on the work of the Joint Commission on the Place of Mathematics in the Secondary Schools.

The Mathematical Association held two sessions on Thursday afternoon and Friday morning as well as a joint session with the other organizations early Thursday afternoon. The Association is indebted to the Program Committee under the chairmanship of Professor C. C. MacDuffee for a fine program; this follows together with abstracts of some of the papers, numbered in accordance with their place on the program.

Joint Session of Sections A and K of the American Association, The American Mathematical Society, The Mathematical Association, The Econometric Society, The Institute of Mathematical Statistics

At this session Professor G. C. Evans gave his retiring vice-presidential address as Chairman of Section A on "Mathematical progress in theoretical economics." This address will appear in the *Bulletin of the American Mathematical Society*.

First Session of the Association

1. "Stieltjes integrals" by Professor T. H. Hildebrandt, University of Michigan.

2. "The successive iterates of the Stieltjes transform expressed in terms of the elementary functions" by Professor D. V. Widder, Harvard University.

1. Professor Hildebrandt gave a survey of some of the definitions of integrals of the Stieltjes type, comparing ranges of applicability, interrelations and properties. His paper will appear in an early issue of this Monthly.

2. As presented by Professor Widder, the Stieltjes transform changes a given function $f(x)$ into a new function $f_1(x)$ by the formula $f_1(x) = \int_0^\infty K(x, t)f(t)dt$, where $K(x, t)$ is $(x+t)^{-1}$. This transformation was used by T. J. Stieltjes in his theory of continued fractions and hence bears his name. If we use the same

transformation to change $f_1(x)$ into a new function $f_2(x)$, it turns out that $f_2(x)$ can be expressed in terms of $f(x)$ by the same integral formula given above except that $K(x, t)$ is now equal to $(\log x - \log t)(x-t)^{-1}$. This function is called the first iterate of $(x+t)^{-1}$.

By applying the Stieltjes transform to $f_2(x)$ we obtain a new function $f_3(x)$ which will correspond to a new value of $K(x, t)$ called the second iterate of $(x+t)^{-1}$. We continue in this way obtaining a sequence of iterates. This paper proved that all can be simply expressed in terms of the functions $\log x$, $\log t$, x, and t.

The paper was of interest, apart from the technical result obtained, on account of the methods it employed. These methods involve the theory of the Laplace transform, which is like the Stieltjes transform except that $(x+t)^{-1}$ is replaced by e^{-xt}, and the classical theory of Mittag-Leffler concerning the expansion of trigonometric functions in series of rational fractions.

SECOND SESSION OF THE ASSOCIATION

1. Annual business meeting and election of officers.

2. "A fundamental problem in the theory of numbers" by Professor H. W. BRINKMANN, Swarthmore College.

3. "On convexity" by Professor L. L. DINES, Carnegie Institute of Technology.

4. "On a recent advance in statistical inference" by Professor H. L. RIETZ, University of Iowa.

1. A report of this meeting is given on page 136.

2. Let $f(x)$ be a polynomial with integral coefficients. The various prime factors of all the integers obtained by setting x equal to any integer are called the "divisors" of $f(x)$. The problem dealt with in the paper of Professor Brinkmann is the characterization of these divisors for a given polynomial. Several special problems were mentioned; some of these are classical and some are of more recent date.

3. Certain properties associated with an intuitional notion of convexity were noted by Professor Dines and their mutual relationships precisely determined. The extension of the notion of convexity to spaces other than ordinary euclidian space was considered briefly. This paper will appear in an early issue of this MONTHLY.

4. The paper by Professor Rietz appears in the present number of this MONTHLY.

MEETINGS OF THE BOARD OF TRUSTEES

Eleven members of the outgoing and the incoming Board were present at the Indianapolis meetings.

The following thirty-seven persons were elected to membership on applications duly certified:

C. B. BARKER, Jr., A.M.(New Mexico) Instr., Univ. of New Mexico, Albuquerque, N. M.

W. E. BLEICK, Ph.D.(Johns Hopkins) Actuarial Dept. Genl., Prudential Ins. Co., Newark, N. J.

ANGELINE J. BRANDT, A.M.(Michigan) Instr.,

Wheaton Coll., Wheaton, Ill.

E. J. CAMP, Ph.D.(Chicago) Asso. Prof., Macalester Coll., St. Paul, Minn.

H. C. CHRISTOFFERSON, Ph.D.(Columbia) Prof., Miami Univ., Oxford, Ohio

C. R. CLARK, A.M.(Michigan) Teacher, McKinley High School, Washington, D.C.

H. S. M. COXETER, Ph.D.(Cambridge) Asst. Prof., Univ. of Toronto, Toronto, Ont., Can.

A. W. DAVIS, M.S.(Iowa State Coll.) Instr., State School of Mines, Rapid City, S. D.

LOUIS DE LASALLE, Ph.D.(Catholic Univ.) Prof., St. Mary's Coll., Winona, Minn.

A. K. EITTREIM, M.S.(Iowa) Teacher, High School, Ironwood, Mich.

H. S. GRANT, Ph.D.(Pennsylvania) Asst. Prof., Rutgers Univ., New Brunswick, N. J.

J. A. GREENWOOD, Ph.D.(Missouri) Instr., Duke Univ., Durham, N. C.

G. M. HARNER, M.S.(Marquette) Instr., Marquette Univ., Milwaukee, Wis.

JULIA M. HAWKES, Ph.D.(Michigan) Prof., Math. and Astr., Doane Coll., Crete, Nebr.

FRITZ JOHN, Ph.D.(Göttingen) Asst. Prof., Univ. of Kentucky, Lexington, Ky.

L. A. KNOWLER, Ph.D.(Iowa) Instr., Hunter Coll., New York, N. Y.

H. D. LANTZ, A.M.(Columbia) Teacher, High School, Far Rockaway, N. Y.

H. L. LEE, A.M.(Tennessee) Instr., Univ. of Tennessee, Knoxville, Tenn.

S. J. LORING, B.S.(Mass. Inst. of Tech.) Structural Engr., Chance Vought Aircraft Co., East Hartford, Conn.

F. E. LOWANCE, Ph.D.(Duke) Asst. Prof., Wofford Coll., Spartanburg, S. C.

H. A. LUTHER, Ph.D.(Iowa) Instr., A. and M. Coll. of Texas, College Station, Tex.

W. L. MASSEY, A.M.(Duke) Instr., Univ. of Chattanooga, Chattanooga, Tenn.

L. E. MEHLENBACHER, Ph.D.(Michigan) Prof. Head of Dept., State Teachers Coll., Flagstaff, Ariz.

KARL MENGER, Ph.D.(Vienna) Prof., Univ. of Notre Dame, Notre Dame, Ind.

R. H. MOORMAN, A.M.(Vanderbilt) Teacher, Wallace Univ. School, Nashville, Tenn.

ETHEL C. MUGGLI, A.M.(North Dakota) Head of Dept., State Teachers Coll., Dickinson, N. D.

GORDON PALL, Ph.D.(Chicago) Asst. Prof., McGill Univ., Montreal, P.Q., Can.

I. E. PERLIN, Ph.D.(Chicago) Instr., Armour Inst. of Tech., Chicago, Ill.

RUTH M. PETERS, Ph.D.(Radcliffe) Prof., Math. and Physics. Lake Erie Coll., Painesville, Ohio

MAXWELL READE, A.M.(Harvard) Fellow in Math., Rice Inst., Houston, Tex.

HARRIET REES, Ph.D.(Chicago) Instr., Univ. of Utah, Salt Lake City, Utah

N. S. RISLEY, B.S.(Case) Asso., Case School of Appl. Sci., Cleveland, Ohio

ARTHUR SARD, Ph.D.(Harvard) Instr., Queens Coll., Flushing, N. Y.

ZENS L. SMITH, M.S.(Chicago) Asst. Prof., Univ. of Chicago, Chicago, Ill.

LOUISE THOMPSON, A.M.(Columbia) Registrar and Instr., Shorter Coll., Rome, Ga.

H. C. WIEDEMAN, B.S.(Buffalo) Instr., Univ. of Denver, Denver, Colo.

C. W. WILLIAMS, A.M.(Maryland) Instr., Washington and Lee Univ., Lexington, Va.

The financial report of the Secretary-Treasurer for the year 1937, and a report by a firm of certified public accountants which appears at the close of this report, were presented and accepted.

After certain modifications of the details of the Putnam Competition Plan, it was accepted by vote of the Trustees. These details were approved also the following week by the trustees of the Putnam Fund.

The Trustees voted (1) to nominate Professor J. H. Taylor as a representative on the board of the American Documentation Institute, (2) to nominate Mrs. Juna Lutz Beal to membership in the Society for 1938 under our sustaining membership, (3) to accept the report of progress from the Committee on Testing and to continue the work of this committee, (4) to request the officers of the Association to make contact with the officers of the Society with respect to the

part which the Association might take in the International Congress in 1940, (5) to nominate Professor F. D. Murnaghan as representative of the Association on the National Research Council for a three-year term from July 1, 1938.

The following were appointed associate editors of the MONTHLY for the year 1938, as nominated by Professor Moulton, Editor-in-Chief:

W. F. Cheney	R. E. Gilman	F. W. Owens
Otto Dunkel	C. A. Hutchinson	Helen B. Owens
B. F. Finkel	J. R. Musselman	R. G. Sanger
Tomlinson Fort	H. L. Olson	David Eugene Smith
T. C. Fry		F. E. Wood

ANNUAL BUSINESS MEETING

The annual business meeting and election of officers was held Friday morning. The Secretary announced the names of those who had been elected to membership at the meeting of the Trustees. He also reported the deaths of the following members:

R. P. BAKER, Professor of mathematics, State University of Iowa. (August 13, 1937)
R. D. BEETLE, Professor of mathematics, Dartmouth College. (July 9, 1937)
H. Y. BENEDICT, President, University of Texas. (May 10, 1937)
J. D. BOND, Professor of mathematics, University of Tennessee. (November 10, 1937)
L. P. CHEBOTAR, New York, N. Y. (August 28, 1937)
S. A. COREY, Des Moines, Iowa. (September 30, 1936)
C. B. HALDEMAN, Ross, Ohio. (November 3, 1937)
H. O. HANSON, Mutual Life Insurance Company. (May 3, 1937)
J. C. D. HARDING, Instructor in mathematics, University of Delaware. (May 28, 1937)
R. R. HITCHCOCK, Professor of mathematics, University of North Dakota. (March 10, 1937)
L. M. HOSKINS, Professor emeritus of mathematics, Stanford University. (September 8, 1937)
C. A. ISAACS, Professor of mathematics, State College of Washington. (August 12, 1937)
S. I. JONES, Life and Casualty Insurance Company. (March 28, 1937)
H. E. SLAUGHT, Professor emeritus of mathematics, University of Chicago. (May 21, 1937)
J. N. VEDDER, Professor of mathematics, Union College. (December 26, 1936)

The Association adopted the two amendments which were presented in the MONTHLY for November 1937.

Professor J. A. G. Shirk mentioned briefly the advantages of the organization, Kappa Mu Epsilon.

The result of the election of officers for 1938 was as follows:

Vice-Presidents for 1938: H. E. BUCHANAN, Tulane University; E. J. MOULTON, Northwestern University.

Additional members of the Board of Trustees, to serve until January 1941: WILLIAM BETZ, Rochester (New York) Public Schools; A. B. COBLE, University of Illinois; ARNOLD DRESDEN, Swarthmore College; J. H. WEAVER, Ohio State University.

REPORT OF THE SECRETARY-TREASURER AS TREASURER, DECEMBER 10, 1937

RECEIPTS

Balance Dec. 14, 1936		$7,103.06
1936 indiv. dues	$458.25	
1937 indiv. dues	6,910.22	
1937 inst. dues	604.97	
1937 subscriptions	1,023.20	
Initiation fees	212.00	
Advertising	481.00	
Authors' reprints	253.22	
Sale copies of MONTHLY	177.16	
Sale First Carus Mon.	13.75	
Sale Second Carus Mon.	10.00	
Sale Third Carus Mon.	21.25	
Sale Fourth Carus Mon.	12.50	
Sale Fifth Carus Mon.	133.99	
Sale Archibald's Outline of Hist. of Math.	104.60	
Annals subscriptions	7.50	
Duke Journal subscriptions	8.00	
Sale Rhind Papyrus	243.00	
Life membership fees	101.18	
Drury Coll. int. Hardy Fund	120.00	
Refund from Editor-in-Chief	7.30	
Sale Idaho Power Co. Bonds	2,100.00	
Int. Peoples Bkg. Co.	22.04	
Int. Cleveland Trust Co.	67.58	
Int. Genl. End. Fund	789.86	
Int. Carus Fund	131.25	
Int. Chace Fund	239.01	
Int. Chauvenet Fund	15.00	
Payment from restricted Carus Fund	49.70	
Payment from restricted Chace Fund	2.20	14,319.73
Total 1937 receipts to date		$21,422.79
Total expenditures		15,922.91
Balance to end of 1937 business		$ 5,429.88
Received on 1938 business		741.45
Book balance Dec. 10, 1937		$ 6,171.33

EXPENDITURES

Publisher's bills (Nov. '36–Oct. '37)		$5,545.54
Reprints		301.60
Register expense		213.87
Manager's office		83.61
Editor-in-Chief's office 1936		53.26
Editor-in-Chief's office 1937		592.50
Committee on membership		118.68
Committee on Testing		20.01
Carus Monograph Committee		155.00
Secretary-Treasurer's office		
Postage	$419.90	
Bond	11.26	
Safety deposit	4.40	
Office supplies	134.57	
Express, tel., etc.	108.84	
Clerical work	2,008.15	
Printing	293.55	
Bank tax	27.69	3,008.36
Annals subvention		150.00
Duke Journal subvention		200.00
Expense sections from init. fees		154.20
North Carolina meeting		62.14
State College meeting		102.19
Indianapolis meeting		27.50
Forwarded *Annals* subscriptions		10.50
Paid *Annals* subscriptions		15.00
Sust. memb. in Amer. Math. Soc.		100.00
Refund subscriptions		16.65
Storage back copies MONTHLY		30.00
Insurance back copies MONTHLY		6.40
Paid back copies MONTHLY		22.70
Paid B. F. Finkel int. Hardy Fund		120.00
Library expense chiefly binding		200.64
Excess cost HOLC Bonds		8.65
Excess cost Bethlehem Steel Bonds		2.92
Cost Montana Power Bonds		1,845.99
Transfer to Genl. End. Fund		2,000.00
Transfer to Chace Fund		675.00
Transfer to Carus Mon. Fund		150.00
Total expenditures		$15,992.91
Checking account		529.64
Savings account		25.00
Oberlin Savgs. Bk. acct. restricted		916.30
Peoples Banking Co. savgs. acct.		1,143.10
Cleveland Trust Co. savgs. acct.		3,557.29
Bank balance Dec. 10, 1937		$ 6,171.33

Exhibit of the Funds of the Association

Carus Monograph Fund

Balance Dec. 14, 1936		$6,475.04
Receipts: Sales	$ 191.49	
Interest	174.43	365.92
		$6,840.96
Expense acct. Carus Committee		155.00
		$6,685.96
Certificate of deposit	$1,760.49	
C. & O. 3½% Refunding Mortgage Bonds Series D, 1996	2,000.00	
3⅛% U. S. Treasury Bond of 1946–49	1,000.00	
3% HOLC Bond 1944–52	1,000.00	
U. S. Savings Bonds	150.00	
Cash in bank, restricted, certificate of participation	546.70	
Cash in bank, unrestricted	228.77	
Balance Dec. 10, 1937		$6,685.96

Arnold Buffum Chace Fund

Balance Dec. 14, 1936		$6,833.16
Receipts: Sale Papyrus	$ 243.00	
Interest	242.24	485.24
		$7,318.40
3⅛% U. S. Treasury Bonds 1946–49	$2,000.00	
3% HOLC Bond 1944–52	1,000.00	
U. S. Savings Bonds	1,425.00	
Western United Gas & Electric Co. 5½% Bonds Series A, 1955	2,500.00	
Certificate of deposit	132.29	
Cash in bank, restricted, certificate of participation	24.20	
Cash in bank, unrestricted	236.91	
Balance Dec. 10, 1937		$7,318.40

Chauvenet Prize Fund

Balance Dec. 14, 1936		$ 657.94
Interest		15.00
		$ 672.94
3% HOLC Bond 1944–52	$ 500.00	
Cash in bank, unrestricted	172.94	
Balance Dec. 10, 1937		$ 672.94

Life Membership Fund

Liability on life memberships as of Jan. 1, 1937	$ 733.72
Life membership payments	101.18
	$ 834.90
To be transferred to current funds, surplus	30.07
Liability on life memberships as of Jan. 1, 1938	$ 804.83

GENERAL ENDOWMENT FUND

Balance Dec. 14, 1936		$16,200.00
Transferred from current funds		2,000.00
		$18,200.00
3½% U. S. Treasury Bonds 1944–46	$1,000.00	
3½% U. S. Treasury Bonds 1943–45	1,000.00	
3% HOLC Bonds 1944–52	5,500.00	
Land Trust Certificate, Hotel Cleveland Building Site	700.00	
3¾% Montana Power Co. First Mortgage Bonds, 1966	2.000.00	
5% Texas Power & Light Co. First Mortgage Gold Bond, 1956	1,000.00	
3½% C. & O. Refunding Mortgage Bond Series C, 1996	1,000.00	
3½% Pennsylvania R. R. Co. Bonds Series C, 1970	2,000.00	
3½% Bethlehem Steel Co. Consol. Mortgage Bonds, 1966	2,000.00	
Oberlin Savings Bank savings account	2,000.00	
Balance Dec. 10, 1937		$18,200.00

It may be pointed out that the Trustees transferred two thousand dollars to the General Endowment Fund in the form of bonds of the Bethlehem Steel Company and that two thousand dollars invested in the Idaho Power Company were replaced by a like amount in Montana Power Company bonds.

Of the funds on hand, indicated in the first division of this financial report, $228.77 belongs to the Carus Monograph Fund, $236.91 to the Arnold Buffum Chace Fund, $172.94 to the Chauvenet Prize Fund, while $804.83 is held as a Life Membership Fund, representing the liability on life memberships already paid for, as of date January 1, 1938.

When the accounts were closed Dec. 10, 1937, there remained on the total business for 1937 the following items:

BILLS RECEIVABLE		BILLS PAYABLE	
1937 individual dues	$200.00	Publisher's bills (Nov.–Dec. 1937)	$1,150.00
Advertising	100.00	Printing *Register*	450.00
	$300.00	President's office	30.00
		Editor-in-Chief's office	150.00
		Secretary-Treasurer's office	400.00
		Subsidy *Annals*	50.00
		Subsidy *Duke Journal*	50.00
		Carus Monograph Fund	228.77
		Chace Fund	236.91
		Chauvenet Prize Fund	172.94
		Life Membership Fund	804.83
		Init. fees due to sections	980.00
			$4,703.45

If to the balance on 1937 business shown in this report, $5,429.88, there be added the estimated bills receivable, $300.00, and there be subtracted the estimated bills payable, $4,703.45, there results an estimated final balance on 1937 business of approximately $1,025.00. Since two thousand dollars of the current

funds were transferred by vote of the Trustees to the General Endowment Fund during the past year, the result shows that the Association made a small profit in the year's business.

W. D. CAIRNS, *Secretary-Treasurer*

REPORT OF THE CERTIFIED PUBLIC ACCOUNTANT

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)
STATEMENT OF FUND ASSETS—DECEMBER 10, 1937

Current fund:			
Deposits in banks, savings accounts		$4,725.39	
Deposits in banks, checking accounts		529.64	
Deposits in bank subject to withdrawal restrictions		916.30	
		$6,171.33	
Less, Due to:			
Life Membership Fund	$804.83		
Carus Monograph Fund	228.77		
Arnold Buffum Chace Fund	236.91		
Chauvenet Prize Fund	172.94	1,443.45	
Current Fund balance (exclusive of current accounts payable estimated at $3,260, less current accounts receivable estimated at $300)			$4,727.88
General Endowment Fund:			
Investments at par:			
U. S. Treasury Bonds, 3¼% 1944–46		1,000.00	
U. S. Treasury Bonds, 3¼% 1943–45		1,000.00	
Home Owners' Loan Corporation Bonds, 3% 1944–52		5,500.00	
The Montana Power Company, 1st and Refunding Bond, 3¾% 1966		2,000.00	
Texas Power & Light Company, 1st and Refunding Bond, 5% 1956		1,000.00	
The Chesapeake & Ohio Railway Company, Refunding Mortgage Bond, Series C, 3¼% 1996		1,000.00	
The Pennsylvania Railroad Company, General Mortgage Bond, Series C, 3¼% 1970		2,000.00	
Bethelehem Steel Corporation, Consolidated Mortgage Bond, Series E, 3¼% 1966		2,000.00	
Hotel Cleveland Site Land Trust Certificates		700.00	
Deposits in bank, savings account		2,000.00	
General Endowment Fund balance			$18,200.00
Life Membership Fund:			
Due from Current Fund		$804.83	
Life Membership Fund balance			$804.83
Carried forward			$23,732.71

Brought forward		$23,732.71
Carus Monograph Fund:		
Investments at par:		
U. S. Treasury Bonds, 3¼% 1946–49	$1,000.00	
Home Owners' Loan Corporation Bonds, 3% 1944–52	1,000.00	
U. S. Savings Bonds, 1947	150.00	
The Chesapeake & Ohio Railway Company, Refunding Mortgage Bond, Series D, 3¼% 1996	2,000.00	
	$4,150.00	
Certificate of deposit	1,760.49	
Deposits in bank subject to withdrawal restrictions	546.70	
Due from current fund	228.77	
Carus Monograph Fund balance		6,685.96
Arnold Buffum Chace Fund:		
Investments at par:		
U. S. Treasury Bonds, 3¼% 1946–49	$2,000.00	
Home Owners' Loan Corporation Bonds, 3% 1944–52	1,300.00	
U. S. Savings Bonds, 1945	750.00	
U. S. Savings Bonds, 1947	375.00	
Western United Gas & Electric Company, 30 year Gold Bond, Series A, 5½% 1955	2,500.00	
	$6,925.00	
Certificate of deposit	132.29	
Deposits in bank subject to withdrawal restrictions	24.20	
Due from current fund	236.91	
Arnold Buffum Chace Fund balance		$7,318.40
Chauvenet Prize Fund:		
Home Owners' Loan Corporation Bonds, 3% 1944–52	$500.00	
Due from current fund	172.94	
Chauvenet Prize Fund balance		$672.94
Total all funds		$38,410.01

DETAILS OF CURRENT FUND EXPENSES

For the period from December 15, 1936 to December 10, 1937

Printing and subscriptions		$ 5,911.99
Office salaries and expenses:		
Secretary-Treasurer	$3,008.36	
Other officers	722.07	3,730.43
Subventions, meetings and expenses of sections		696.03
Committee expenses		138.69
Membership fee		100.00
Insurance and storage		36.40
Library expense		200.64
Register expense		213.87
Miscellaneous		131.57
		$11,159.62

STATEMENT OF FUND ACCOUNTS AND CASH RECEIPTS AND DISBURSEMENTS

For the period from December 15, 1936 to December 10, 1937

Classification of Funds	Total	Current	General Endowment	Life Membership	Carus Monograph	A. B. Chace	Chauvenet Prize
Fund balances, per Treasurer's report, December 15, 1936	$36,522.23	$ 5,622.37	$16,200.00	$733.72	$6,475.04	$6,833.16	$657.94
Cash receipts:							
Dues, individuals, year 1937	6,910.22	6,910.22	—	—	—	—	—
Dues, institutions, year 1937	604.97	604.97	—	—	—	—	—
Dues, individuals, year 1936	458.25	458.25	—	—	—	—	—
Dues, life memberships	101.18	—	—	101.18	—	—	—
Subscriptions	1,023.20	1,023.20	—	—	—	—	—
Interest on investments	1,311.15	879.48	—	—	174.43	242.24	15.00
Advertising	481.00	481.00	—	—	—	—	—
Initiation fees	212.00	212.00	—	—	—	—	—
Authors' reprints	253.22	253.22	—	—	—	—	—
Proceeds from sale of Rhind Papyrus	243.00	—	—	—	—	243.00	—
Proceeds from sale of Carus Monographs	191.49	—	—	—	191.49	—	—
Proceeds from sale of MONTHLY	177.16	177.16	—	—	—	—	—
Miscellaneous	240.10	240.10	—	—	—	—	—
Dues and subscriptions, year 1938	741.45	741.45	—	—	—	—	—
Proceeds from sale of investments	2,100.00	2,100.00	—	—	—	—	—
Total receipts	15,048.39	14,081.05	—	101.18	365.92	485.24	15.00
	51,570.62	19,703.42	16,200.00	834.90	6,840.96	7,318.40	672.94
Cash disbursements:							
Carus Monograph Committee	155.00	—	—	—	155.00	—	—
Expenses, details annexed	11,159.62	11,159.62	—	—	—	—	—
Purchase of investments	1,845.99	1,845.99	—	—	—	—	—
Total disbursements	13,160.61	13,005.61	—	—	155.00	—	—
Balances before transfers between funds	38,410.01	6,697.81	16,200.00	834.90	6,685.96	7,318.40	672.94
Transfers to or *from* funds	—	*1,969.93*	2,000.00	*30.07*	—	—	—
Fund balances, December 10, 1937	$38,410.01	$ 4,727.88	$18,200.00	$804.83	$6,685.96	$7,318.40	$672.94

To the Board of Trustees
The Mathematical Association of America (Inc.) Oberlin, Ohio
Dear Sirs:

We have examined the accounts and records of

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

for the purpose of verifying the assets as at December 10, 1937, and submit herewith Statement of Fund Assets which, in our opinion, presents the financial position of the Association at that date on the basis of stating investments at par.

We also include in our report Statement of Fund Accounts and Cash Receipts and Disbursements and Details of Current Fund Expenses for the period from December 15, 1936 to December 10, 1937, as recorded on the books.

Very truly yours,
LYBRAND, ROSS BROS. & MONTGOMERY

THE FALL MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The fall meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the Johns Hopkins University, Baltimore, Maryland, on December 4, 1937. The Chairman, Professor Gillie A. Larew, Randolph-Macon Woman's College, presided over both sessions, morning and afternoon. Six papers were read at the morning session, and in the afternoon, at the invitation of the Section, Professor T. Y. Thomas of Princeton University delivered a lecture on "Fields of parallel vectors."

There were fifty-six in attendance including the following thirty-eight members of the Association: O. S. Adams, Clara L. Bacon, N. H. Ball, G. A. Bingley, Archie Blake, C. C. Bramble, Abraham Cohen, L. S. Dederick, Alexander Dillingham, J. A. Duerksen, J. H. Edmonston, E. J. Finan, Almeda J. Garland, Michael Goldberg, Isabel Harris, F. E. Johnston, Sidney Kaplan, L. M. Kells, A. E. Landry, C. L. Leiper, Florence P. Lewis, S. B. Littauer, T. W. Moore, W. K. Morrill, F. D. Murnaghan, C. H. Rawlins, Jr., J. N. Rice, A. W. Richeson, R. E. Root, M. A. Scheier, J. L. Stearn, T. Y. Thomas, John Tyler, C. H. Wheeler III, G. T. Whyburn, John Williamson, Oscar Zariski, R. T. Zoch.

The following seven papers were read:

1. "A simple method of integration" by Professor John Tyler, U. S. Naval Academy.

2. "Some points in the teaching of the calculus" by Dr. S. B. Littauer, U. S. Naval Academy.

3. "Kasner's invariant" by Dr. George Comenetz, St. John's College, introduced by Professor G. A. Bingley.

4. "Representation of rational numbers by homogeneous forms" by Dr. O. F. G. Schilling, Johns Hopkins University, introduced by Dr. Oscar Zariski.

5. "Linear combinations of three successive orthogonal polynomials" by Dr. D. L. Netzorg, Johns Hopkins University, introduced by Dr. Oscar Zariski.

6. "The mechanical transformation of Boolean expressions" by Dr. Archie Blake, U. S. Coast and Geodetic Survey.

7. "Fields of parallel vectors" by Professor T. Y. Thomas, Princeton University.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. Let u, I, and R be functions of x satisfying the relation (1) $du/dx = aI(x) + R(x)$. Integrate both sides of this equation, solve for $\int I(x)dx$ and we have (2) $\int I(x)dx = u/a - \int R(x)\ dx/a$. From these two equations Professor Tyler obtained the following rules for integrating the expression $I(x)$. *Rule* 1. Determine by inspection a product of two functions that has *one term* in its derivative of *like form* as the expression $I(x)$ to be integrated. This product or expression is the function u outside the integral. *Rule* 2. To determine the unknown denominator a, differentiate the expression u and reduce to the form given in equation (1). The denominator a is the coefficient of I, and $(-R/a)$ will be the integrand of the new integral. Professor Tyler illustrated the method with several algebraic and trigonometric examples.

2. Dr. Littauer was concerned with the fact that it is difficult to find a text on calculus which presents its material in a naturally motivated fashion. Instances were cited of popular and distinguished texts which present the concepts of integration and of Taylor's series in confusing and non-motivated fashion, if not in an actually unsound way. Well-known, simple and natural developments of these topics were given. The question was raised: Shall the writing of texts remain a haphazard undertaking? Opinions are invited from the members of the Association on the suggestion that the Association is the proper agency to take responsibility for the publication of undergraduate texts under the direction and authorship of a committee of competent scholars who are willing to devote full time for a limited period to the production of a given text. Profits from publication can be used to subsidize advanced expositions which are not profitable.

3. This paper appeared in full in the February number of the MONTHLY.

4. Dr. Schilling considered a class of homogeneous forms of degree n in n^2 undeterminates, namely, the reduced norm forms of general elements lying in the maximal orders of normal simple algebras over the rational number field. He proved that each integer can be obtained as the value of such a form for a suitable specialization of the undeterminates by integers, provided that the degree of the algebra is greater than two and that it has no infinite ramification. It can be shown that all ideals in the maximal orders of the algebras under consideration are principal ideals. Thus, in particular, all irreducible ideals dividing a given prime in a fixed maximal order are generated by one element. The reduced norm of a suitable generator of such an irreducible component is equal to the given prime.

5. Dr. Netzorg considered the roots $r_1, \cdots, r_n$ of

$$Q(x) = P_n(x) + AP_{n-1}(x) + BP_{n-2}(x) = 0 \qquad (A, B \text{ real}),$$

where the P's are members of a system of polynomials orthogonal with respect to a distribution function $\psi(x)$. He showed that when $B < \lambda_n$ all the r's are real. Here λ_n, which is greater than zero, is the coefficient in the recurrence formula $P_n(x) = (x - c_n)P_{n-1}(x) - \lambda_n P_{n-2}(x)$. Regarding the Cotes's coefficients $H_1, \cdots, H_n$ in the mechanical quadrature formula which uses $r_1, \cdots, r_n$ as base points and $\psi(x)$ as distribution function, it was shown that if $B < \lambda_n$, then all H's are positive; while if $B = \lambda_n$, then exactly one of the H's equals zero; and if $B > \lambda_n$ and all the r's are real, then exactly one of the H's is negative. It is possible to prove, for instance, that of the $3n-6$ zeros of the Wronskian of P_n, P_{n-1}, and P_{n-2}, exactly $n-2$ are real, and that these separate the zeros of $P_{n-1}(x)$.

6. An efficient method of studying any given expression in finite Boolean algebra is by reduction to syllogistic form—that is, to a polynomial such that any monomial included in it is included in one of its terms—and especially to the simplified or minimal syllogistic form. Dr. Blake showed how these reductions and others of importance can be performed mechanically in various ways. Mechanical methods can be used also, with some increase in complication, when the class of expressions is augmented by the admittance of symbols representing infinite sums and products. But an essential difficulty arises: certain important canonicalizations lead not to a single expression but to an infinite sequence of expressions, and can be pursued mechanically only to the limits of the machine.

7. Professor Thomas presented a descriptive account of the theory of fields of parallel vectors, including some recent work on this problem, and illustrated it by the parallel displacement of a vector on the surface of a cone.

MICHAEL GOLDBERG, *Secretary*

THE FALL MEETING OF THE MICHIGAN SECTION

The third fall meeting of the Michigan Section of the Mathematical Association of America was held at Wayne University on Saturday, December 4, 1937. The chairman of the Section, Professor V. G. Grove, presided.

The attendance was about sixty, including the following twenty-nine members of the Association: W. L. Ayres, W. D. Baten, J. B. Brandeberry, C. J. Coe, A. H. Copeland, S. E. Crowe, P. S. Dwyer, N. C. Fisk, J. D. Fitzpatrick, K. W. Folley, R. E. Gaskell, T. N. E. Greville, V. G. Grove, T. H. Hildebrandt, L. S. Johnston, C. E. Love, E. D. McCarthy, D. D. Miller, A. L. Nelson, L. F. Ollmann, H. H. Pixley, L. C. Plant, E. D. Rainville, L. J. Rouse, M. E. Shanks, C. J. Shires, E. R. Sleight, Fern Welker, J. B. Winslow.

Forty-nine persons attended the luncheon at noon at Hotel Webster Hall. A short business meeting was held at this time, and Professor P. S. Dwyer was elected secretary of the Section to serve the remainder of the term of Professor C. C. Craig who is in England.

The following papers were presented at the morning and afternoon sessions:

1. "Some problems of municipal pensions" by Professor J. B. Winslow, University of Toledo.

2. "Some remarks concerning Bernstein's article which appeared in the Mathematische Annalen, Band 97, 1926" by Professor W. D. Baten, Michigan State College.

3. Papers presented by the Committee on Undergraduate Interests.

a) "A theoretical lens surface" by Mr. Carl King, Wayne University.

b) "A new series for deriving Euler's numbers" by Eugene F. Trombley, University of Detroit.

c) "The eccentricities of the sections of a cone" by Dominic Toffolo, Wayne University.

d) "A pendulum problem" by Mr. Stewart Pollock, Albion College. The paper was presented by Franklin Waite, Albion College.

e) "The derivation of the coefficient of correlation between corresponding sample averages" by Miriam M. Geboo, Michigan State College.

4. "A decomposition of a simply ordered set" by Professor K. W. Folley, Wayne University.

5. "Polygon and polynomial" by Professor N. H. Anning, University of Michigan.

6. "On the rational solutions of a certain minimum problem" by Professor Ben Dushnik, University of Michigan, introduced by the Secretary.

7. "An abstract function space" by Dr. E. W. Paxson, Wayne University, introduced by Professor A. L. Nelson.

8. "Some generalizations of the associative law" by Rev. B. A. Hausman, S.J., University of Detroit, introduced by Professor L. S. Johnston.

9. General Session.

Abstracts of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. New pension systems are formed today upon sound actuarial principles in almost every field except municipal pensions. In a few instances this has been done, but in general it requires the remaking of an old system, since municipal pensions have been paid to firemen and policemen for seventy-five years. Professor Winslow summarized the principal problems as (1) lack of contributions by members in service, (2) length of service required to qualify for a pension, (3) lack of a minimum age limit before a pension may be received, (4) what percentage relation a pension should bear to the regular salary, (5) in what manner should widows receive protection, (6) to what extent should children share in a pension, (7) how to merge separate boards for each group into one, and (8) irregularity of adding members to the active forces.

2. The remarks of Professor Baten concerned an article, "Sur l'extension du théorème limite du calcul des probabilités aus sommes de quantités dépendentes," by Serge Bernstein. They show how to generalize proofs of certain

theorems and present applications of some of the theorems given by Bernstein. A frequency distribution is obtained for the sum of n variables.

3. (a) The fact that spherical lenses do not perfectly focus parallel rays to a point suggests the problem: What is the general equation of the refracting surface that will bring parallel rays to a point focus? The surface is found by Mr. King to be a quadric surface such as would be generated by revolving a conic about its transverse axis. The eccentricity of the conic is exactly the reciprocal of the index of refraction of the surface, and the focal point of the light rays lies at the focus of the conic. The more general problem would be to derive the surface necessary to refract rays from a source at a finite distance to a focus. This can be solved in a similar manner, but results in an awkward quartic equation.

(b) Mr. Trombley exhibited a recurrence formula for deriving Euler's Numbers derived by application of Leibniz's Theorem for successive differentiation. This method of derivation seems materially shorter than the conventional one shown in elementary texts.

(c) A formula was derived by Mr. Toffolo giving the eccentricity of a conic section as a function of the generating angle of the cone and the angle of inclination made by a plane with the axis of the cone.

(d) The distances of the centers of oscillation and percussion from the center of an ellipse were determined by Mr. Pollock such that the distance between them will be the equivalent length of a simple pendulum, thus making the period of oscillation the same.

(e) Miss Geboo gave a proof that the correlation coefficient between the variables in a parent population is equal to the correlation coefficient between all possible corresponding sample means. This was done by expressing the summations $\sum\bar{x}\bar{y}$, $\sum\bar{x}^2$, $\sum\bar{y}^2$, $\sum\bar{x}$, $\sum\bar{y}$ in terms of the original measurements and substituting into the formula for the correlation coefficient between sample means. The ellipses found by cutting the normal correlation surface by planes corresponding to constant frequencies were discussed. The probability that a point taken at random will fall within such an ellipse was determined and applied to testing significance between sets of correlated variables.

4. A simply ordered set M which contains an everywhere dense η_α subset may be decomposed into $2^{\aleph_\alpha}$ disjoint subsets, each of power $<2^{\aleph_\alpha}$, and such that the sum of every $2^{\aleph_\alpha}$ of these subsets has at least one point in common with every perfect subset of M. From this result Professor Folley showed that the generalized hypothesis of the continuum is equivalent to the following statement: The set M may be decomposed into disjoint subsets, each of power at most $\aleph_\alpha$, such that the sum of any class of more than $\aleph_\alpha$ of them has at least one point in common with every perfect subset of M.

5. By the Lill process (E. Lill, *Nouvelles Annales de Mathématiques*, 1867, p. 359, 1868, p. 363) an "orthogon" is associated with any polynomial with real numerical coefficients. Professor Anning showed how complex numbers and

Horner's (Ruffini's) method could be applied for simplifying the construction, use, and explanation of the Lill orthogon.

6. Professor Dushnik considered a minimum problem of the type encountered in an elementary course in the differential calculus, and, assuming the *given* quantities to be rational numbers, inquired when the *required* quantities (*i.e.*, the answer to the problem) will also be rational numbers. The result makes use of the rational solutions of the equation $x^2+y^2=z^2$.

7. Dr. Paxson discussed the concept of a linear topological space L, and indicated how the space of continuous functions on a closed real interval to the space L may have assigned to it a special convex topology, under which the function space is of the same linear topological type as the support space L. He showed how bicompactness of a pseudo-local nature is reflected from one space to the other, this notion being of importance in the application of fixed point theorems to existence questions for differential equations phrased in non-metric spaces.

8. Mr. Hausman considered generalizations of the associative postulate of the theory of discrete groups. A method of approach to the problem was outlined, various generalizations old and new were given, together with the principal properties of the resulting quasi-groups.

9. Dr. Max Coral discussed the treatment of mathematical induction in college algebra texts. Professor L. S. Johnston discussed the use of superimposed sheets of cellophane to illustrate complicated figures. Professor V. G. Grove called attention to a mistake in the discussion of radius of curvature in Granville's Calculus. Mr. D. K. Kazarinoff discussed the problem of the motion of the center of mass in a system of two particles moving along two lines. Professor N. H. Anning, Professor W. D. Baten, and Professor L. S. Johnston commented briefly on topics.

W. L. AYRES, *Acting Secretary*

ON A RECENT ADVANCE IN STATISTICAL INFERENCE*

H. L. RIETZ, University of Iowa

1. Introduction. In so far as I have been able to learn, courses in statistical methods as part of collegiate mathematics were practically non-existent in this country thirty-five years ago [1].† That a striking change has occurred in this situation is obvious from the fact that courses in this subject are given currently by one or more departments in nearly all our liberal arts colleges. In fact, it has perhaps become only too much the fashion to give courses in statistical methods in at least three departments of a college—mathematics, education, and economics.

These collegiate courses in statistics are much concerned with the distributions of one or more classes of numerical items called a *population* or universe of discourse. Various averages of the items of a sample are used to estimate mean values, indexes, and parameters which characterize the population although the term "parameter" may possibly not be used in the most elementary courses. In the present paper, the term "parameter" denotes a number used to specify some characteristic of a population. For example, in the normal frequency function

$$f(x) = \frac{1}{\sqrt{2\pi\theta}} e^{-(x-m)^2/2\theta}$$

for a population of x's, the population mean m and the variance θ are the parameters. An appropriate estimate of the mean m or of the variance θ by the use of a random sample of observed values of x is called a *statistic*.

As another and mathematically even more elementary illustration of a parameter than those in the normal frequency function we may name the binomial distribution given by the terms in the expansion of $(q+p)^n$. This is a discontinuous distribution with frequencies defined only at the non-negative integral numbers, x, of occurrences in n trials where the parameter p is the probability of an occurrence in any one trial, and $q=1-p$.

Judging by the textbooks in use together with various outlines of courses, and lack of substantial mathematical prerequisites for the courses, it is fairly obvious that nearly all of these collegiate courses in statistical methods avoid sampling theory, at least so far as proving theorems is concerned. The books do, however, touch sampling fluctuations at least lightly in attempting to explain the meaning of the probable error of an average calculated from a sample of data or in the estimation of a parameter of a population by means of an inquiry based on a sample.

In explaining the meaning of the probable error of a statistic, one of the

* Read at the Indianapolis meeting of the Mathematical Association on December 31, 1937, at the invitation of the program committee. Presented for the Slaught Memorial Volume.

The paper is designed to be of special interest to teachers of collegiate courses in statistics.

† Numbers in brackets refer to references at the end of the paper.

usual types of definition is essentially the following: The probable error of a statistic, s, is a positive number, E_s, such that the chances are even that the population parameter of which s is an estimate from the sample, will fall within the interval $s-E_s$ to $s+E_s$.

This definition contains an inference about the values of a population parameter on the basis of information obtained from a random sample drawn from the population.

Formulas for E_s, in terms of observed data, when s may represent any one of a considerable number of statistics, say an arithmetic mean or a correlation coefficient, are usually listed for convenient application in the numerous textbooks for teaching the courses in statistics.

Under the definition stated above, it is noteworthy that these formulas depend on a fundamental assumption whose validity has long been in doubt. The assumption in question is to the effect that initially, that is, before our drawings of a sample are made, in our lack of knowledge about the distribution of possible values of an unknown parameter, say of θ, we may assume the existence distribution of θ to be constant.

Historically, this type of assumption is of more than ordinary interest. Bayes, in the *Philosophical Transactions* for 1763 [2], seems to have contributed a new set of ideas by applying the notion of probability not only to the probable chance results that will follow from *a priori* situations, but to the probable situations which antedated the results.

Consider the following simple and very special example: Each of a group of 3 urns contains 5 white balls and 2 black balls. Each of a group of 2 urns contains 1 white ball and 4 black balls. One ball is drawn and found to be black. Find the probability that it came from the first of the two groups.

By postulating that each urn is equally likely to be chosen we have for the first and second groups of urns, the "existence probabilities" 3/5 and 2/5 respecively. By next assuming that within an urn each ball is equally likely to be drawn, we have, for the first and second groups, the probabilities $p_1=2/7$ and $p_2=4/5$ respectively. Then Bayes's formula gives as the chance, q, that the black ball drawn came from the first group, the value

$$q = (\tfrac{3}{5})(\tfrac{2}{7})/(\tfrac{3}{5}\cdot\tfrac{2}{7} + \tfrac{2}{5}\cdot\tfrac{4}{5}).$$

In such a special example, the existence probabilities are commonly accepted as known rather than as unknown, and not much contention seems to exist about the validity of the solution. However, there has been contention and outspoken criticism, within the past fifty years, on the part of a few mathematicians, as to the solutions of very special examples similar to that cited above. This may be well illustrated by the following example [3] used by Professor George Chrystal in an argument against the validity of Bayes's formula.

"A bag contains three balls, each of which is either white or black, all possible numbers of white being equally likely. Two at once are drawn at random and prove to be white; what is the chance that all the balls are white?"

In Chrystal's argument, it is stated that "any one who knows the definition of mathematical probability, and who considers this question apart from the inverse rule, will not hesitate for a moment to say that the chance is $\frac{1}{2}$." This view taken by Chrystal was based mainly on what he called "the point of view of practical common sense." But from Bayes's theory, we get

$$\frac{\frac{1}{2}\cdot 1}{\frac{1}{2}\cdot 1 + \frac{1}{2}\cdot\frac{1}{3}} = \frac{3}{4}$$

for the chance in question.

Professor E. T. Whittaker [4] has shown in a very clear exposition that Chrystal's criticism is not well founded in this special case and that Bayes's formula is applicable.

Whittaker first applied the so-called common-sense argument of Chrystal to essentially the same example, but with the numbers increased so as to intensify certain effects to a degree that will impress common-sense judgment. To do this, he considers a bag containing 1,000,001 balls, each of which is either white or black, all possible numbers of white being equally likely *a priori*. Then 1,000,000 balls are conceived to be drawn at random, and found to be all white. What is the probability that the remaining ball is white?

Chrystal's doctrine would still give the answer $\frac{1}{2}$, but the observed fact, that the drawings of 1,000,000 balls have taken place without revealing a black ball, furnishes an irresistable common-sense verdict against the correctness of an answer in the neighborhood of $\frac{1}{2}$. Perhaps it should be added that after thus disposing of Chrystal's criticism in relation to this special example, Whittaker establishes directly the validity of the result given by Bayes's formula by considering a large number of bags, say 12,000,000, each constituted in accord with Chrystal's statement.

Although the doctrine of Bayes is appropriate for certain special examples, such as we have cited, it seems that certain eminent mathematicians, including Laplace, accepted as an assumption a highly generalized form of what Bayes postulated in special cases. In the treatment of Laplace, the idea was developed in connection with the "doctrine of insufficient reason" which tended toward the notion that initially cases may be considered to be equally probable unless we have some reason to think they are not. Under this doctrine, it became conventional to assume in the proofs of the formula for the probable error of a statistic, to which attention is called above, that initially the possible values of an unknown parameter may be assumed to be uniformly distributed. Thus, the "existence" distribution of probabilities would be assumed constant.

The invalidity of this assumption in many applied problems of statistical interest may be seen clearly in cases of a continuous distribution function with a derivative.

Suppose that our initial assumptions relating to a parameter θ were such that θ would initially be distributed in accord with a continuous frequency function, $\phi(\theta)$, which has a derivative at each point within its possible range on θ, say

from $\theta=\alpha$ to $\theta=\beta$. Next, suppose $\phi(\theta)$ were restricted to be constant throughout the range of θ. Then it is well known that the distribution of a simple non-linear function of θ would not be a constant. For example, the distribution of $z=\theta^n$ ($n\neq 1, \theta$ real and non-negative) would not be constant, but would be distributed in accord with a frequency function $(z^{(1-n)/n})/n$. But if θ is a population parameter, it seems fairly obvious that the logical character of our theory should usually, if not always, be such as to enable us to use a power of θ instead of θ as a parameter if we found it convenient to do so.

2. Fiducial theory. The preceding lengthy introduction would be inexcusable except that it is designed to lead up to the important fact that, although in the usual statistical inquiry by sample, the true value of the population average or parameter θ is unknown and remains unknown, there are cases in which precise statements can be made in terms of probabilities about the bounds within which a parameter θ lies without making an assumption about the initial distribution of the possible values of θ. It has been only about seven years since R. A. Fisher [5] initiated some important ideas in this connection to which interesting contributions have been made by several mathematical statisticians [6–10].

It is the main object of the present paper to bring certain elements of this advance concerned with statistical inference to the attention of college teachers of statistics, and to give a few illustrations.

For simplicity, consider a case of a single parameter, θ, in which we know the frequency function of the statistic, t, to be given by an integrable function

$$y_t = f(t, \theta), \tag{1}$$

where the values of t obtained from observation may be assumed to be good estimates of θ.

An illustration may be found in studies on a normal parent population of x's with a frequency function

$$y = \frac{1}{\sqrt{2\pi\theta}} e^{-(x-m)^2/2\theta} \tag{2}$$

in which m is the assigned population mean, and θ is the unknown population variance usually denoted by σ^2.

In this illustration, suppose t is an estimate of θ from a random sample consisting of n variates, $x_1, x_2, \cdots, x_n$, by applying the formula

$$t = \frac{1}{n-1} \sum_{i=1}^{i=n} (x_i - \bar{x})^2, \tag{3}$$

where $\bar{x}$ is the arithmetic mean of $x_1, x_2, \cdots, x_n$.

Suppose we know (1) in such form that it is possible to calculate a table of values of the probabilities that the statistic, t, will fall into an assigned interval selected on a possible range (a, b) for any assigned value of θ within the possible range (α, β) of θ.

Next, for illustration, select a positive number ϵ, say $\epsilon = .005$, on which to base a certain level of confidence about values of θ to be expressed in terms of probabilities.

As our main problem may be clarified by a geometrical representation, conceive of corresponding values of t and θ obtained in an extensive statistical experiment as represented by rectangular coordinates within the rectangle bounded by lines $t=a$, $t=b$, $\theta=\alpha$, and $\theta=\beta$ (Fig. 1).

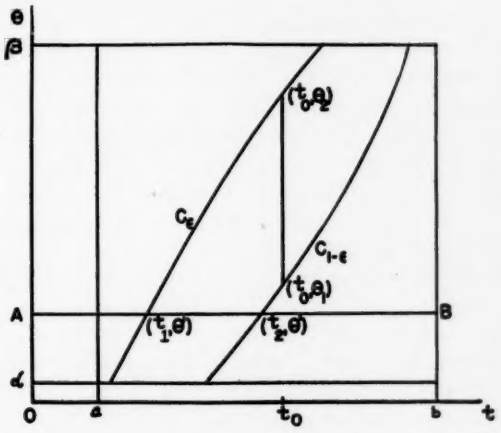


Fig. 1

Consider an arbitrary assignment for θ, say that $\theta=\theta'$ is the true value of θ. This gives the line AB (Fig. 1). Since the distribution of the statistic t is assumed to be known for each assigned value of θ, we may locate on the line AB two points, t_1 and t_2 $(t_1 \leqq t_2)$ such that ϵ is equal to the probability that t obtained from a random sample will yield a value of t less than or equal to t_1, and similarly ϵ is the probabiliy that such a sample will yield a value greater than or equal to t_2. Then we have an interval on AB from t_1 to t_2 such that $1-2\epsilon$ is the probability that the random sample will yield a value within this interval.

More formally stated, we may introduce a function $F(t, \theta)$ defined as the definite integral of $f(t, \theta)$ in (1) from $t=a$ to t. That is,

$$F(t, \theta) = \int_a^t f(t, \theta)dt,$$

for any arbitrarily assigned real value of θ on its range α to β.

Then $F(a, \theta)=0$, $F(b, \theta)=1$, $F(t_1, \theta')=\epsilon$, $F(t_2, \theta')=1-\epsilon$, $(0<\epsilon<1)$.

By considering all possible assignments of θ, in its possible range (α, β), the locus of our set of lower values of t, illustrated by t_1 on the line AB, will give a continuous curve which we mark with C_ϵ in Figure 1, the subscript ϵ being used to remind us that ϵ is the probability that a random value of t for $\theta=\theta'$ will fall

below or at t_1. Similarly, our set of upper values of t, illustrated by t_2 on AB, give a curve which we mark with $C_{1-\epsilon}$.

If t is a good estimate of θ, its value usually, if not always, increases with θ for all possible values. Thus, we shall restrict our further considerations to cases in which we may assume that t increases as θ increases and vice versa. More precisely we are concerned with one-valued monotone increasing functions represented by the two curves marked C_ϵ and $C_{1-\epsilon}$. The region bounded by these two curves and the lines $\theta=\alpha$ and $\theta=\beta^*$ has been called by Neyman the confidence belt with confidence coefficient equal to $1-2\epsilon$.

Next, consider the set of points, (t, θ), that would be obtained in Figure 1 in carrying out an extensive statistical experiment for which we seek a degree of accuracy in the long run, indicated by the value we assign to ϵ. Then it is fairly obvious that the confidence belt is so constructed that $1-2\epsilon$ is the expected relative frequency with which points, (t, θ), will lie inside the confidence belt, and 2ϵ is the expected relative frequency with which such points will lie outside the confidence belt or on its boundary, whatever the nature of the initial distribution function of the parameter θ may be.

Conceive of drawing a large number of sets of random samples of n items each from a population consisting either of an infinite supply or of a finite supply with replacements, and that one of these samples, taken at random, yields a value of $t=t_0$ for a certain *statistic*, then the line $t=t_0$ parallel to the θ-axis would fail to intersect the boundaries of the confidence belt, in two points, in at most a small fractional part (less than 2ϵ) of the total number of sets of drawings. Denote the ordinates of the points in which the line $t=t_0$ cuts the curves $C_{1-\epsilon}$ and C_ϵ by θ_1 and θ_2 respectively (Figure 1). These boundary values of θ are called *fiducial limits* of θ that correspond to $t=t_0$ and the interval θ_1 to θ_2 is called the fiducial interval for $t=t_0$. It is important to emphasize that the statement that $1-2\epsilon$ is the probability that a value of θ taken at random will fall into the confidence belt is to be associated with the whole belt, that is, with results of repeated application of a sampling procedure to all values of t met with in an extensive statistical experiment, and not merely with an assigned t. The probability that (θ, t) falls within the confidence belt may differ for different assignments of t, but in the long run of statistical experience, the expected relative frequency of points within the confidence belt is $1-2\epsilon$. By choosing ϵ to be small, the probability is nearly 1 that the parameter θ lies within the confidence belt.

The theory of confidence belts and fiducial intervals finds its main application in the testing of a certain hypothesis for possible rejection under the assumption that it is true. Such an hypothesis has been termed a *null hypothesis*. If, for a given ϵ, the null hypothesis is rejected due to the value of t found from actual data, the value of t is said to be significant at the level of probability equal to 2ϵ. On the other hand, a value of t from observed data which does not reject the null hypothesis is said to be non-significant.

* The α and β are the extreme values of θ.

To illustrate further, let us now return to the problem mentioned above which consists in finding fiducial limits of the variance $\theta=\sigma^2$ for estimates $t=s^2=\sum_{i=1}^{i=n}(x_i-\bar{x})^2/(n-1)$ from samples of n drawn from a normal distribution.

It is known that the sampling frequency function [11–13] of the variable s^2/σ^2 is

$$(4)\qquad \frac{(n-1)\left(\frac{(n-1)s^2}{2\sigma^2}\right)^{(n-3)/2} e^{-(n-1)s^2/2\sigma^2}}{2\Gamma\left(\frac{n-1}{2}\right)}.$$

By letting

$$(5)\qquad \frac{(n-1)s^2}{\sigma^2}=\chi^2,$$

we get the χ^2 distribution

$$F(\chi^2)d(\chi^2)=\frac{\left(\frac{\chi^2}{2}\right)^{(n-3)/2} e^{-\chi^2/2}}{2\Gamma\left(\frac{n-1}{2}\right)}d(\chi^2).$$

By the use of easily available tables [14], we may determine the two critical values, χ_1^2 and χ_2^2 of χ^2 for which $\int_0^{\chi^2}F(\chi^2)d\chi^2$ is equal to an assigned ϵ and to the corresponding $1-\epsilon$ respectively. The fiducial limits of $\theta=\sigma^2$ are then $(n-1)s^2/\chi_1^2$ and $(n-1)s^2/\chi_2^2$.

It may be noted that these limits are at once completely expressible in terms of the observations. In other words, the distribution of s/σ is independent of unknown parameters, and may be calculated from the known number of items, n, in the sample.

Consider next a case in which the variations of the statistic are not continuous.

The type of problem under consideration in which the observed x can assume only integral values is well illustrated by a binomial distribution as treated by C. J. Clopper and E. S. Pearson [7].

Consider the separation of a class of items (population) into two sub-classes and no more by drawing at random a sample of n items from a population when we are concerned with the unknown fractional part, p, of the items of the population that have a certain property P. Suppose a random sample shows x individuals with the property P, and $n-x$, without it. Then $t=x/n$ is a statistic that is, an estimate of p. For example, the population may be conceived to consist of an infinite supply of white and black balls thoroughly mixed in which p is the unknown fractional part of the balls that are white. Again, the population may be conceived to be a class of persons that differ with respect to the presence

or absence of a certain character, W, and p the unknown fractional part of these persons in whom the character in question is present. Obviously, many actual examples of data leading to binomial distributions may be readily cited.

The parent distribution with which we are concerned in these illustrations may be represented by the terms in the expansion of

$$(q + p)^n \tag{6}$$

with p and q unknown, where $p+q=1$.

We may think of x, the number of observed occurrences arising from n trials, and the unknown p as the rectangular coördinates (x, p) of points in a rectangle as shown in Figure 2.

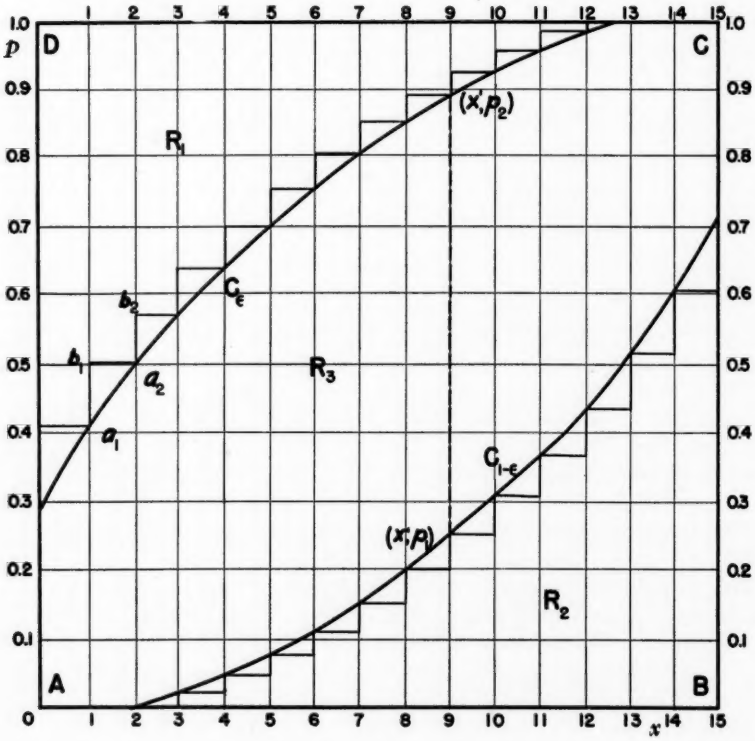


Fig. 2

We follow C. J. Clopper and E. S. Pearson [7] rather closely in the following treatment, but use as an illustration $n=15$ instead of $n=10$, and $\epsilon=0.005$ instead of their 0.025.

Let $S(p, n; 0 \cdots x)$ denote the sum of the first $x+1$ terms of (6) expanded,

and $S(p, n; x \cdots n)$ the sum of the last $n-x+1$ terms. Then, although it is not ordinarily possible to choose values of x_1 and x_2 such that $S(p, n; 0 \cdots x_1)$ and $S(p, n; x_2 \cdots n)$ equal exactly an assigned ϵ, say $\epsilon = 0.005$, it is clearly possible to choose x_1 and x_2 so that

$$S(p, n; 0 \cdots x_1) \leqq 0.005 < S(p, n; 0 \cdots x_1 + 1) \quad \text{and} \tag{7}$$

$$S(p, n; x_2 \cdots n) \leqq 0.005 < S(p, n; (x_2 - 1) \cdots n). \tag{8}$$

Now conceive of the determination of such positive integral values of x_1 and x_2 throughout the whole range of values of p from 0 to 1.

In Figure 2, the lower "corner" points of the upper sort of stairway configuration have the coördinates, (x, p), for which the probability

$$S(p, n; 0 \cdots x_1) = 0.005 \tag{9}$$

except for computational errors usually involving interpolation. The smooth curve C_ϵ is drawn through these points. Similarly, the upper "corner" points of the lower stairway configuration (Figure 2) have coördinates, (x, p), for which the probability

$$S(p, n; x_2 \cdots n) = 0.005. \tag{10}$$

The smooth curve $C_{1-\epsilon}$ is drawn through these points.

For the calculation of the values of the sums of terms of the binomial to get values of x_1 and x_2 that satisfy (7) and (8), tables published in one of the *Medical Research Council's Reports* [15] were found convenient. These tables give the values directly of the sum for any number of terms of the binomial $(q+p)^n$ for $p = 0.025, 0.05, 0.075, 0.10, 0.15, \cdots, 0.90, 0.925, 0.95, 0.975$, and for $n = 5, 10, 15, 20, 30$.

To exhibit the situation geometrically, the coördinates, (x, p), of points on the vertical segments a_1b_1, a_2b_2, $\cdots$ of the upper stairway diagram of Figure 2, satisfy (7). That is, the probability is equal to or less than 0.005 that a trial consisting of drawing a random sample of 15 will give coördinates, (x, p), of points in the region R_1 above or on the upper stairway configuration (Figure 2). Similarly, the coördinates, (x, p), of points on the vertical segments of the lower stairway diagram, satisfy (8). That is, the probability is equal to or less than 0.005 that a trial in question will give coördinates, (x, p), of points in the region R_2 below or on the lower stairway configuration.

Expressed in another form, it may now be inferred, in the long run of statistical experience involved in the continued drawing of samples of 15, from any binomial population whatever, that we should expect at least 99% of the points, (x, p), to lie within the region, R_3, of the rectangle bounded by the two stairways, since not more than $\frac{1}{2}\%$ of the points are on or above its upper stairway boundary and not more than $\frac{1}{2}\%$ are on or below its lower stairway boundary.

It may be observed that no point (x, p) can fall within the area bounded by either of the curves, C_ϵ or $C_{1-\epsilon}$, and the corresponding stairway configuration. The curves are more convenient than the stairway configurations for interpolation for intermediate values of n. The lozenge shaped area in Figure 2, bounded

by the curves C_ϵ and $C_{1-\epsilon}$, and sides of our rectangle $ABCD$, may then be taken as the confidence belt for the binomial parameter, p, with drawings of 15 items per sample.

Fiducial limits, p_1 and p_2 of p that correspond to an assigned x, say to $x=x'$ for example, may be represented by the ordinates of two points on the boundaries of the confidence belt for which $x=x'$. Thus, we may denote by (x', p_1) and (x', p_2) the coördinates of the points that represent the lower and upper fiducial limits respectively. The interval on the ordinate $x=x'$, between these two points is called the fiducial interval.

If we conceive of thus determining the fiducial interval for each observed value of x in which we are interested, the prediction that the unknown probability, p, will lie within the fiducial interval corresponding to an observed x will be correct in the long run of statistical inquiry, in at least $100(1-2\epsilon)$ per cent of the cases, and wrong in not more than 200ϵ per cent of the cases. The degree of confidence in our predictions about the probable values of p will thus depend on the level of probability, ϵ, which we may choose at will.

From our illustrations, it is fairly obvious that by the construction of confidence belts for various levels of probability, and for various numbers of items in a sample, it is possible to greatly improve the rationale underlying statistical inference about a population parameter, provided we know the distribution function of the statistic that may be appropriately used as an estimate of the parameter in question.

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ON THE REDUCTION OF A REPRESENTATION TO CLASSICAL CANONICAL FORM

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The matrix a_s^r may be used to determine a representation* by means of the equations

$$\bar{x}^i = a_\alpha^i x^\alpha. \tag{1}$$

The vectors x^i and $\bar{x}^i$ are regarded as being referred to the same coördinate system and as set in correspondence by the equations (1).

The vectors x^i and $\bar{x}^i$ when referred to a new base become y^i and $\bar{y}^i$,

$$y^i = p_\alpha^i x^\alpha, \qquad \bar{y}^i = p_\alpha^i \bar{x}^\alpha. \tag{2}$$

These equations will be said to define a transformation of coördinates.

Under the transformation (2) the representation (1) becomes

$$\bar{y}^i = \bar{a}_\alpha^i y^\alpha, \tag{3}$$

$$\bar{a}_s^r = a_\sigma^\rho p_\rho^r q_s^\sigma, \tag{4}$$

$$p_\alpha^r q_s^\alpha = p_s^\alpha q_\alpha^r = \delta_s^r. \tag{5}$$

Equations (4) are the transformation equations for a mixed tensor. These equations are more familiar in matrix notation. Regarding the upper index as determining the row of the square matrix, and the lower, the column, they are

$$\bar{a} = paq, \qquad pq = 1. \tag{6}$$

The matrices a and $\bar{a}$ are said to be similar. The representation (1) is expressed in classical canonical form when the transformation p of (6) is so chosen that a takes the form

$$\bar{a} = \begin{pmatrix} \lambda & 1 & & & & & & & \\ & \lambda & 1 & & & & & & \\ & & \cdot & & & & & & \\ & & & 1 & & & & & \\ & & & \lambda & & & & & \\ & & & & \mu & 1 & & & \\ & & & & & \mu & 1 & & \\ & & & & & & \cdot & & \\ & & & & & & & 1 & \\ & & & & & & & \mu & \cdot \\ & & & & & & & & \cdot \end{pmatrix}, \tag{7}$$

* Dickson: Modern Algebraic Theories, chap. 5. The terms, representation, collineation, linear correspondence, and transformation, are often used synonymously.

the elements not given being all zero's. The λ's, μ's $\cdots$ are roots of the characteristic equation $|a-\lambda|=0$, and relations such as $\lambda=\mu$ are not excluded.

Hermitian matrices under the unitary group are illustrative of quantities that transform by (4) or (6), but since their canonical form is diagonal, they are too special to be of interest here.

The theory of invariant factors or elementary divisors gives a means for determining a canonical form, but does not determine the transformation matrix p of (6). This is a remarkable feature of that theory. The purpose of the present paper is to devise a means for determining this transformation matrix p.

The vanishing of the characteristic determinant

$$|a_s^r - \lambda\delta_s^r| = 0, \tag{8}$$

is the condition that a set of functions h^ρ exist satisfying

$$h^\rho(a_\rho^r - \lambda_i\delta_\rho^r) = 0, \tag{9}$$

where λ_i is one of the n roots of (8). These will be different from zero when $|a_s^r| \neq 0$, as we shall assume.

Arrange the roots $\lambda_1 \cdots \lambda_n$ in an order such that repeated roots are together in the sequence. Assume that $h_{1|}^r$ is a solution of (9) corresponding to λ_1. Adjoin $n-1$ other vectors to this solution to form a set of n independent vectors $h_{s|}^r$. When these n vectors are taken for the base vectors in a new coördinate system y so that they become $\delta_{s|}^r$, equations (2) relate these vectors,

$$\delta_{i|}^r = p_\alpha^r h_{i|}^\alpha. \tag{10}$$

By the introduction of new quantities $H_r^{i|}$, defined by

$$H_\alpha^{r|}h_{i|}^\alpha = \delta_i^r, \tag{11}$$

it follows that

$$p_s^r = H_s^{r|}, \quad q_s^r = h_{s|}^r. \tag{12}$$

Since the tensor equations (9) hold in the new coördinates, when $\lambda_i=\lambda_1$ and $h_{1|}^\rho$ is $\delta_{1|}^\rho$, we have

$$\delta_{1|}^\rho(\bar{a}_\rho^r - \lambda_1\delta_\rho^r) = 0, \qquad \bar{a}_1^r = \lambda_1\delta_1^r, \tag{13}$$

that is, the elements of the first column are $\lambda_1, 0, \cdots, 0$.

The matrix will therefore be of the form

$$\begin{pmatrix} \lambda_1 & a_2 \cdots a_n \\ 0 & b_2 \cdots b_n \\ \cdot & \cdot \quad \cdot \\ \cdot & \cdot \quad \cdot \\ 0 & c_2 \cdots c_n \end{pmatrix}.$$

In the subspace $x^2, \cdots, x^n$, the new equations of the representation are determined by the matrix obtained by striking out the first row and first column of the above matrix. The number λ_2 will be a root of the characteristic equation and the corresponding solution may be indicated by $h^r_{2|}$, $r=2, \cdots, n$. For $h^1_{2|}$, zero or any convenient number may be assigned. Let the vector $h^r_{2|}$, $r=1, \cdots, n$ be chosen for a new x^2 coördinate axis, so that it becomes the vector $\delta^r_{2|}$. In this transformation the characteristic vector $\delta^r_{1|}$ remains on the x_1 axis and other arbitrarily chosen independent vectors are chosen to determine the transformation indicated by equations (10) and (12). As in (13) $\delta^r_{1|}$ is a characteristic vector, so the elements of the first column will be $\lambda_1, 0, \cdots, 0$. The vector $\delta^r_{2|}$ is an eigen-vector in the subspace $x^2, \cdots, x^n$ at least. The second column of the transformed matrix will be $a, \lambda_2, 0, \cdots, 0$. In this manner all elements below the principal diagonal are reduced to zero's and the diagonal elements are the λ_i. The reduction is more rapid if as many characteristic vectors as possible are chosen from the space and from each subspace as coördinate axes for each transformation.

Further reduction may be effected by the use of the following elementary transformations:

p_1 = the interchange of the ith and jth rows;
p_2 = the multiplication of the ith row by k;
p_3 = the addition of the ith row to the jth row, $i>j$.

The matrix corresponding to these operations p_a followed by p_b may be found by operating on the unit matrix by left multiplication. The resulting matrix may be designated by $p_b p_a$. The inverse of the elementary matrix p will be that elementary matrix q which operating on columns will restore p to the identity:

q_1 = the interchange of the ith and the jth columns;
q_2 = the division of the ith column by k;
q_3 = the subtraction of the jth column from the ith column, $i>j$.

Accordingly, the operations p_i and q_i on a are easily seen to be expressible by matrices as $p_i a q_i$ in accordance with (6).

The method of operation with these elementary transformations may be most easily explained by a simple illustration. Consider the matrix,

$$a = \begin{pmatrix} \lambda & a & b & e & f \\ & \lambda & c & g & h \\ & & \lambda & i & j \\ & & & \mu & d \\ & & & & \mu \end{pmatrix},$$

where $\lambda \neq \mu$, and all missing elements are zero's.

With p_2, $k=(\lambda-\mu)/i$, $a_1=p_2 a q_2$. The element i is therefore replaced by $\lambda-\mu$. Then by the transformation $p_3(i, j=4, 3)$, $a_2=p_3 a_1 q_3$ the element i transforms to zero. Similarly, g and e may be transformed to zero without effecting any zeros below the principal diagonal. Similarly j, h, f may be successively reduced

to zero. The next operations deal with the blocks along the principal diagonal. If a and b are not both zero, both may be reduced to 1's by the operation p_2, then one to zero by p_3. A like treatment applies to columns, hence in the general case, such a block may be transformed so that each row and column above the diagonal has not to exceed a single 1, all elements below the diagonal remaining zero. By p_1 this block may be reduced to the canonical form. A numerical illustration is given in the following paper.

LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

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The linear differential equations

$$\frac{dx^i}{dt} = a_\alpha^i x^\alpha, \qquad i = 1, \cdots, n, \ \alpha \text{ summed from } 1, \cdots, n, \tag{1}$$

were first completely treated by Weierstrass.* A more recent treatment is given by Nyswander.† The earlier method of Weierstrass is generally regarded as being more tedious than later methods. The solution used here bears a similarity to that of Weierstrass and is believed to have advantages in theory and practice over the other methods.

Using the matrix notation equations (1) are

$$\frac{dx}{dt} = ax. \tag{2}$$

If a transformation on x

$$y = px, \qquad x = qy, \qquad pq = 1, \tag{3}$$

is made, equation (2) for constant p's gives the transformed equations

$$\frac{dy}{dt} = \bar{a}y, \qquad \bar{a} = paq. \tag{4}$$

It has been shown how the matrix p may be determined so that a takes the canonical form corresponding to that determined by the matrix a of equation (7) of the preceding paper. These equations (4) are readily integrated to obtain a solution y. From (3) $x = qy$.

The non-homogeneous equation corresponding to (2) is

$$\frac{dx}{dt} = ax + \theta(t). \tag{5}$$

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* Weierstrass, Berlin Abhandlungen, 1875; Werke, vol. 2, pp. 75–76.

† J. A. Nyswander, American Journal of Mathematics, vol. 47, 1925, pp. 257–276. F. R. Moulton, Differential Equations, chap. 15.

Multiplying this matrix equation by the matrix p, using (3) and (4.2), we obtain

$$\frac{dy}{dt} = \bar{a}y + p\theta. \tag{6}$$

In the normal form these are readily integrated to give

$$y = y(t) + F(t), \tag{7}$$

the $y(t)$ involving the arbitrary constants and $F(t)$ being the particular solution.

From (7) by multiplication by q we obtain

$$x = qy + qF, \tag{8}$$

which completely solves the nonhomogeneous case.

We shall determine the matrices p and q upon which our solution depends for a numerical set that are solved in detail by F. R. Moulton (*Differential Equations*, p. 280).

$$\frac{dx^1}{dt} = x^1 - \frac{1}{4}x^2 + \frac{1}{2}x^3 + \frac{3}{4}x^4,$$

$$\frac{dx^2}{dt} = -\frac{1}{8}x^1 + x^2 - \frac{3}{8}x^3 - \frac{1}{2}x^4,$$

$$\frac{dx^3}{dt} = \frac{1}{2}x^1 + \frac{3}{4}x^2 + x^3 - \frac{1}{4}x^4,$$

$$\frac{dx^4}{dt} = -\frac{3}{8}x^1 - \frac{1}{2}x^2 - \frac{1}{8}x^3 + x^4.$$

The characteristic determinant has $\lambda = 1$ as a quadruple root. For $\lambda = 1$ this determinant is of rank 2. The first two rows are used to determine two characteristic vectors $h^r_{1|} = (-3, 2, 1, 0)$ and $h^r_{2|} = (-4, 3, 0, 1)$. The annexation of two other independent vectors $h^r_{3|} = (1, 0, 0, 0)$, $h^r_{4|} = (0, 1, 0, 0)$ determines the matrix (12) of the preceding paper.

$$q^r_{0s} = h^r_{s|} = \begin{pmatrix} -3 & -4 & 1 & 0 \\ 2 & 3 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad p^s_{0r} = H^{s|}_r = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 3 & 4 \\ 0 & 1 & -2 & -3 \end{pmatrix}. \tag{9}$$

From these and (4) we obtain

$$\bar{a}_0 = p_0 a q_0 = 1/8 \begin{pmatrix} 8 & 0 & 4 & 6 \\ 0 & 8 & -3 & -4 \\ 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 8 \end{pmatrix}. \tag{10}$$

The following elementary operations p on the rows;

a_1 = multiplication of last row by 6, and the second row by 3/2,
a_2 = addition of the second row to the first row,
a_3 = multiplication of the second row by 1/9,
a_4 = subtraction of the first row from the second row,
a_5 = multiplication of third row by $-1/16$ and the fourth row by $-1/72$,
a_6 = interchange of the second and third rows,

and the corresponding inverse operations q on columns, reduce the matrix a to

$$\bar{a} = paq = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

In this equation

$$p = p_1 p_0,\ p_1 = a_6 \cdots a_1. \tag{12}$$

The matrix p_1 is found by applying the elementary operations on the unit matrix, as

$$p_1 = \begin{pmatrix} 1 & 3/2 & 0 & 0 \\ 0 & 0 & -1/16 & 0 \\ -1 & -4/3 & 0 & 0 \\ 0 & 0 & 0 & -1/12 \end{pmatrix}. \tag{13}$$

From (9) and (13), with (12), we obtain

$$p = \begin{pmatrix} 0 & 0 & 1 & 3/2 \\ -1/16 & 0 & -3/16 & -1/4 \\ 0 & 0 & -1 & -4/3 \\ 0 & -1/12 & 1/6 & 1/4 \end{pmatrix}, \quad q = \begin{pmatrix} 0 & -16 & 3 & 0 \\ 2 & 0 & 0 & -12 \\ -8 & 0 & -9 & 0 \\ 6 & 0 & 6 & 0 \end{pmatrix}. \tag{14}$$

The equations (4), (11) now integrate to give

$$\begin{aligned} y^1 &= c_1 e^t + c_2 t e^t, & y^2 &= c_2 e^t, \\ y^3 &= c_3 e^t + c_4 t e^t, & y^4 &= c_4 e^t. \end{aligned} \tag{15}$$

Equations (3), (12) give

$$\begin{aligned} x^1 &= (-16c_2 + 3c_3)e^t + 3c_4 t e^t, \\ x^2 &= (2c_1 - 12c_4)e^t + 2c_2 t e^t, \\ x^3 &= (-8c_1 - 9c_3)e^t + (-8c_2 - 9c_4)t e^t, \\ x^4 &= (6c_1 + 6c_3)e^t + (6c_2 + 6c_4)t e^t. \end{aligned} \tag{16}$$

By setting all of the constants but c_i equal to zero, $i = 1, \cdots, 4$, fundamental solutions are found to be:

$$x^1 = (0t + 0)e^t,\ (0t - 16)e^t,\ (0t + 3)e^t,\ (3t + 0)e^t,$$
$$x^2 = (0t + 2)e^t,\ (2t + 0)e^t,\ (0t + 0)e^t,\ (0t - 12)e^t,$$
$$x^3 = (0t - 8)e^t,\ (-8t + 0)e^t,\ (0t - 9)e^t,\ (-9t + 0)e^t,$$
$$x^4 = (0t + 6)e^t,\ (6t + 0)e^t,\ (0t + 6)e^t,\ (6t + 0)e^t.$$

MATRIX APPROXIMATION AND LATENT ROOTS

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1. Introduction. Among the mathematical problems arising out of recent factorial theories in psychology, one of the most interesting is that of approximating to a given matrix by another of lower (preassigned) rank. The problem comes out of the attempt to pick out a set of primary abilities, or factors, supposed to contribute linearly in the performance of any given task. Thus when a set of tests is given to a group of individuals, the measure of the individual's accomplishments (his score) in a given test is taken to be a linear combination of the extent to which he possesses the factors requisite to the performance of the task imposed. A perfect description is not to be expected because of the inevitable presence of errors of one form or another.

The problem amounts to expressing a matrix A of numbers, obtained empirically, as a sum,

$$A = B + C,$$

where B is assumed to contain the "true" or "significant" theoretical values, and C contains the errors of measurement and whatever additional terms may be regarded as irrelevant. Evidently the mathematical problem is not defined until suitable conditions are imposed upon the matrices B and C, or at least upon C. Since C is to consist of quantities to be neglected, it is natural to ask for some type of non-negative *departure function* [12],* vanishing when A and B are identical, to be minimized under suitable restrictions. In the simplest case, which is to be considered here, Eckart [2] suggested taking as the *departure function* the *sum of the squares of the elements of* C. It turns out that there is always a solution and that $BC' = 0$, where the prime denotes the transpose. Hence the approximating vectors are orthogonal to the residuals. It turns out, further, that the same solution also minimizes the sum of the squares of the elements of CC' under the restriction that $BC' = 0$.

For Kelley's theory [6] this simple problem is adequate, and its solution, to be outlined here, is essentially complete. Additional complications are presented, however, by the Spearman single-factor theory [9] and by Thurstone's more general theory [10], due to the expectation of so-called specific factors, each operating in one test only. Thus while Kelley's theory throws all errors indiscriminately into C, Thurstone allows some of them to go into a theoretically

* See references at the end of the paper.

accounted-for matrix $\overline{C}$. But the additional freedom proves to be mathematically embarrassing, and no satisfactory departure function has yet been defined and shown to have a (generally) unique solution.

2. The theoretical solution. Any matrix

$$A = (a_{i\alpha}), \qquad \begin{pmatrix} i = 1, \cdots, n \\ \alpha = 1, \cdots, \nu \geqq n \end{pmatrix}, \tag{1}$$

may be thought of as defining a projection of vectors in an n-space S upon a ν-space Σ, or as defining the conjugate transformation projecting the vectors of Σ upon S. If e_i are orthogonal unit vectors in S and ϵ_α orthogonal unit vectors in Σ, then the one transformation projects e_i upon $\Sigma_\alpha a_{i\alpha}\epsilon_\alpha$ and the other projects ϵ_α upon $\Sigma_i a_{i\alpha} e_i$.

Multiplication of A on the right by an orthogonal matrix corresponds to a rotation of axes in Σ; multiplication on the left by such a matrix corresponds to a rotation in S. Either operation we shall refer to briefly as a *transformation of* A. Since the departure function of two matrices A and B is equal to the sum of the squares of the differences of the vectors of A and those of B, it is evident that:

THEOREM 1. *If $\overline{A}$ is any transform of a matrix A, and $\overline{B}$ the same transform of B, then the departure function of $\overline{A}$ and $\overline{B}$ is the same as that of A and B.*

Eckart [1] has shown that any matrix can be transformed into a diagonal matrix. That is to say:

THEOREM 2. *If A is any matrix, then there exist orthogonal matrices W and Ω such that*

$$WA\Omega' = \begin{pmatrix} D_\rho & 0 \\ 0 & 0 \end{pmatrix}, \tag{2}$$

where

$$D_\rho = [d_1, \cdots, d_\rho] = \begin{pmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & d_\rho \end{pmatrix}, \tag{3}$$

where Ω' denotes the transpose of Ω, and where ρ is the rank of A. Moreover W and Ω can always be so chosen that

$$d_1 \geqq d_2 \geqq \cdots \geqq d_\rho > 0. \tag{4}$$

In brief sketch the proof runs as follows: Since A is of rank ρ, the transformation it defines projects S into a ρ-dimensional subspace of Σ. Hence we may choose the vectors ϵ_α so that $\epsilon_1, \cdots, \epsilon_\rho$ are in this subspace while $\epsilon_{\rho+1} \cdots \epsilon_\nu$ are orthogonal to it. This amounts to transforming A by a right orthogonal matrix.

Likewise the conjugate transformation defined by A projects Σ upon a ρ-dimensional subspace of S, and the vectors e_i may be so chosen that $e_1, \cdots, e_\rho$ lie in this subspace. If we suppose these transformations made, A has the form

$$A = \begin{pmatrix} A_\rho & 0 \\ 0 & 0 \end{pmatrix}, \tag{5}$$

where A_ρ is non-singular. Then, leaving $\epsilon_{\rho+1}, \cdots, \epsilon_n$ and $e_{\rho+1}, \cdots, e_n$ fixed, it is known [8] that A_ρ can be transformed to the form D_ρ.

We may now prove the following fundamental theorem [2]:

THEOREM 3. *If a matrix A, of rank ρ, is diagonalized as in Theorem 2, then the best rank r ($<\rho$) approximation to A is*

$$B = W'\begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix}\Omega, \tag{6}$$

where the matrix

$$D_r = [d_1, \cdots, d_r] \tag{7}$$

is the diagonal matrix formed from the r largest diagonal elements of D_ρ. The matrix B is unique unless $d_r = d_{r+1}$.

This is equivalent to saying that A and its best fit B can be diagonalized simultaneously, and that the non-zero diagonal elements of B are thus the largest diagonal elements of A. For, suppose that B has been diagonalized while A has not, and let A and B have the form

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \tag{8}$$

$$B = \begin{pmatrix} \Delta_r & 0 \\ 0 & 0 \end{pmatrix}. \tag{9}$$

Then the matrix

$$\begin{pmatrix} \Delta_r & A_2 \\ 0 & 0 \end{pmatrix} \tag{10}$$

has the same rank as B and is certainly a better fit to A unless $A_2 = 0$. Hence $A_2 = 0$, and, for a like reason, $A_3 = 0$.

If $A_1 \neq \Delta_r$, then a better fit to A could be obtained by replacing Δ_r by A_1, in B, contrary to the assumption that B is the best fit. Hence $A_1 = \Delta_r$.

The form of B is unchanged by transformations represented by

$$\begin{pmatrix} I_r & 0 \\ 0 & w \end{pmatrix}, \qquad \begin{pmatrix} I_r & 0 \\ 0 & \omega' \end{pmatrix}$$

on the left and the right, where I_r is the identity matrix of rank r, and w and ω'

are arbitrary orthogonal matrices of suitable rank. These transformations of A give it the form

$$\begin{pmatrix} \Delta_r & 0 \\ 0 & wA_4\omega' \end{pmatrix}$$

so that by Theorem 2, w and ω can be chosen to diagonalize A_4. It is then evident that Δ_r must contain the r largest elements of D_ρ.

If W and Ω are the orthogonal matrices of Theorem 2, then evidently

$$WAA'W' = \begin{pmatrix} D_\rho^2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{11}$$

$$\Omega A'A\Omega' = \begin{pmatrix} D_\rho^2 & 0 \\ 0 & 0 \end{pmatrix}, \tag{12}$$

where the first matrix is $n \times n$ and the second $\nu \times \nu$. This with the preceding theorem gives [2]:

THEOREM 4. *If B is the rank-r matrix which best fits A, then BB' and $B'B$ are the rank-r matrices which best fit AA' and $A'A$ respectively. Moreover, the elements d_i^2 are the latent roots of the matrices AA' and of $A'A$, and the matrices W and Ω are matrices of their principal vectors.*

3. Methods of calculation. In theory the approximation problem is now solved, but there remains the problem of finding suitable methods of calculation. Two methods will be developed, one due to Hotelling [4, 5] and one to Horst [3], which, besides their utility, serve to throw interesting light upon the problem. Underlying these two methods is the principle stated in the following theorem, now readily obtainable:

THEOREM 5. *If W and Ω are orthogonal matrices which diagonalize a matrix A of rank ρ as in Theorem 2, and if W_r and Ω_r are the matrices whose first r rows are identical with the first r rows of W and Ω respectively, but whose last r rows are zero, then the matrix B_r of rank r which gives the best approximation to A is expressible in the form*

$$B_r = W_r'W_rA = A\Omega_r'\Omega_r. \tag{13}$$

The subscript r on B_r is introduced now in order that we may speak of approximations of different ranks. To derive the first form, then, note that in equation (6) nothing is changed if we replace W by W_r. But if we do this, we may, without changing the results, include the remaining $\rho - r$ elements in the diagonal matrix, and have

$$B_r = W_r' \begin{pmatrix} D_\rho & 0 \\ 0 & 0 \end{pmatrix} \Omega = W_r'WA. \tag{14}$$

But, in view of the zero elements of W_r', we may drop the last $n - r$ rows of W:

$$W_r' W = W_r' W_r. \tag{15}$$

This gives the desired result.

Let w_i be the vector in the ith row of W. Then if

$$R = AA', \tag{16}$$

we know that

$$w_i R = d_i^2 w_i, \tag{17}$$

and any vector $x = \sum x_i w_i$ is transformed by R into a vector whose component along w_i is d_i^2 times that of the original vector x. Hence if $x_1 \neq 0$ and $d_1 > d_2$ repeated transformations by R yields a vector xR^m approaching w_1 in direction. If $d_1 = d_2 = \cdots = d_\sigma > d_{\sigma+1}$, then these transformations project x into vectors xR^m whose components in the space of $w_1, \cdots, w_\sigma$ become relatively greater. But $w_1, \cdots, w_\sigma$ can then be replaced by any orthogonal combination of themselves,* and we may therefore suppose, again, that xR^m approaches the vector w_1 in direction. Hence for large values of m we have approximately

$$xR^{m+1} = d_1^2 xR^m. \tag{18}$$

Thus d_1^2 is obtainable approximately from (18) and w_1 is the unit vector along either member of (18).

The matrix

$$R_1 = R - d_1^2 w_1' w_1 \tag{19}$$

is diagonalized by W to the form $[0, d_2^2, \cdots]$. Hence the same procedure may be applied to R_1 as to R, in order to obtain d_2^2 and w_2. It is not necessary, however, to start again with R_1 and raise it to the mth power, since the formula

$$R_1^m = R^m - d_1^{2m} w_1' w_1 \tag{20}$$

is readily verified. This justifies Hotelling's method for calculating latent roots and principal vectors:

THEOREM 6. *Let $R = AA'$ be diagonalized to the form $D^2 = [d_1^2, \cdots, d_\rho^2, 0, \cdots, 0]$ by the orthogonal matrix W whose row-vectors are denoted by w_i. Let, further, $d_1 \geqq d_2 \geqq \cdots \geqq d_\rho > 0$. If x is any vector not orthogonal to w_1, then the sequence of vectors xR^m approaches w_1 in direction, and d_1^2 is given approximately by equation* (18). *The vector w_2 and the latent root d_2^2 are obtainable then from R_1 defined in* (19), *and so on, sequentially. In succeeding steps, the matrix R in* (19) *and* (20) *is replaced by R_{i-1}, d_1 by d_i, and w_1 by the row-vector w_i.*

It may be pointed out that if two of the roots are nearly equal, the process will converge only slowly.

Hotelling's method gives the approximating matrix B_r by equation (13) only after the matrix W_r has been calculated. However, it yields a method for cal-

* This is evident from Eckart's proof [1] of our Theorem 2.

culating, in addition, the latent roots at the same time. In principle it utilizes r arbitrary vectors x, chosen successively and projected into the principal vectors w. Horst's method starts with r independent vectors and makes the entire projection all at once (though in successive approximations of course) without yielding either latent roots or principal vectors directly. It depends upon the following theorem:

THEOREM 7. *If W is an orthogonal matrix which diagonalizes $R=AA'$ in such a way that successive diagonal elements are arranged in descending order of magnitude; if $d_r > d_{r+1}$; if W_r is obtained from W by omitting the last $n-r$ rows; and if, finally, P is a matrix of rank r, then the sequence of matrices*

$$(R^mP)[(R^mP)'(R^mP)]^{-1}(R^mP)'$$

has as its limit with increasing m the matrix $W_r'W_r$ whenever W_rP is non-singular.

The last condition will be recognized as generalizing Hotelling's requirement that the arbitrary vector x shall not be orthogonal to w_1.

Since

$$R = W'D^2W,$$

it follows that

$$R^m = W'D^{2m}W.$$

Then the matrix whose limit is sought can be written

$$W'D^{2m}WP[P'W'D^{4m}WP]^{-1}P'W'D^{2m}W.$$

If W_rP is non-singular we may write

$$WP = \begin{pmatrix} I \\ Q \end{pmatrix} W_rP,$$

where Q is some matrix. Hence

$$D^{2m}WP = \begin{pmatrix} D_r^{2m} \\ D_{n-r}^{2m}Q \end{pmatrix} W_rP,$$

where D_{n-r} is obtained from D by dropping the first r diagonal elements. Hence

$$\begin{aligned}[(R^mP)'(R^mP)]^{-1} &= [(W_rP)'(D_r^{4m} + Q'D_{n-r}^{4m}Q)(W_rP)]^{-1} \\ &= (W_rP)^{-1}(D_r^{4m} + Q'D_{n-r}^{4m}Q)^{-1}(W_rP)'^{-1}.\end{aligned}$$

Hence

$$(R^mP)[(R^mP)'(R^mP)]^{-1}(R^mP)' = W'\begin{pmatrix} D_r^{2m} \\ D_{n-r}^{2m}Q \end{pmatrix}(D_r^{4m} + Q'D_{n-r}^{4m}Q)^{-1}(D_r^{2m},\ Q'D_{n-r}^{2m})W.$$

This expression is unchanged if we replace the matrix $(D_r^{2m},\ Q'D_{n-r}^{2m})$, wherever it appears, by

$$D_r^{-2m}(D_r^{2m}, Q'D_{n-r}^{2m}) = (I_r, D_r^{-2m}Q'D_{n-r}^{2m}).$$

But the elements of $D_r^{-2m}Q'D_{n-r}^{2m}$ are obtained from those of Q' by multiplying by $d_i^{-2m}d_j^{2m}$ for $i \leqq r$ and $j > r$. Hence if $d_r > d_{r+1}$, all such factors tend to zero with increasing m, and we have, therefore, for the required limit,

$$W'\begin{pmatrix} I_r \\ 0 \end{pmatrix} I_r(I_r, 0)W = W_r'W_r,$$

as was to be proved. Thus if the vectors of P actually have r independent vector projections upon the space of B, the matrix $W_r'W_r$ can be obtained by Theorem 7 and hence B itself by Theorem 5.

In Horst's own presentation, his procedure was described, without proof of convergence, in somewhat different terms. Starting with a matrix P of rank r he considered a problem suggested by Mosier [7], *viz.*, that of finding a matrix X such that PX is the best least-squares fit to A among all right multiples of P. This matrix is given by Mosier as

$$X = (P'P)^{-1}P'A. \tag{21}$$

Calling this matrix P_1, Horst then asks for the matrix Y such that YP_1 is the best least-squares fit among all left multiples of P_1. Such a matrix YP_1 is manifestly at least as good a fit as PP_1. Taking transposes, a formula similar to (21) can be written. But calling this matrix P_2, it can be shown that

$$P_2P_1 = (RP)[(RP)'(RP)]^{-1}(RP)'A. \tag{22}$$

The process as described above is thus clearly indicated.

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SOLUTION OF THE GENERAL MAGIC SQUARE

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1. Introduction. The general magic square may, perhaps, be best described by its diagram:

$a_{1\,1}$	$a_{1\,2}$		$a_{1\,n-1}$	$a_{1\,n}$
$a_{2\,1}$	$a_{2\,2}$		$a_{2\,n-1}$	$a_{2\,n}$
$a_{n-1\,1}$	$a_{n-1\,2}$		$a_{n-1\,n-1}$	$a_{n-1\,n}$
$a_{n\,1}$	$a_{n\,2}$		$a_{n\,n-1}$	$a_{n\,n}$

The symbol a_{rs} is employed to designate the cell in the rth row, sth column. A square of the nth order is thus seen to consist of n^2 cells. The conditions for a general magic square are simply that the a's denote any algebraic quantities whose sum, in any row, column or either major diagonal, adds to the same number.*

If, in particular, the a's consist of the consecutive integers from 1 to n^2, we have the case of the "ordinary" magic square. Although the latter problem presents many difficulties, the former is relatively simple, and, indeed, a construction for its general solution, if $n \geqq 5$, will be given in the present paper. The cases $n=3$ and 4, however, which do not fit this construction, are treated separately.

Considering a square of the nth order, then, we find $2n+2$ linear conditions involving n^2 quantities. That at most $2n$ conditions are independent is, moreover, easily shown. For, if we denote by S the sum of all the a's, it is obvious that if their sum in $n-1$ separate rows or columns is always S/n, the sum in the remaining row or column will be S/n. Thus the complete solution of the general magic square should involve at least n^2-2n arbitrary constants. It will be found that exactly that number are involved.

2. The third order magic square. This order possesses the property that, if we denote by S the sum of any row, then the center cell $a_{2\,2}$ equals $S/3$. To prove this, note that

$$a_{1\,1}+a_{2\,2}+a_{3\,3}=S, \qquad a_{2\,1}+a_{2\,2}+a_{2\,3}=S, \qquad a_{3\,1}+a_{2\,2}+a_{1\,3}=S,$$

and

$$a_{1\,1}+a_{2\,1}+a_{3\,1}=S, \qquad a_{3\,3}+a_{2\,3}+a_{1\,3}=S.$$

* On occasion the diagonal properties have been dispensed with. Thus see D. N. Lehmer, Transactions of the American Mathematical Society, vol. 31, 1929, pp. 529–551.

Hence by adding the first three equations and subtracting the last two we have

$$3a_{2\,2} = S.$$

If we now take $a_{1\,1}$, $a_{2\,2}$, $a_{1\,3}$ equal to $k+a$, k, $k+b$ respectively, we obtain the solution

(I)

$k+a$	$k-a-b$	$k+b$
$k-a+b$	k	$k+a-b$
$k-b$	$k+a+b$	$k-a$

The latter is the general formula, since the quantities $a_{1\,1}$, $a_{2\,2}$, $a_{1\,3}$ are perfectly arbitrary, and the conditions on the remaining cells are both necessary and sufficient for a solution.

It is noteworthy that the solution (I) is unaltered (aside from rotation and reflexion) by the substitution of $-a$ for a, or of $-b$ for b, or by the interchange of a and b. Hence we shall regard $a \geqq b$, where both are positive. Thus $k+a+b$ is always the largest, $k-a-b$ the smallest number in the set (I). For example, if $n=3$, the ordinary magic square consists of the numbers 1 to 9 inclusive. Then we must have

$$k+a+b=9, \qquad k-a-b=1.$$

Whence $k=5$, $a=3$, $b=1$, yielding as the only ordinary magic square of the third order

8	1	6
3	5	7
4	9	2

3. The fourth order magic square.

The characteristic property of this order is that if S denotes the sum of any row, then the sum of the four central elements is S; that is,

$$a_{2\,2} + a_{2\,3} + a_{3\,2} + a_{3\,3} = S.$$

The proof follows that given in section 2.

If now we take $a_{1\,1}$, $a_{1\,2}$, a_{13}, $a_{2\,1}$, $a_{2\,2}$, $a_{2\,3}$, $a_{3\,2}$, $a_{3\,3}$ arbitrary, we again find that the necessary conditions on the remaining cells will also prove sufficient for a solution; which will therefore be the general one. For simplicity, we write these a's as $A-a$, $C+a+c$, $B+b-c$, $D+a-d$, B, C, A, D respectively, to obtain

(II)

$A-a$	$C+a+c$	$B+b-c$	$D-b$
$D+a-d$	B	C	$A-a+d$
$C-b+d$	A	D	$B+b-d$
$B+b$	$D-a-c$	$A-b+c$	$C+a$

This formula was first given by E. Bergholt,* who attempted to prove it general by an argument similar to that given in the last paragraph of section 1. However, he stated without proof that the 8 conditions involved in this general magic square are independent.

4. The higher order magic squares. For the general magic square of the nth order, $n>4$, we construct the following diagram:

(III)

$a_{1\,1}$	$a_{1\,2}$	$a_{1\,3}$		$a_{1\,v-1}$	$a_{1\,v}$	$a_{1\,v+1}$		$a_{1\,n-2}$	$a_{1\,n-1}$	$a_{1\,n}$
$a_{2\,1}$	$a_{2\,2}$	$a_{2\,3}$		$a_{2\,v-1}$	$a_{2\,v}$	$a_{2\,v+1}$		$a_{2\,n-2}$	$a_{2\,n-1}$	b_1
$a_{3\,1}$	$a_{3\,2}$	$a_{3\,3}$		$a_{3\,v-1}$	$a_{3\,v}$	$a_{3\,v+1}$		$a_{3\,n-2}$	$a_{3\,n-1}$	b_2
$a_{n-2\,1}$	$a_{n-2\,2}$	$a_{n-3\,3}$		$a_{n-2\,v-1}$	$a_{n-2\,v}$	$a_{n-2\,v+1}$		$a_{n-2\,n-2}$	$a_{n-2\,n-1}$	b_{n-3}
b_{2n-2}	$a_{n-1\,2}$	$a_{n-1\,3}$		$a_{n-1\,v-1}$	b_{2n-1}	$a_{n-1\,v+1}$		$a_{n-1\,n-2}$	$a_{n-1\,n-1}$	b_{n-1}
b_{2n-3}	b_{2n-4}	b_{2n-5}		b_{2n-v-1}	b_{2n}	b_{2n-v-2}		b_{n+1}	b_n	b_{n-2}

Let

$$R_i = \text{the sum of the } a\text{'s in the } i\text{th row},$$

$$C_i = \text{the sum of the } a\text{'s in the } i\text{th column},$$

$$D_1 = \sum_1^{n-1} a_{rr},$$

$$D_2 = \sum_1^{n-1} a_{r\,n-r+1},$$

$$S = \text{the sum of all the } a\text{'s},$$

the a's being arbitrary, and the b's being defined as follows:

(IV)

$$b_1 = R_1 - R_2, \qquad b_{2n-v-2} = R_1 - C_{v+1},$$

$$b_2 = R_1 - R_3, \qquad b_{2n-v-1} = R_1 - C_{v-1},$$

$$\cdots\cdots\cdots \qquad \cdots\cdots\cdots$$

$$b_{n-3} = R_1 - R_{n-2}, \qquad b_{2n-4} = R_1 - C_2,$$

$$b_{n-2} = R_1 - D_1, \qquad b_{2n-3} = R_1 - D_2,$$

$$b_{n-1} = R_1 - C_n - \sum_1^{n-2} b_i, \qquad b_{2n-2} = R_1 - C_1 - b_{2n-3},$$

$$b_n = R_1 - C_{n-1}, \qquad b_{2n-1} = R_1 - R_{n-1} - b_{n-1} - b_{2n-2},$$

$$b_{n+1} = R_1 - C_{n-2}, \qquad b_{2n} = R_1 - C_v - b_{2n-1}.$$

* Nature, vol. 83, 1910, p. 368.

We note first that in this construction b_{2n-1} is to occupy any cell in the $n-1$st row up to but not including either diagonal. Such an arrangement is impossible in the cases $n \leqq 4$, and the latter were hence excluded.

Now by definition (IV), the square (III) will be magic if the lone condition,

$$b_{n-2} + b_n + b_{n+1} + \cdots + b_{2n-3} + b_{2n} = R_1,$$

is satisfied.

We may write

$$b_{n-1} = D_1 + \sum_{1}^{n-2} R_i - C_n - (n-2)R_1 = D_1 + S - R_{n-1} - (n-2)R_1 - C_n,$$

$$b_{2n-2} = D_2 - C_1,$$

$$b_{2n-1} = (n-1)R_1 + C_1 + C_n - D_1 - D_2 - S,$$

$$b_{2n} = D_1 + D_2 - C_1 - C_v - C_n - (n-2)R_1 + S.$$

Then, substituting in the left-hand member and combining terms, we have

$$R_1 - (C_1 + C_2 + \cdots + C_n) + S.$$

But

$$C_1 + C_2 + \cdots + C_n = S.$$

Hence the square is magic and (III) is the general solution.

As an illustration, our solution, for $n = 5$, may be diagrammed as

				b_1
				b_2
b_8		b_9		b_4
b_7	b_6	b_{10}	b_5	b_3

When arbitrary numbers are inscribed in the blank cells, the b's are immediately determined, if they are taken in the order indicated by their subscripts; and the resulting square will be magic. The method thus has the merit of yielding quickly magic squares of any desired order.

Note by the Editor. For additional work on magic squares, the reader may refer to A. L. Candy's *Construction, classification and census of magic squares of even order*, which was reviewed in this MONTHLY, vol. 44, 1937, page 528.

RECENT PUBLICATIONS

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All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Modern Higher Algebra. By A. A. Albert. (University of Chicago Science Series, edited by G. A. Bliss, H. H. Barrows, and Preston Kyes.) Chicago, University of Chicago Press, 1937. 14+319 pages. $4.00.

The Elements of Analytic Geometry. By H. E. Buchanan and G. E. Wahlin. New York, Farrar and Rinehart, 1937. 9+256 pages. $2.25.

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REVIEWS

General Mathematics for Students of Business. By William S. Schlauch. New York, F. S. Crofts and Co., 1936. 4+393 pages. $3.75.

The preface states that this book is a text in general mathematics for students who wish to prepare themselves for work in the mathematical theory of investments, finance, insurance, business calculations, statistics, and budgeting. In this respect the author has succeeded in including those topics of mathematics from algebra, trigonometry, calculus, and finite differences which are fundamental in the study of business and statistics courses and which require a mathematical technique as a prerequisite.

The material included covers a wide range of topics and the author has certainly included the majority of those topics which would be required in the study of most business subjects. Two features which stand out in the text are: The inductive procedure followed in presenting new topics; and the practical application of each section of theory in developing business formulas and rules for calculation and in the solving of such problems as arise in business and finance. Each section of theory is followed by a large number of carefully selected illustrative examples and also a long list of well graded problems for solution.

On the whole the subject matter is well presented and the definitions and formulas are clearly expressed. In one or two instances, however, the reviewer feels that the definitions might have been improved; for example, the definitions of *variable* and *constant* on page 185. In the definition of a constant there is no distinction made between arbitrary and absolute constants; while in the definition of a variable the use of the term *number* may be confusing to the student.

The tables are not extensive but seem adequate for the purpose of supplying the text. It seems unfortunate to the reviewer that the author did not follow the text with an index of the subject matter.

A. W. RICHESON

Geschichte der Elementar-Mathematik. By Johannes Tropfke. Dritter Band, Proportionen, Gleichungen. Dritte Auflage. Berlin and Leipzig, Walter de Gruyter and Co., 1937. 4+239 pages.

Eight years have passed since this reviewer called attention (this MONTHLY, vol. 38) to the first volume of the third edition of Dr. Tropfke's series of histories of elementary mathematics. At that time attention was also called to the fact that nearly a third of a century had elapsed since the first edition appeared. Today we can look back four years more to the date of publication of the second volume of the third Auflage, and cannot but wonder at the persistence of the man who was then an Oberstudiendirektor in the Kirschner-Schule in Berlin and is at present what we may call an Emeritus Professor if not an Emeritus Director. To undertake the revision of three volumes, and to look forward to a

similar revision of four more, requires a combination of faith, hope, and perseverance that is at least very unusual. It is to be hoped that Dr. Tropfke has now the time and the health to carry on his plan of completing the remaining volumes of the entire series which he has so well begun.

As all readers of his works and of the reviews which appear in such periodicals as this MONTHLY will recall, his plan was to publish seven volumes covering the topics of (1) Rechnen, (2) Allgemeine Arithmetik, (3) Proportionen, Gleichungen (the one here under review), (4) Geometrie, (5) Trigonometrie, (6) Analysis, and (7) Stereometrie. To carry out this plan he has worked studiously, consulting many of the important histories of the subject and, particularly in the present number, showing a reasonable degree of accuracy in his quotations and his use of proper names. He has followed his original project of providing a table of contents (Inhalt) for each book, reserving a complete index to be placed at the end of Book VII—a plan of which this writer does not approve but for which Dr. Tropfke has no doubt a reason which he feels will justify the arrangement and which European writers often adopt. To wait until Vol. VII appears in the probably rather long distant future, in order to find a desired item in Vol. I, is not pleasant to contemplate.

The present volume devotes twenty-two pages to proportion, the following 176 to equations, and some forty to appendices relating to the growth of modern symbolism and to the history of cubic equations. In general Dr. Tropfke's sources are German, but they show some familiarity with a few English, French, and Italian writers and, of course, with Latin and Greek texts. He seems to have taken more pains in spelling and quoting than were shown in the first two volumes of the third edition, although he still adheres to the Italian *abbaci* instead of using the *abaci* which appears more frequently in the early Latin manuscripts. In the case of Widman, as the man himself seems to have preferred to spell his name, he not only uses the spelling Widmann but employs one form on page 20, and the other form only two pages later.

Of course such small differences are of little moment, at least to any reader well versed in the original texts, but it is to be hoped that they will not be so numerous in Vol. IV as they have been in the preceding ones.

Turning from the somewhat trivial habit of reviewers, that of unimportant fault-finding, we may well consider as more important the question of the usefulness of the book. In this respect it may safely be said that Vol. III, like its predecessors, will be of great value to the student in this important part of mathematics—the region of algebra which treats of proportion, symbolism, and the history of the growth of the equation. That the series appears before the recent discoveries of Neugebauer, Gandz, Thureau-Danjin, and other students of the early achievements of the Sumerians and Babylonians is unfortunate, but such misfortunes are the fate of historians in any line. We meet with a similar incident in connection with the discoveries within the past year relating to numerals in the newly discovered remains in southern Palestine, which are more concerned with the first volume than with the third.

In comparing Dr. Tropfke's series with other works on the history of elementary mathematics it should be borne in mind that it is composed of seven volumes as against only one or two in other attempts to tell the story of the subject. This permits of a much more elaborate treatment of every phase of elementary mathematics, with the certainty of increases in the chance for errors, and it will be strange if the number shall not turn out to be rather large.

The series however should have place in the libraries of all our colleges and universities, both for its clear treatment of the growth of the equation and the symbolism of the subject, and for the bibliographical material which is furnished in the large number of footnotes. We should all congratulate Dr. Tropfke on the progress already made in his great undertaking, and express the hope of his completion of the four remaining volumes in the near future.

DAVID EUGENE SMITH

Introduction to Mathematics. By Hollis R. Cooley, David Gans, Morris Kline, and Howard E. Wahlert of the Department of Mathematics, Washington Square College, New York University. Houghton Mifflin Company, 1937. 18+634 pages. $3.75.

This book is intended as a survey course for college freshmen to correct wrong impressions that many students have about mathematics and its relation to other fields of knowledge. This is stated clearly by the authors in their preface as follows: "Most students who do not specialize in mathematics leave its study without having acquired any real understanding of the character of the subject or its relations to the sciences, the arts, philosophy, and to knowledge in general. Too often they have been taught little more than a variety of techniques in special branches of mathematics and thus have acquired a narrow, distorted, and hence incorrect view of mathematics." "Even students who specialize in mathematics usually acquire no proper understanding of the nature of the subject or of its relations to other fields of knowledge until late in their college years. This book attempts to remedy this unfortunate situation by showing that mathematics has much more to offer serious minded college students than mere training in memorizing formulas and manipulating symbols." It is the opinion of the reviewer that the authors have done an excellent piece of work in presenting this material to clear up such misunderstandings on the part of students.

The principal objectives of the book as stated by the authors are:

1. To show how some of the fundamental ideas of mathematics have their sources in physical experience.

2. To show how, from these ideas, mathematics builds broad, logical theories which have wide applications in the physical, biological, and social sciences, in the arts, and in philosophy.

3. To show that mathematics is not merely a collection of methods useful in the sciences, but a vast unified system of reasoning which possesses many of the characteristics of a fine art.

4. To acquaint the student with the logical situation of a mathematical system and thus provide him with a standard of exact reasoning which should help him to achieve a more critical attitude towards conclusions arrived at in other fields.

5. To show that science and philosophy are indebted to mathematics for many precise concepts, such as velocity, motion and infinity.

6. To open the student's mind to the fact that the development of mathematics from ancient to modern times has been an important factor in the development of civilization.

In the organization of subject matter and method of treatment the authors have departed widely from the traditional method. Most of the subject matter having to do largely with techniques has been omitted and other topics such as number and number systems, fundamental concepts in algebra and geometry, types of equations in algebra, the function concept and its application, concept of a limit, non-euclidean geometry have been stressed. The material is organized so that it presents a somewhat unified treatment in which the various topics, such as numbers and number systems for example are considered largely from the standpoint of their historical development.

It is written in an interesting narrative style which reads very smoothly and not only should it hold the interest of the student but it should stimulate him to want to delve further into the subject matter by reading some of the many excellent references given at the end of each chapter.

The book is divided into four parts and each part into several chapters. Part I deals with the development of algebra and geometry from their basic concepts starting with the concept of a whole number and ending with the unification of algebra and geometry by means of coördinate geometry.

Practically all of the material included in Part I has been studied by the student in his high school courses, yet the presentation is so different that it will all seem new and interesting to him and he will have a much clearer understanding of some of the important fundamental concepts and operations of algebra, plane geometry, and coördinate geometry.

Part II develops the function concept and its applications. This part will seem new to the college freshman although he has already studied some functional relationships and graphs of functions. It is presented here in a very simple yet complete manner. Beginning with the definition of variables and constants, and the meaning of a function, the authors consider various types of functions such as algebraic, logarithmic, exponential, trigonometric, and their graphs, also methods of finding the formulas for useful functions. They also discuss functions used for prediction and point out the conditions under which they may be used. This part of the book is especially well done. It contains a wide variety of functions and a great many applications. The authors are careful to state clearly the conditions under which derived functions may be used.

While it is not the main purpose of the book to develop techniques, there are

in this chapter exercises enough to give the student considerable knowledge of and facility in drawing graphs of functions and in obtaining equations to express functional relationship.

Chapter XI, The Use of Functions for Prediction, and Chapter XII, Functions as Quantitative Descriptions, are excellent. The material is carefully selected and the student is made to understand the practical use made of functions of various types.

Part III is devoted to the limit concept. It begins with an elementary discussion of sequences and limit of a sequence of numbers, then the meaning of limit of a function is explained and defined. The limit concept is applied to the problem of rate of change of a function, first by use of speed and then by other applications leading to the derivative of a function.

The formula for finding a derivative of a function is obtained for a function of one type only, namely ax^n where n is a positive integer, and sums of such functions. The derivative is applied to problems in maxima and minima of functions and a set of problems given.

Limits are also used in defining such concepts as instantaneous speed, length of a curve, area bounded by a closed curve, etc., also to the sum of an infinite series. The use of the infinite series is illustrated in explaining some of the ancient paradoxes of Zeno.

While this portion deals with a concept that is usually considered very difficult, it is presented here in such a clear and elementary manner that the freshmen should have no difficulty in understanding it.

In Part IV the authors discuss: Infinite classes, Chapter XVII; non-euclidean geometry, Chapter XVIII; the new understanding of the nature of mathematics to which they lead, Chapter XIX; relativity, Chapter XX; and the broad relations of mathematics to other fields of knowledge, Chapter XXI. These topics are much more difficult than those presented in the first three parts of the book, and although the authors have succeeded in presenting them in a comparatively simple and helpful manner, the writer believes that the last three chapters would be over the heads of most freshmen.

Throughout the book the authors have used well selected illustrations as a means of making their explanations clear, and have used adequate cross references coördinating the various parts. The explanations are given in such complete detail that nothing is left for the student to supply or to be illumined by the instructor. In this respect it is practically a self teaching book that anyone interested in mathematics will enjoy reading. The book is complete with a summary, references to other books on subjects discussed, and suggestions for further study given at the end of each chapter.

There is ample material in the book for a course three hours a week for a year. At the end of that time the student will have learned a great deal about mathematics and its applications to other fields but will not have developed many techniques for carrying out the operations. Much could be done in this

respect if the book were used as the basis for a four or five hour a week course and were supplemented by more exercises to develop techniques.

The mechanical set-up and type used in printing are excellent throughout.

W. E. WILBUR

Lezioni di Analisi Matematica. By Amedo Agostini. Livorno, R. Accademia Navale, 1936, 8+336 pp.

This book represents an excellent introductory course of analysis which has been given for some years at the Italian Naval Academy. The purpose of it is primarily to prepare students for applied mathematics. The treatment is clear and as rigorous as can be expected in a book of such a character, the examples are well chosen, and the problems are instructive. The historical references in this work, however, are frequently misleading. Thus, on page 232 discussing the history of the solution of algebraic equations, the author speaks of Dal Ferro, Ferrari, Ruffini, but fails to mention Abel and Galois. Again, the names of Newton and Leibniz occur in connection with various details, but their general role in the development of the calculus is not indicated, while Cavalieri and Toricelli receive much prominence.

W. SEIDEL

Plane Trigonometry. 10+288+23 pages. $1.80. *Plane and Spherical Trigonometry.* 10+336+27 pages. $2.00. By J. B. Rosenbach, E. A. Whitman, and D. Moskovitz. Boston, Ginn and Company, 1937.

Instead of being "just another trigonometry" this text is an especially appealing one. It covers all the usual topics adequately, it is well set up, and has characteristics that should make it particularly welcome to the critical teacher.

Its good points include a paragraph of historical introduction, sound definitions with the function of the general angle introduced early, unusually clear explanation of interpolation, and splendid instruction on approximations and degree of accuracy.

Among its features not found in many trigonometry texts are: conversion tables in degrees and radians, sufficient stress on linear and angular velocity, area of segment and sector, a clear cut summary of the solution of triangles, well presented instructions for compounding curves, presentation of graphical solution of equations, and clear instructions on the principal values of inverse trigonometric functions. Among the deviations from the methods and notations of the usual text we note, in particular, the proof of sin $(\alpha+\beta)$ and cos $(\alpha+\beta)$ formulas by use of an area formula and the law of cosines respectively, the introduction of A_s for $180°-A$, reference angle and coreference angle.

The plane trigonometry concludes with a good chapter for beginners on complex numbers and De Moivre's theorem. Numerous examples are given in both the plane and spherical trigonometries and at the end of each book answers are given for odd numbers.

MARY E. WELLS

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 462 East Foster Ave., State College, Pa.

BOOKS FOR CLUBS

This list is continued from the January issue.

Bibliographies

H. *A suggested list of mathematical books for junior college libraries*, this MONTHLY, vol. 32, 1925, pp. 462–68.

I. *Mathematics curriculum bibliography*, by Anna Stafford. *School Science and Mathematics*, vol. 37, 1937, pp. 414–15.

General

12. *An Invitation to Mathematics*, by A. Dresden. New York, Henry Holt and Company, 1936. 12+453 pages. $2.80. Review in this MONTHLY, vol. 44, p. 324. An excellent reference book in any club library as a source of material for a year's program on fundamental concepts of mathematics.
13. *Special Topics in Theoretical Arithmetic*, by J. Bowden. Garden City, New York, 1936. 6+217 pages. $2.50 Review in this MONTHLY, vol. 44, p. 327. A book in which leaders of mathematics clubs should find some interesting suggestions.
14. *A Mathematician Explains*, by Mayme I. Logsdon. Chicago, University of Chicago Press, 1935. 12+175 pages. $2.00. Review in this MONTHLY, vol. 44, p. 528. An excellent book for clubs, as it contains unusual, interesting information which should stimulate the younger readers to more intensive study of mathematics.
15. *Construction, Classification and Census of Magic Squares of Even Order*, by A. L. Candy. Ann Arbor, Michigan, Edwards Brothers, Inc., 1937. 6+192 pages. $1.00. Review in this MONTHLY, vol. 44, p. 528. A helpful book for those interested in magic squares, one of the most popular club topics.
16. *Famous Problems of Elementary Mathematics*, by Felix Klein. Translated by W. W. Beman and D. E. Smith. New York, Ginn and Company, 1910. Most excellent source for a series of programs for a club.

CLUB REPORTS
1936–37

Pi Mu Epsilon, University of California at Los Angeles

Director, J. R. Gorman; Vice-Director, Lucille M. Donovan; Secretary, Estelle Mazziotta; Treasurer, W. E. Mason. The annual prize examination sponsored by the chapter was won by Harry Loss. The chapter held its annual Christmas and beach parties, its initiations and six program meetings. Topics discussed included: Orthogonal polynomials; Theory of singular points; Concepts of infinity and dimensionality; Axioms of projective geometry; Central collineations.

Mathematics Club, Elmira College

This group uses mathematical games and stunts to enliven its social meetings. Topics for regular meetings included: Magic squares; Ahmes papyrus; History of the calendar.

Kappa Mu Epsilon, State Teachers College of Nebraska

President, Lucille Wright; Vice-President, M. Davenport; Secretary, Ruth Wagner; Treasurer, R. Anderson. This lively chapter in addition to social meetings and banquets held monthly meetings for reading papers on mathematics. Topics discussed included: Measurement of time, its history and methods of correction; A new number system; Future of mathematics; Applied mathematics; Constant e; Mathematics tests for entrance to United States Naval Academy; Organization of a mathematics club in high school; Mathematics in music. A debate, Resolved that the duo-decimal system is preferable to the decimal system of numbers, created much interest.

Mathematics Club, New York State College for Teachers

Topics discussed included: Fundamental concepts; Singular numbers; Algebraic systems; The Simson line; Paradoxes of mathematics; Pythagorean theorem; Magic squares. A Christmas party and a picnic added variety to the year's program.

Pi Mu Epsilon, Iowa State College

Director, Phyllis Christy; Vice-Director, C. Amick; Secretary, Alice Churchill; Treasurer, Madelyn K. Cornwall; Librarian, Elaine Leffingwell; Faculty Adviser, Dr. D. L. Holl. The chapter sponsors a mathematics club in which many junior college students appear on the programs. An initiation banquet was held by the chapter alone. The prize given to the student completing calculus with the highest average in freshman and sophomore mathematics and in general scholarship was won by Lee Nicol. Topics for the ten club meetings in addition to several old favorites included: Projections of a quadrangle; Properties of conics in perspective; Light takes the shortest path; Electricity for a lemon squeezer; Farmer chases the pig; Impact of billiards by triangles; Reciprocation; Caustics; Sieve of Eratosthenes; Illustrating models of parabaloid of revolution, intersecting cylinders, and projections of a quadrangle. *Flatland Fantasy* was presented at one meeting.

Mathematics Club, University of Cincinnati

President, E. E. Yelton; Vice-President, P. M. Pepper; Secretary-Treasurer, S. J. Lawrvill. Topics discussed included: Probability as applied to horse betting; Cryptic operations; The derivation of Kepler's Laws of Motion from Newton's equations; Caustic curves of the circle and parabola; Curve fitting for additional data; Biomathematics; Functional equations with illustrations. The annual picnic closed the year's program.

Mathematics Club, Haverford College

President, A. P. Leib; Secretary, B. C. Lintz. Topics discussed included: Solution of geometrical problems by inversion; Maps; Extension of Descartes's rule to the complex domain; Application of matrices to problems of arrangement; Theory and application of continued fractions.

Michigan State College Mathematics Club

President, L. Liesenring; Vice-President, R. Tesar; Secretary-Treasurer, Bonnietta Miller; Faculty Adviser, Dr. J. E. Powell. The first and last meetings of the year were open air picnics. At the bi-monthly meetings the average attendance was fifty. Topics presented included: Geography of the moon; Development and proposed revision of the calendar; Works of Kepler and Newton; Astronomical results of Kepler's and Newton's Laws; Measurement and weight of planets; Planimeters, polar and simple; Mathematical theory of polar and jack knife planimeters; Some properties of numbers; Newton, Archimedes, Napier, Gauss, Lobachevsky, and Sylvester; History of mathematics; Modern mathematics.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 321. *Proposed by V. Thébault, Le Mans, France.*

In what system of enumeration may a four-digit number of the form *aabb* be the square of a number of the form *cc*, provided furthermore that if expressed in the decimal system, *a* and *b* consist of the same two digits in opposite orders?

E 322. *Proposed by J. E. Trevor, Cornell University.*

A manufacturer makes all possible sizes of brick-shaped blocks of molded building material, which are such that the lengths of the edges are integral multiples of the unit of length, and that the number of units in the total length of the twelve edges of a block is equal to two-thirds of the number of units of volume in the block. How many and what sizes does he make?

E 323. *Proposed by J. R. Musselman, Western Reserve University.*

If D, E, and F are the midpoints of the sides BC, CA, and AB respectively of the triangle ABC, show that the centers of the nine-point circles of triangles AFE, BDF, and CED form a triangle homothetic with triangle ABC in the ratio of 1:2.

E 324. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

Right triangles with integer sides proportional to 3:4:5 are well known. The sides of such a triangle form an arithmetic progression. Show that these constitute the only right triangles having integer sides in arithmetic progression, and furthermore that no right triangle exists having integer sides in geometric progression or in harmonic progression.

E 325. *Proposed by W. W. Bigelow, Beloit College, Wisconsin.*

Show that the radius of the sphere which can displace the most liquid from a filled conical wineglass is given by the formula, $r=hn/(n^2+n-2)$, where h is the altitude of the cone and n is the cosecant of its generating angle.

E 326. *Proposed by J. Rosenbaum, Bloomfield, Connecticut.*

Find all positive integer solutions, with $x<y$, of $x^2+y^2+xy=20461$.

SOLUTIONS

E 288 [1937, 478]. *Proposed by W. B. Campbell, Drexel Institute of Technology.*

If $\cos(a+bi)=p+iq$, where a, b, p, and q are real, find explicit expressions for all permissible values of $\cos a$, $\sin a$, $\sinh b$, in terms of p and q, classifying where necessary, and showing that they all lead to real values for a and b.

Solution by J. H. Butchart, Phillips University, Oklahoma

By the series definition of $\cos(a+bi)$ we may show that $\cos(a+bi)$ equals $\cos a \cosh b - i \sin a \sinh b = p+iq$. Upon equating real and imaginary parts separately and eliminating successively b and a, we have the following quadratics in $\sin^2 a$ and $\sinh^2 b$:

$$\sin^4 a + (p^2 + q^2 - 1)\sin^2 a - q^2 = 0,$$
$$\sinh^4 b + (1 - p^2 - q^2)\sinh^2 b - q^2 = 0.$$

Each of these equations has one positive real root, so we may solve and get

$$\sin a = \pm (1/\sqrt{2})[-k + (k^2 + 4q^2)^{1/2}]^{1/2},$$
$$\sinh b = \pm (1/\sqrt{2})[+k + (k^2 + 4q^2)^{1/2}]^{1/2},$$

where $k=p^2+q^2-1$. It is easily seen that $\sin^2 a < 1$ for all real values of p and q, and that $\cos a = \pm(1/\sqrt{2})[2+k-(k^2+4q^2)^{1/2}]^{1/2}$. It is also seen that there are no restrictions on p and q from the value of $\sinh b$, for $\cosh b$ is always greater than 1 in absolute value.

Also solved by E. P. Starke and the proposer.

E 289 [1937, 478]. *Proposed by A. A. Bennett, Brown University.*

Show that the Pappus configuration in the plane (consisting of three triads of points, the triads by pairs being triply perspective from the points of the remaining triad), comprises nine points, P_{xy}, and nine lines, p_{uv}, $(x, y, u, v = 0, 1, 2)$, where the notation may be so assigned that P_{xy} is incident to p_{uv} if and only if $ux \equiv v+y \pmod 3$.

Solution by J. H. Butchart, Phillips University, Oklahoma

The congruence, $ux \equiv v+y \pmod 3$, gives us the table of incidences:

p_{00}	p_{01}	p_{02}	p_{10}	p_{11}	p_{12}	p_{20}	p_{21}	p_{22}
P_{00}	P_{02}	P_{01}	P_{00}	P_{02}	P_{01}	P_{00}	P_{02}	P_{01}
P_{10}	P_{12}	P_{11}	P_{11}	P_{10}	P_{12}	P_{12}	P_{11}	P_{10}
P_{20}	P_{22}	P_{21}	P_{22}	P_{21}	P_{20}	P_{21}	P_{20}	P_{22}

It is clear from the table, or from the relation itself, that this scheme is satisfactory. There are $9 \cdot 3 \cdot 2 \cdot 2$ ways in which the configuration can be labeled. This notation is suggestive, for the triads of points are P_{0i}, P_{1i}, $P_{2i}(i=0, 1, 2)$. If P_{0i}

is to be the center of perspectivity, the corresponding points of the other two triads are P_{2u} and P_{2v}, where $i+u+v\equiv 0 \pmod 3$.

E 290 [1937, 478]. *Proposed by Meyer Karlin, Student, Yeshiva College.*

If $S_k=1+x+x^2\cdots+x^k$, decompose the fraction, $1/(S_{2n}S_{2n+1}S_{2n+2})$, into the algebraic sum of three fractions (also proper) whose denominators are respectively S_{2n}, S_{2n+1}, and S_{2n+2}.

Solution by E. P. Starke, Rutgers University.

Into the identity,

$$x(x^{2n}-1)(x^{2n+2}-1)(x^{2n+3}-1)+x(x^2-1)(x^{2n+1}-1)(x^{2n+3}-1)$$
$$-(x^{2n+3}+x^4-x^3-1)(x^{2n+1}-1)(x^{2n+2}-1)\equiv(1-x^2)(1-x)^2$$

substitute $x^{2n+1}-1\equiv(x-1)S_{2n}$, $x^{2n+2}-1\equiv(x-1)S_{2n+1}$, $x^{2n+3}-1\equiv(x-1)S_{2n+2}$, and divide through by $(1-x^2)(x-1)^2\,(S_{2n}S_{2n+1}S_{2n+2})$ to get

$$\frac{x(x^{2n}-1)/(1-x^2)}{S_{2n}}-\frac{x}{S_{2n+1}}+\frac{(x^{2n+3}+x^4-x^3-1)/(x^2-1)}{S_{2n+2}}\equiv\frac{1}{S_{2n}S_{2n+1}S_{2n+2}}$$

which may be written in the form

$$\frac{-\sum_{k=1}^{n}x^{2k-1}}{S_{2n}}-\frac{x}{S_{2n+1}}+\frac{x^2+1+\sum_{k=1}^{n}x^{2k+1}}{S_{2n+2}}\equiv\frac{1}{S_{2n}S_{2n+1}S_{2n+2}}.$$

Also solved by the proposer.

E 291 [1937, 478]. *Proposed by V. Thébault, Le Mans, France.*

With the digits, 1, 2, 3, 4, 5, 6, 7, 8, and 9, used once each, form a perfect square which is divisible by 99, and show that the solution is unique.

Solution by J. E. Burnam, Hardin-Simmons University, Texas.

Let N^2 be the required number. Since $123456789<N^2<987654321$, we have $11036<N<31427$. Since N^2 is divisible by 99, N must contain the factors 3 and 11, and hence be a multiple of 33. With a table of the squares of numbers less than 1000 we can rapidly eliminate most of the multiples of 33 between 11036 and 31427. Consider the square of the number composed of the last three digits of each potential N, and the square of the number composed of its first three digits, and discard all N's for which there appears the digit 0, or any duplication of digits. By this process there will remain less than fifty possible values of N. Among these are just two which meet the conditions of the problem. They are 19569 and 29106, whose squares are respectively 382945761 and 847159236. The solution is not unique.

The first of these two solutions was also found by W. E. Buker, Wm. Douglas, C. W. Trigg, and the proposer.

E 292 [1937, 478]. *Proposed by K. W. Miller, Utilities Research Commission, Inc., Chicago.*

Verify the following trigonometric identities:

(a) $\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\frac{\cos k\theta}{\cos^k\theta} = 0$ if n is odd, but $= (-1)^{n/2+1}\tan^n\theta$ if n is even.

(b) $\sum_{k=0}^{n}(-1)^{k+1}\binom{n}{k}\cos^k\theta\cos k\theta = (-1)^{(n+1)/2}\sin^n\theta\sin n\theta$ if n is odd,

$= (-1)^{n/2+1}\sin^n\theta\cos n\theta$ if n is even.

Here $\binom{n}{k}$ represents a binomial coefficient, as usual.

Solution by E. P. Starke, Rutgers University.

Raising to the nth power both sides of the identity

$$i\tan\theta \equiv -1 + (\cos\theta + i\sin\theta)/\cos\theta$$

and employing De Moivre's theorem, we have

$$i^n\tan^n\theta = \sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}(\cos k\theta + i\sin k\theta)/\cos^k\theta.$$

The real part of the right member is

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\cos k\theta/\cos^k\theta;$$

while the real part of the left member is 0 if n is odd, and is $(-1)^{n/2}\tan^n\theta$ if n is even. Equating these, we have relations equivalent to (a).

For (b) we use similarly the identity

$$-1 + \cos\theta(\cos\theta + i\sin\theta) \equiv i\sin\theta(\cos\theta + i\sin\theta).$$

The real part of the nth power of the left member is

$$\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\cos^k\theta\cos k\theta,$$

while for the right member the real part is $(-1)^{(n+1/)2}\sin^n\theta\sin n\theta$ if n is odd, and is $(-1)^{n/2}\sin^n\theta\cos n\theta$ if n is even. Again, these may be equated and the results are immediately reducible to the proposed form.

If in the above discussion, imaginary parts are equated, the results are

(a′) $\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\frac{\sin k\theta}{\cos^k\theta} = (-1)^{(n+1)/2}\tan^n\theta$ if n is odd, but $= 0$ if n is even.

(b′) $\sum_{k=0}^{n}(-1)^{k}\binom{n}{k}\cos^k\theta\sin k\theta = (-1)^{(n+1)/2}\sin^n\theta\cos n\theta$ if n is odd,

$= (-1)^{n/2}\sin^n\theta\sin n\theta$ if n is even.

E 293 [1937, 479]. *Proposed by J. H. Butchart, Phillips University, Enid, Oklahoma.*

Construct three circles through a point P so that the sum of the directed segments cut off by the circles on any line through P is zero.

Solution by W. B. Clarke, San Jose, California,

Construct two circles with radii $r_1 \neq r_2$, internally tangent at P, and a third circle with radius $r_3 = r_1 + r_2$, externally tangent to the first two circles at P. Then diameter $d_3 = d_1 + d_2$. Any straight line through P results in similar figures in the three circles, and forms chords k_1, k_2, and k_3 proportional to the respective diameters, and hence $k_3 = k_1 + k_2$.

Also solved by E. P. Starke and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

Correction. In 3841 [1937, 543] after A_1, B_1, C_1, D_1 in the fourth line insert: Let P' be the isogonal conjugate of P with respect to the tetrahedron $A_1B_1C_1D_1$. The fifth line should then read "the point P'" instead of "a point P'."

3863. *Proposed by S. B. Townes, University of Oklahoma.*

The vertices of a simplex in n dimensions are O, P_1, P_2, $\cdots$, P_n. Let $OP_i = (a_{ii})^{1/2}$ and the cosine of the angle between OP_i and OP_j be $a_{ij}/(a_{ii}a_{jj})^{1/2}$, $i, j = 1, 2, \cdots, n$. Show that r_n, the radius of the circumscribed hypersphere, is given by

$$r_n^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ii}a_{jj}A_{ij}/4\,|\,a_{ij}|\,,$$

where A_{ij} is the cofactor of a_{ij} in the symmetric determinant $|a_{ij}|$. This result in different notation, for $n = 2$, appears in (10) page 160 of Dickson's *Studies in the Theory of Numbers.*

3864. *Proposed by H. D. Grossman, New York City.*

Prove that the condition that $ax^2 + bx \pm c$ are both factorable, where a, b, c are integers whose G. C. D. is unity, is $ac = rs(r^2 - s^2)$, $\pm b = r^2 + s^2$, where r and s are relatively prime integers.

3865. *Proposed by H. D. Grossman, New York City.*

Prove that the general solution in integers of $ab=cd$, $a+b=c-d$, where a, b, c, d are integers whose G. C. D. is unity, is $a=r(r-s)$, $b=s(r+s)$, $c=r(r+s)$, $d=s(r-s)$, where r and s are relatively prime integers. It will be observed that $a-d$, $b+c$, $ac-bd$, $bc-ad$ are each perfect squares. If a is unity in the preceding problem, the roots of the given two polynomials are the a, b, c, d in this second problem.

3866. *Proposed by J. M. Feld, New York City.*

Show that the equation of the circumcircle of the triangle whose sides have the equations $L_i \equiv a_i x+b_i y+c_i=0$, $i=1, 2, 3$, can be written in the form

$$(a_1^2 + b_1^2)(a_2b_3)L_2L_3 + (a_2^2 + b_2^2)(a_3b_1)L_3L_1 + (a_3^2 + b_3^2)(a_1b_2)L_1L_2 = 0,$$

where $(a_ib_j)=a_ib_j-a_jb_i$.

3867. *Proposed by V. Thébault, Le Mans, France.*

Given a hexagon $A_1A_2A_3A_4A_5A_6$ whose consecutive sides are mutually perpendicular, show that: (1) The diagonals A_1A_4, A_2A_5, A_3A_6 meet in a point M. Let Δ_1 and Δ_4, Δ_2 and Δ_5, Δ_3 and Δ_6 be the perpendiculars to A_1A_4, A_2A_5, A_3A_6 at their corresponding extremities; and let B_1, B_2, B_3, B_4, B_5, B_6 be the intersections of (Δ_6, Δ_1), (Δ_1, Δ_2), (Δ_2, Δ_3), (Δ_3, Δ_4), (Δ_4, Δ_5), (Δ_5, Δ_6). The hexagon $B_1B_2B_3B_4B_5B_6$ is inscriptible in a circle through M. (3) The areas of the A and B hexagons are equal.

SOLUTIONS

3774 [1936, 189]. *Proposed by J. Rosenbaum, Bloomfield, Conn.*

If $A_1A_2 \cdots A_n$ is a regular polygon with unit radius, the locus of a point P such that

$$\sum_{i=1}^{n} (PA_i)^{2r} = K,$$

is a circle, provided r is a positive integer less than n and K is not less than n.

Show also that when

$$K = \frac{1 \cdot 3 \cdot 5 \cdots (2r-1)2^r n}{r!},$$

the locus is the circumcircle of the polygon.

Solution by W. V. Parker and R. A. Hefner, Georgia School of Technology.

Let the polar coördinates of A_i be $(1, 2i\pi/n)$. The equation of the locus is then

$$\sum_{i=1}^{n} \left[\rho^2 + 1 - 2\rho \cos\left(\theta - \frac{2i\pi}{n}\right)\right]^r = K \tag{1}$$

or

$$(2) \qquad \sum_{p=0}^{r} \frac{r!}{(r-p)!p!}(-2)^p(\rho^2+1)^{r-p}\rho^p \sum_{i=1}^{n} \cos^p\left(\theta - \frac{2i\pi}{n}\right) = K.$$

If p is a positive integer less than n,

$$\sum_{i=1}^{n} \cos^p\left(\theta - \frac{2i\pi}{n}\right) = 0, \qquad \text{if } p \text{ is odd},$$

$$= \frac{p!n}{2^p(p/2)!(p/2)!}, \qquad \text{if } p \text{ is even}.$$

(See Hobson, *Plane Trigonometry*, fifth edition, pages 54 and 90.)

Since only even values of p occur in (2) we may write it

$$(3) \qquad \sum_{s=0}^{[r/2]} \frac{r!n}{(r-2s)!s!s!}(\rho^2+1)^{r-2s}\rho^{2s} = K,$$

where $[r/2]$ is the largest integer less than or equal to $r/2$.

Since all terms in the left member of (3) are even powers of ρ with positive coefficients and its constant term is n, it is satisfied by one and only one positive value of ρ^2, provided K is greater than n. Hence the locus is a circle if r is a positive integer less than n and K is greater than n. It is a point circle for $K=n$.

If in (1), we set $\rho=1$ and $\theta=0$, we have

$$K = \sum_{i=1}^{n}\left(2 - 2\cos\frac{2i\pi}{n}\right)^r = 2^{2r}\sum_{i=1}^{n}\sin^{2r}\frac{i\pi}{n}$$

$$= \frac{(2r)!n}{r!r!} = \frac{1\cdot 3\cdot 5\cdots(2r-1)2^r n}{r!},$$

if r is a positive integer less than n. (Hobson as above.)

Additional Discussion. If the radius of the polygon is a instead of unity, we must have $K>a^{2r}n$ in order for the locus to be a circle. In order for this circle to be the circumcircle we must supply the factor a^{2r} in the above expression for K.

It is interesting to note also that if $S_1, S_2, \cdots, S_n$ are the sides of a regular polygon circumscribed about a circle of radius a, the locus of P such that

$$\sum_{i=1}^{n}(PS_i)^{2r} = K, \qquad (PS_i \text{ is distance from } P \text{ to side } S_i),$$

is a circle, provided r is a positive integer less than $n/2$ and K is greater than $a^{2r}n$. The locus is the inscribed circle of the polygon if

$$K = \frac{1\cdot 3\cdot 5\cdots(4r-1)na^{2r}}{(2r)!}.$$

We may write the equation of the locus in polar form as

$$\sum_{i=1}^{n}\left[\rho\cos\left(\theta-\frac{2i\pi}{n}\right)-a\right]^{2r}=K,$$

By the same reasoning as in the preceding problem, this is

$$\sum_{s=0}^{r}\frac{(2r)!n}{(2r-2s)!s!s!2^{2s}}a^{2r-2s}\rho^{2s}=K, \qquad 2r<n,$$

and hence is a circle if K is greater than $a^{2r}n$.

If we set $\rho=a$ and $\theta=0$, we have

$$K=a^{2r}\sum_{i=1}^{n}\left[\cos\frac{2i\pi}{n}-1\right]^{2r}=(2a)^{2r}\sum_{i=1}^{n}\sin^{4r}\frac{i\pi}{n}$$

$$=\frac{a^{2r}(4r)!n}{2^{2r}(2r)!(2r)!}=\frac{1\cdot3\cdot5\cdots(4r-1)na^{2r}}{(2r)!},$$

for the circle inscribed in the polygon.

If a regular polygon of n sides $S_1, S_2, \cdots, S_n$ is circumscribed about a circle of radius a and another regular polygon with vertices $A_1, A_2, \cdots, A_n$ is inscribed in the same circle, the locus of a point P such that

$$\sum_{i=1}^{n}(PA_i)^{4r}=(2a)^{2r}\sum_{i=1}^{n}(PS_i)^{2r}$$

is the given circle, provided r is a positive integer less than $n/2$.

It is evident that the loci in the preceding problems are independent of the relative positions of the polygons and hence the expression on either side of this equation is unaltered by rotating either polygon about its center.

From the preceding problems we see that the equation of this locus is

$$\sum_{s=0}^{r}\frac{(2r)!na^{2s}\rho^{2s}}{(2r-2s)!s!s!}\left[(\rho^2+a^2)^{2r-2s}-(2a^2)^{2r-2s}\right]=0.$$

It is evident that $(\rho^2+a^2)-2a^2=\rho^2-a^2$ is a factor of each term of this sum. After this factor is removed from each term the remaining factor of the term contains only even powers of ρ with positive coefficients. Therefore, this equation is not satisfied by any other positive values of ρ^2, and its only real solution is $\rho^2=a^2$.

Solved also by E. F. Allen and C. V. L. Smith.

3775 [1936, 190]. *Proposed by D. T. Sigley, University of Kansas City.*

Prove that the order g of an abstract finite group G which involves $6k$ (k a positive integer) sets of conjugate operators is of the form $6n$.

Solution by the Proposer.

From the theory of group characteristics it is known (see Miller, Blichfeldt,

and Dickson, *Finite Groups*, 1916, Ex. 3, p. 265, and Th. 22, p. 271) that the order g of a group involving s complete sets of conjugate operators satisfies a diophantine relation of the form

$$g_1^2 + g_2^2 + g_3^2 + \cdots + g_s^2 = g,$$

where g_i divides g.

1. To prove that g is even, assume that g is odd. Then every g_i, $(i=1, 2, 3, \cdots, 6k)$ is odd, and hence the sum of $6k$ odd numbers is an odd number, which is a contradiction.

2. To prove that g is divisible by 3 assume the contrary. Each g_i, $(i=1, 2, 3, \cdots, 6k)$, is of the form $3n \pm 1$. But $(3n \pm 1)^2 \equiv 1$, mod 3, from which it results that $6k \equiv g$, mod 3, which is a contradiction. We have accordingly proved that g is of the form $6n$.

Note. The case for $k=1$ was communicated to the proposer by R. D. Carmichael.

3776 [1936, 190]. *Proposed by E. P. Starke, Rutgers University.*

Determine all triangles whose sides are relatively prime integers and such that one angle is double another.

Solution by H. L. Olson, Kalamazoo, Mich.

It will be shown that there are infinitely many triangles such that one angle is twice another and whose sides are integers having unity for their highest common divisor, but that there are no such triangles for which each pair of sides are relatively prime integers.

Let angle $B=2A>0$: then

$$(1) \quad a:b:c = \sin A:\sin 2A:\sin 3A = 1:2\cos A:4\cos^2 A - 1, \quad 1 > \cos A > 1/2.$$

Since $\cos A$ is rational, we set $\cos A = p/q$, where p and q are relatively prime integers $2q > 2p > q$. We now have $a:b:c = q^2:2pq:4p^2-q^2$. The three integers to the right of the equality sign have no common divisor but unity if q is odd. If q is even and prime to p, we set $q=2r$ and

$$(2) \qquad a = r^2, \quad b = pr, \quad c = p^2 - r^2; \quad 2r > p > r > 1, \quad \cos A = p/2r.$$

These are necessary where p and r are relatively prime integers; and from (1) we see that they are sufficient. The two cases of isosceles triangles are excluded; for in one case $A=45°$ and $\cos A$ is irrational, and in the other, $\cos A$ must satisfy

$$4\cos^2 A - 2\cos A - 1 = 0$$

and is also irrational. It then follows that the two triangles given by r, p and r', p' are the same if and only if $r=r'$, $p=p'$. The formulas (2) then give each such triangle without repetition. For a given r there are $\phi(r)$ distinct triangles,

where $\phi(r)$ means the number of integers less than r and prime to r, including unity.

Solved also by H. T. R. Aude, W. F. Cheney, Jr., R. A. Hefner, C. H. Lady, W. V. Parker, A. Pelletier, J. Rosenbaum, S. B. Townes, S. Vatriquant, and the proposer.

Editorial Note. The majority of the solvers, including the proposer, interpreted the problem as meaning that the three integers for the sides have unity for their highest common divisor. Lady showed that there exists no triangle with each pair of sides relatively prime; this fact follows from the above solution since for a and b to be relatively prime r must be unity, and this is shown to be impossible. Also the majority of the solvers derived first the condition $b^2=a(a+c)$. Aude listed the twenty triangles each of whose sides is less than 100. Cheney made the following remarks: There are such triangles which have integral areas, for example, for $r=5$ and $p=6$ or 8. There will exist such Heronian triangles for each odd value of r which is the sum of two relatively prime squares. See *Heronian Triangles* by Cheney in this MONTHLY, 1929, pages 22–28.

3777 [1936, 190]. *Proposed by A. D. Wallace, University of Virginia.*

Show that the total curvature along a characteristic line of a surface is equal to the square of the ratio of the arc length on the spherical image to the arc length on the surface.

Note. Along an asymptotic line the total curvature is minus the square of the ratio.

Solution by M. L. Vest, Davis and Elkins College, Elkins, West Virginia.

The equation of the characteristic lines of a surface, when referred to any lines whatever as parametric, is

$$[D(GD - ED'') - 2D'(FD - ED')]du^2 + 2[D'(GD + ED'') - 2FDD'']dudv + [2D'(GD' - FD'') - D''(GD - ED'')]dv^2 = 0.$$

Let us refer the surface to the lines of curvature as parametric. The necessary and sufficient condition that this be true is that $F=D'=0$. Referred to these lines, the equation of the characteristic lines becomes

$$D'' = D\left(\frac{du}{dv}\right)^2. \tag{1}$$

Making use of relationship (1), we find the total curvature of the surface along a characteristic line to be

$$K = \frac{DD'' - D'^2}{H^2} = \frac{D^2}{EG}\left(\frac{du}{dv}\right)^2.$$

The square of the ratio of the arc length on the spherical image to the arc length on the surface is, when referred to the lines of curvature, given by

$$\left(\frac{d\sigma}{ds}\right)^2 = \frac{\mathcal{E}du^2 + \mathcal{G}dv^2}{Edu^2 + Gdv^2}.$$

For a point of the characteristic line

$$\mathcal{E} = \frac{1}{H^2}\left[GD^2 - 2FDD' + ED'^2\right] = \frac{D^2}{E},$$

$$\mathcal{G} = \frac{1}{H^2}\left[GD'^2 - 2FD'D'' + ED''^2\right] = \frac{D^2}{G}\left(\frac{du}{dv}\right)^4,$$

and the ratio becomes

$$\left(\frac{d\sigma}{ds}\right)^2 = \frac{D^2}{EG}\left(\frac{du}{dv}\right)^2 = K,$$

and the proposition is thereby demonstrated.

Again, the equation defining the asymptotic lines of a surface

$$Ddu^2 + 2D'dudv + D''dv^2 = 0$$

becomes

$$D'' = -D\left(\frac{du}{dv}\right)^2 \tag{2}$$

when referred to the lines of curvature. Making use of this relationship in the same manner as was done above, we find

$$K = -\frac{D^2}{EG}\left(\frac{du}{dv}\right)^2 = -\left(\frac{d\sigma}{ds}\right)^2,$$

proving the note affixed to the above problem.

Solved also by C. E. Springer and the proposer.

Editorial Note. With more detail the proof can be made quite elementary and more comprehensive. Let $\mathbf{r}$ be the vector to a point P on the convex surface S on which the lines of curvature are the parametric curves; let $\mathbf{n}$ be the unit vector normal to S directed from the concave to the convex side; and let $\mathbf{r}_1 = \partial\mathbf{r}/\partial u$, $\mathbf{r}_2 = \partial\mathbf{r}/\partial v$, $\mathbf{n}$ form a right-hand system of vectors. If $d\phi$ is the angular displacement of $\mathbf{n}$ for $v = \text{const.}$, then $\mathbf{n}_1 du = d\phi(\mathbf{r}_1 du/ds) = K_1\mathbf{r}_1 du$, or $\mathbf{n}_1 = K_1\mathbf{r}_1$, where K_1 is the curvature of the normal section through the unit tangent $\mathbf{r}_1 du/ds$. For any direction through P we have

$$\begin{aligned} d\mathbf{r} &= \mathbf{r}_1 du + \mathbf{r}_2 dv = \mathbf{t}_1 x + \mathbf{t}_2 y, \qquad x = \sqrt{\mathbf{r}_1^2}\,du, \qquad y = \sqrt{\mathbf{r}_2^2}\,dv, \\ d\mathbf{n} &= \mathbf{n}_1 du + \mathbf{n}_2 dv = \mathbf{t}_1 K_1 x + \mathbf{t}_2 K_2 y, \end{aligned} \tag{1}$$

where $\mathbf{t}_1$, $\mathbf{t}_2$ are orthogonal unit vectors. Since the slopes of $d\mathbf{r}$ and $d\mathbf{n}$ are y/x and K_2y/K_1x, we have

$$\tan\theta = \frac{K_2 - K_1}{K_1\dfrac{x}{y} + K_2\dfrac{y}{x}}, \quad K_1 > 0, \quad K_2 > 0, \quad K_1 \neq K_2, \tag{2}$$

where θ is one of the angles between $d\mathbf{r}$ and $d\mathbf{n}$. The two terms added in the denominator have the same sign and we may take them as positive, and their product is the constant for P, $K_1K_2=K$, the Gaussian curvature of S at P. Hence the denominator is a minimum when the two terms are equal, as we know from the inequality between the arithmetic and geometric means. A curve on S such that at each of its points the acute angle between its tangent and $d\mathbf{n}$ is a maximum is called a characteristic curve. It satisfies the equation $(y/x)^2 = K_1/K_2$, or the differential equation

$$K_1\mathbf{r}_1^2du^2 - K_2\mathbf{r}_2^2dv^2 = 0. \tag{3}$$

After multiplying this equation by K_1-K_2, we have by use of (1)

$$(d\mathbf{n})^2 - K(d\mathbf{r})^2 = 0. \tag{4}$$

This proves the theorem of the problem after the obvious insertions in (4). Moreover, this equation applies for any parameters whatever, and hence it is the equivalent of the complicated equation at the beginning of the above solution. If the surface has a negative K, that is, if K_1 and K_2 have opposite signs, then it is possible for $d\mathbf{n}$ to be perpendicular to $d\mathbf{r}$, and the curve locus of such points is called an asymptotic line. In this case we easily find for the differential equation of these curves

$$K_1\mathbf{r}_1^2du^2 + K_2\mathbf{r}_2^2dv^2 = 0, \quad \text{or} \quad (d\mathbf{n})^2 + K(d\mathbf{r})^2 = 0,$$

and this gives the theorem in the note to the problem.

We now add some remarks about characteristic curves. If $d\mathbf{r}$ defines a certain tangent direction from P, and if $\delta\mathbf{r}$ is perpendicular to $d\mathbf{n}$, we say that the second direction is conjugate to the first. This relation is reciprocal, for it is easily shown that $\delta\mathbf{r}\cdot d\mathbf{n} = d\mathbf{r}\cdot\delta\mathbf{n}$. It is then clear that we may say that characteristic curves are such that at each point the acute angle between the tangent to the curve and its conjugate direction is a minimum. We readily find from (1) the expression

$$\delta\mathbf{r} = K_2y\mathbf{t}_1 - K_1x\mathbf{t}_2. \tag{5}$$

Denote by k_1 the normal curvature of S in the direction $d\mathbf{r}$ and by k_2 that for $\delta\mathbf{r}$. Then we have

$$k_1 = \frac{d\mathbf{r}\cdot d\mathbf{n}}{d\mathbf{r}\cdot d\mathbf{r}} = \frac{K_1x^2 + K_2y^2}{x^2 + y^2},$$

$$k_2 = K\frac{K_1x^2 + K_2y^2}{K_1^2x^2 + K_2^2y^2}, \qquad K = K_1K_2, \tag{6}$$

where k_2 is obtained from k_1 by replacing x, y by $K_2y, -K_1x$. We easily find that the curves for which at each point $k_1 = k_2$ satisfy (4), and they are therefore the characteristic curves. We next consider the curves for which k_1k_2 is an extreme. After a slight reduction we find that

$$\frac{K}{k_1k_2} = 1 + \frac{(K_1 - K_2)^2}{\left(K_1\frac{x}{y} + K_2\frac{y}{x}\right)^2}.$$

If $x=0$ or $y=0$, the maximum of k_1k_2 is K, which is given by the principal directions; the curves are the lines of curvature. Using the argument of the arithmetic and geometric mean we see that the minimum for k_1k_2 is given by the characteristic curves. The values in this case are

$$k_1 = k_2 = 2K/J, \qquad J = K_1 + K_2.$$

At each point of S there are two characteristic curves, and it is obvious from the above that their directions are conjugate. Moreover, from the x, y equations which gave (3) it is also obvious that their directions are bisected by the principal directions. If any two directions from P are equally inclined to a principal direction the expression for k_1 in (6) shows at once that the normal curvatures in these directions are equal. But there is only one pair of such directions which are also conjugate as has been shown.

If it is desired to derive the complicated equation for the characteristic curves, we may use the following:

$$(d\mathbf{r})^2 = ds^2 = Edu^2 + 2Fdudv + Gdv^2,$$

$$-\mathbf{r}_1\cdot\mathbf{n}_1 = D, \qquad -\mathbf{r}_1\cdot\mathbf{n}_2 = -\mathbf{r}_2\cdot\mathbf{n}_1 = D', \qquad -\mathbf{r}_2\cdot\mathbf{n}_2 = D'',$$

$$\begin{vmatrix} \mathbf{r}_1 & \mathbf{r}_2 & -\mathbf{n}_1 & -\mathbf{n}_2 \\ E & F & D & D' \\ F & G & D' & D'' \end{vmatrix} = 0,$$

where we obtain a determinant for $\mathbf{n}_1$ by suppressing the last column of the matrix, while for $\mathbf{n}_2$ we suppress the third column.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

The Mathematics Division of the Society for the Promotion of Engineering Education is planning a series of conferences, part of which will be joint conferences with the Division of Physics. These conferences will be held in connection with the annual meeting of the Society at College Station, Texas, June 27–30, 1938. Professor H. B. Phillips of the Massachusetts Institute of Technology is planning the program.

During recent years the Department of Mathematics of the University of Chicago has placed emphasis on a particular field each Summer Quarter. Analysis was emphasized in the summers of 1935 and 1937, and algebra will be featured by a special seminar in 1938. There will also be a Conference on Algebra beginning at 9:30 A.M. on Tuesday, June 28 and ending on July 1. The program will include the following addresses: L. E. Dickson (Chicago), On the history and nature of the integral quantities of algebras; R. Hull (Illinois), The arithmetic of rational cyclic algebras; A. A. Albert (Chicago), On cyclic algebras; R. Brauer (Toronto), On the structure of normal division algebras; S. MacLane (Chicago), The structure of perfect fields; O. Zariski (Johns Hopkins), Ideal theory and algebraic geometry; S. Lefschetz (Princeton), Formal power series in algebraic geometry; C. C. MacDuffee (Wisconsin and the Institute for Advanced Study), The use of matrices in ideal theory; C. G. Latimer (Kentucky), Ideals in a quaternion ring and Hermitian forms; E. Artin (Notre Dame), Theory of quadratic forms; R. Baer (North Carolina), Group structure and subgroup lattice; J. Williamson (Johns Hopkins), The conjunctive equivalence of pencils of Hermitian and anti-Hermitian matrices; M. H. Ingraham (Wisconsin), Topics in the theory of matrices whose elements belong to a division algebra; O. Schilling (Johns Hopkins), Local class field theory; N. Jacobson (North Carolina), Simple Lie algebras. Housing accommodations near the University have been arranged at a moderate charge for those attending the Conference. Those desiring to reserve accommodations may do so by writing to A. A. Albert, Eckhart Hall, The University of Chicago.

The following appointments to instructorships have been announced:
University of Alabama: L. D. Rodabaugh
George Washington University: Dr. W. C. Mitchell
Montana State College: Dr. Nathan Schwid
University of Pittsburgh: J. O. Blumberg
University of Texas: E. L. Godfrey, Miss H. C. Miller, R. H. Sorgenfrey, R. L. Swain.

Professor H. W. Stager of the Salinas Junior College, died at Palo Alto on December 20, 1937, at the age of fifty-eight, having retired from teaching in 1931. His best known research was embodied in *A Sylow Factor Table* published by the Carnegie Institution in 1916.

ON CONVEXITY*

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1. Introduction. At least two different approaches to an intuitional notion of convexity are common in elementary texts. A convex closed curve is described as one which can be cut by a straight line in only two points. But a convex polygon is described as one no extended side of which will pass through the interior of the polygon. A pupil or teacher, seeking a *logical* justification for the use of the same adjective *convex* in the two instances, might have some difficulty. Justification lies of course in the fact that curve and polygon alike bound convex sets of points. It happens that one characteristic property of such sets suggests the description cited for the convex curve, while a different property suggests the description of the convex polygon.

In this paper attention will be called to certain properties, relative to point sets, which seem to be inherent in any intuitional notion of convexity, and consideration given to the precise interrelationships of these properties. For simplicity, the discussion will be restricted to point sets in a plane; but it may be said in advance that it all applies, with only the most obvious alterations, to point sets in euclidean space of n dimensions. In fact some parts are extensible to more general spaces, as will be indicated near the end of the paper.

Critical study of convex sets seems to have originated with Minkowski [1].† A simplified and elegant treatment of some of Minkowski's ideas was given by Carathéodory [2]. By far the most comprehensive treatment is that by Bonnesen and Fenchel [3], in which is to be found an extensive bibliography to 1934.

2. Five properties which point sets may possess. A point set *may* be

1. Bounded
2. Closed
3. Possessed of inner points
4. Linearly connected
5. Completely supported at its boundary points.

A set is *bounded* if there exists a definite square within which all of its points lie. In analytic terms, there exists a positive constant M such that for each point (x, y) of the set, $|x| < M$ and $|y| < M$.

A set is *closed* if it contains all of its limit points. That is, if $\{(x_n, y_n)\}$ is a sequence of points of the set, and if $\lim_{n\to\infty}(x_n, y_n) = (\bar{x}, \bar{y})$, then $(\bar{x}, \bar{y})$ belongs to the set.

Inner points, the possession of which constitutes property 3, comprise one of three mutually exclusive categories into which any well defined point set divides the points in its plane. Relative to a given set a point will be called: an *exterior* point if it does not belong to the set, a *boundary* point if it belongs

* This paper includes the substance of an address delivered by invitation before the Mathematical Association of America, Dec. 31, 1937, at Indianapolis. Presented for the Slaught Memorial Volume of the MONTHLY.

† Numbers in brackets refer to references at the end of the paper.

to the set but is a limit point of exterior points, an *inner* point if it belongs to the set and is not a boundary point.*

A set is *linearly connected* if membership of points A and B in the set implies membership of all points on the line segment connecting A and B.†

A definition of the fifth property must be preceded by some preliminary discussion regarding the relation of a point set to the lines in its plane. Any line divides a plane in which it lies into two half-planes which may be designated as *closed* or *open* according as they are thought of as including or excluding the points of the line itself. Analytically, the line

$$l(x, y) \equiv ax + by + c = 0$$

determines two closed half-planes, one consisting of those points (x, y) for which $l(x, y) \geqq 0$, the other of those for which $l(x, y) \leqq 0$. The corresponding open half-planes are represented if the equality signs be omitted.

Any point set $\mathfrak{M}$ serves to classify all lines in its plane into three categories with associated names as follows. Relative to the point set $\mathfrak{M}$, a line l is called: (1) a *bounding* line if all points of $\mathfrak{M}$ lie in one of the open half-planes determined by l, (2) a *separating* line if points of $\mathfrak{M}$ lie in each of the open half-planes, or (3) a *supporting* line if all points of $\mathfrak{M}$ lie in one of the closed half-planes and at least one point of $\mathfrak{M}$ lies on l.‡

A set of points is *completely supported at its boundary* if through each of its boundary points there can be passed a supporting line relative to the set.

3. Fundamental theorem on convexity. The theorem to be considered in this section justifies the two different intuitional approaches to the notion of convexity mentioned in our introduction. It was first proved by Minkowski, but his proof was scattered over many pages. Other proofs have since been given by various authors, the one here presented being due essentially to Carathéodory.

THEOREM I. *For a set of points which is bounded and closed and possesses inner points, the two properties linearly connected and completely supported at the boundary are logically equivalent.*

Let $\mathfrak{M}$ be a set of points which is bounded and closed and possesses inner points.

We will first assume that it is linearly connected, and prove that it is com-

* It is to be noted of course that, for a particular point set, one or more of these categories may be vacant. For example the set of points on a line segment has no inner points (in the sense defined), and hence does not possess property 3.

† This property is usually taken alone as characteristic of a convex set. Thus the set consisting of *all* points in a plane would be said to be convex. There would seem to be at least an intuitional advantage in the more directly descriptive term *linearly connected*.

‡ For example, a side of an elementary convex polygon is a segment of a supporting line relative to the set of points within and on the polygon. Likewise, a tangent to a circle is a supporting line relative to the set of points within and on the circle.

pletely supported at the boundary by establishing the following two propositions:

(A) Through each exterior point there passes a bounding line of $\mathfrak{M}$.

(B) From a sequence of bounding lines passing respectively through a sequence of exterior points which has a boundary point B of $\mathfrak{M}$ as limit point, there can be chosen a subsequence of lines, the limiting (in a sense to be defined) line of which is a supporting line of $\mathfrak{M}$ through B.

To prove (A), let E be any exterior point relative to $\mathfrak{M}$. Then, since $\mathfrak{M}$ is bounded and closed, and since the distance from E to a variable point (x, y) of $\mathfrak{M}$ is a continuous function of x and y, this distance has, by a well known theorem of Weierstrass, a definite minimum which is attained at some point M of $\mathfrak{M}$. The line through E, perpendicular to the segment EM is then a bounding line for $\mathfrak{M}$. For otherwise it would either contain or separate points of $\mathfrak{M}$; indeed it would certainly *contain* a point of $\mathfrak{M}$ since $\mathfrak{M}$ is linearly connected. But if it contained a point Q of $\mathfrak{M}$, then all points of the segment MQ would belong to $\mathfrak{M}$, and consideration of the right triangle MEQ shows that M would not be the nearest point of $\mathfrak{M}$ to E.

To prove (B), let the boundary point $B=(\bar{x}, \bar{y})$ of $\mathfrak{M}$ be the limit of a sequence of exterior points $\{E_n\}=\{(x_n, y_n)\}$, and let

$$a_n x + b_n y + c_n = 0 \tag{1_n}$$

be the equation of a bounding line through E_n. We can and will assume that these equations are in normal form, so that $a_n^2+b_n^2=1$ and $|c_n|$ is the distance of the line (1_n) from the origin. Since the sequence of number triples $\{(a_n, b_n, c_n)\}$ is bounded, there exists a number triple (a, b, c) which is the limit of some infinite subsequence

$$\{(a_\nu, b_\nu, c_\nu)\}, \qquad (\nu = n_1, n_2, n_3, \cdots), \tag{2}$$

of $\{(a_n, b_n, c_n)\}$. The triple (a, b, c) serves to determine a line

$$ax + by + c = 0 \tag{3}$$

which is a supporting line of $\mathfrak{M}$ through B.

If the last statement is not obvious, it may be verified analytically by consideration of the function

$$f(u, v, w, x, y) \equiv ux + vy + w$$

of the five variables u, v, w, x, y. Since f is continuous, the limiting value of $f(u, v, w, x, y)$, as $(u, v, w, x, y)\to(a, b, c, \bar{x}, \bar{y})$, is $f(a, b, c, \bar{x}, \bar{y})$. But if the approach is made over the sequence of points $\{(a_\nu, b_\nu, c_\nu, x_\nu, y_\nu)\}$, the value of the function is always zero, since the point $E_\nu=(x_\nu, y_\nu)$ is on the line (1_ν). Therefore $f(a, b, c, \bar{x}, \bar{y})=0$; that is, the line (3) passes through B.

Furthermore the line (3) does not separate points of $\mathfrak{M}$. For if it did there would be points (x', y') and (x'', y'') of $\mathfrak{M}$, such that $f(a, b, c, x', y')$ and $f(a, b, c, x'', y'')$ would have different signs; hence from the continuity of f and the property of the sequence (2), there would be a value of ν for which

$f(a_\nu, b_\nu, c_\nu, x', y')$ and $f(a_\nu, b_\nu, c_\nu, x'', y'')$ would have different signs. But this would imply that the line (1_ν) was a separating line of $\mathfrak{M}$ and so contradict our assumption regarding it. Hence the line (3) is indeed a supporting line of $\mathfrak{M}$ through B.

Conversely, if $\mathfrak{M}$ is completely supported at the boundary it is linearly connected, the proof being relatively simple.

To prove it indirectly, we suppose M and N are points of $\mathfrak{M}$ while a point E on the segment MN does not belong to $\mathfrak{M}$. We join E to any inner point I of $\mathfrak{M}$ not on the line MN. Since E is an exterior point and I an inner point, there must be on the segment EI a boundary point of $\mathfrak{M}$; call it B. By hypothesis there passes through B a supporting line s of $\mathfrak{M}$. And by definition all points of $\mathfrak{M}$ must lie in the same s-determined half-plane, which half-plane must also contain E since it is on the segment MN. But this is impossible, since s certainly separates I and E. The contradiction proves that E must belong to $\mathfrak{M}$, that is, $\mathfrak{M}$ is linearly connected.

It is to point sets which are bounded, closed, possess inner points, and have either (and hence both) of the properties linearly connected and completely supported at the boundary, that the term *convex* can be applied without doing violence to elementary intuitional notions. The boundary points of such a set constitute a convex curve (possibly a polygon). The analog of such a set in space of three or more dimensions is termed a *convex body* and its boundary points constitute a *convex surface.*

4. Two stronger theorems. While Theorem I is pleasing because of its symmetry and its adequacy in the case of intuitionally convex sets, it is not the strongest theorem which can be stated relative to the properties in question. On considering the two halves of the theorem separately and critically, we find that the hypotheses can be weakened without affecting the conclusions. Thus we obtain the two theorems:

THEOREM II. *If a point set is linearly connected it is completely supported at its boundary points.*

THEOREM III. *If a point set is closed and possesses inner points and is com-completely supported at its boundary points, it is linearly connected.*

With reference to Theorem II, we note first that if the point set in question possesses no boundary points it is vacuously true, and if the set consists of a single point it is trivially true. Suppose then that the linearly connected set $\mathfrak{M}$ has a boundary point B and at least one other point M. We denote by $\mathfrak{M}_0$ the subset consisting of those points of $\mathfrak{M}$ which lie within or on the circle with center B and radius BM, and by $\overline{\mathfrak{M}}_0$ the closure of $\mathfrak{M}_0$, that is, the set $\mathfrak{M}_0$ augmented by all of its limit points. Then the set $\overline{\mathfrak{M}}_0$ satisfies the hypothesis of Theorem I except possibly for the possession of inner points, and no use was made of this latter property in the proof of the first half of Theorem I. Furthermore B is a boundary point of $\overline{\mathfrak{M}}_0$, and so we may conclude, as in Theorem I,

that through B there passes a supporting line s of $\overline{\mathfrak{M}}_0$. But then s must also be a supporting line of the original set $\mathfrak{M}$. For if there were points Q and R of $\mathfrak{M}$, one in each of the open half-planes determined by s, all points of the segment BQ would lie in one of the half-planes and those of the segment BR in the other; so the line s would certainly separate points of $\overline{\mathfrak{M}}_0$, contradicting its character as a supporting line of that set.

Theorem III is merely the second half of Theorem I with the assumption of boundedness omitted from the hypothesis. Since no use was made of that property in the proof of that half, the proof is adequate for Theorem III.

5. Complete existential theory of properties 1–5. The five properties listed in §2 are clearly not independent. It is perhaps natural to ask if Theorems II and III express the only implicational relationships between them. The question can be answered only after an exploration of all possibilities. Such an exploration relative to a set of properties leads to what E. H. Moore called their *complete existential theory*. In the case of our five properties the question is the following. Of the $2^5=32$ composite properties representable by the notations

$$(+\,+\,+\,+\,+), (-\,+\,+\,+\,+), \cdots, (-\,-\,-\,-\,+), (-\,-\,-\,-\,-),$$

the ith symbol within parentheses being $+$ or $-$ according as property i is present or absent in the composite property, which of them can be possessed by existent (non-null) point sets?

Theorem II eliminates the possibility of eight combinations characterized by $(\cdots+-)$, and Theorem III eliminates the possibility of two characterized by $(\cdot++-+)$. The remaining twenty-two composite properties are possessed by actually existent point sets, as is shown by the examples listed below. Hence Theorems II and III express the only significant implicational relationships between the properties 1–5. For convenience in writing, the sets are described analytically by conditions upon the coördinates (x, y) of points constituting them. The examples are all simple and can be easily visualized or represented graphically and their characters verified.*

Character	Point set
1. $(+\,+\,+\,+\,+)$	$x^2+y^2\leqq 1$
2. $(-\,+\,+\,+\,+)$	(x, y) unrestricted
3. $(+\,-\,+\,+\,+)$	$x^2+y^2<1$
4. $(+\,+\,-\,+\,+)$	$0\leqq x\leqq 1,\ y=0$
5. $(-\,-\,+\,+\,+)$	$x>0,\ y>0$
6. $(-\,+\,-\,+\,+)$	$x\geqq 0,\ y=0$
7. $(+\,-\,-\,+\,+)$	$0<x<1,\ y=0$
8. $(+\,-\,+\,-\,+)$	$x^2+y^2<1$; and $(2, 0)$
9. $(+\,+\,-\,-\,+)$	$(0, 0)$; and $(1, 0)$
10. $(+\,+\,+\,-\,-)$	$1\leqq x^2+y^2\leqq 2$

* However it may be more amusing and profitable for the reader (with time and interest) to construct his own examples. The search for examples emphasizes the rôles played by the individual properties in our intuitional notion of convexity.

11. $(++---)$ $x^2+y^2=1$; and $(0, 0)$
12. $(+-+--)$ $x^2+y^2\leqq 1$; and $1\leqq x<2$, $y=0$
13. $(+----+)$ $(0, 0)$; and $1<x\leqq 2$, $y=0$
14. $(-++--)$ $x^2+y^2\leqq 1$; and $x\geqq 1$, $y=0$
15. $(-+--+)$ $(0, 0)$; and $x\geqq 1$, $y=0$
16. $(--+-+)$ $(0, 0)$; and $x>1$, y unrestricted
17. $(---++)$ $x>0$, $y=0$
18. $(+----)$ $x^2+y^2=1$; and $1\leqq x<2$, $y=0$; and $2<x\leqq 3$, $y=0$
19. $(-+---)$ $x^2+y^2=1$; and $x\geqq 1$, $y=0$
20. $(--+--)$ $x^2+y^2\leqq 1$; and $1\leqq x<2$, $y=0$; and $x>2$, $y=0$
21. $(----+)$ $(0, 0)$; and $x>1$, $y=0$
22. $(-----)$ $x^2+y^2=1$; and $1\leqq x<2$, $y=0$; and $x>2$, $y=0$.

6. The extension of a set to attain a specified property. Of the five properties listed in §2, properties 2 and 4 are *closure* properties in the sense that they assure, in a set which possesses them, the inclusion of certain subsets of points determined by points of the set and by the property in question. Relative to such a property it is pertinent to ask whether a set which does not have it can be *extended*, by the adjunction of points, so that the extended set will have the property. And if so, what is the least extensive set, containing the given set, which will have the desired property?

Obviously the *least extensive* set containing a set $\mathfrak{M}$ and having a property P must be contained in every such set. It could therefore be nothing other than the greatest common subset of all such sets. If this greatest common subset *has** the property P, it is the desired extended set. It may suitably be called the *P-extension of* $\mathfrak{M}$ and denoted by $\mathfrak{M}_P$. Thus $\mathfrak{M}_2$, the closed extension of $\mathfrak{M}$, is the greatest common subset of all the closed sets containing $\mathfrak{M}$. It is generally known as the closure of $\mathfrak{M}$. And $\mathfrak{M}_4$, the linearly connected extension of $\mathfrak{M}$, is the greatest common subset of all the linearly connected sets containing $\mathfrak{M}$. The composite property, closed and linearly connected, leads to the extension $\mathfrak{M}_{24}$, the greatest common subset of all the closed and linearly connected sets containing $\mathfrak{M}$.

The extensions $\mathfrak{M}_4$ and $\mathfrak{M}_{24}$ have been of use in many connections. A few remarks as to the terminology usually associated with them may be in order. Earlier writers, Carathéodory and F. Riesz for example, merely referred to them as the smallest convex (or smallest closed convex) set containing the given set. Later writers have almost universally adopted the term "convex hull" (konvexe Hülle). Brevity and suggestive terminology are of course appreciated by everyone. But terminology which suggests the *wrong* idea is surely undesirable. The term "hull" probably suggests to most English-speaking persons an *outside covering*, hence it might most appropriately be applied to the aggregate of *boundary* points of a convex set, an aggregate quite different from that

* It should be noted that the greatest common subset of all sets containing $\mathfrak{M}$ and having a given property P does not necessarily have the property P. As an illustration, one may take for $\mathfrak{M}$ the set of points on a line segment and for P the property of possessing inner points.

denoted by $\mathfrak{M}_4$*.

From Theorem II it follows that $\mathfrak{M}_4$ is completely supported at its boundary points. And it is easily seen that if $\mathfrak{M}$ is bounded, closed, or possesses inner points, the same respective properties persist in $\mathfrak{M}_4$.

If all points of $\mathfrak{M}$ lie on a single line, then $\mathfrak{M}_4$ lies on that line; and if in addition $\mathfrak{M}$ is bounded and closed, $\mathfrak{M}_4$ is a closed line segment.

If $\mathfrak{M}$ consists of a finite number of points not on a line, then $\mathfrak{M}_4$ is the set of points within and on the largest polygon having points of $\mathfrak{M}$ for vertices.

7. Two characteristic properties of the linearly connected extension of a bounded and closed set.

THEOREM IV. *If $\mathfrak{M}$ is bounded and closed, $\mathfrak{M}_4$ consists precisely of those points through which pass no bounding lines of $\mathfrak{M}$.*

First, if a point A belongs to $\mathfrak{M}_4$ it cannot be on any bounding line l of $\mathfrak{M}$. For if B is a point of $\mathfrak{M}$ at minimum distance from the bounding line l, the line s through B parallel to l is a supporting line of $\mathfrak{M}$. Since all points of $\mathfrak{M}$ lie in that one of the s-determined closed half planes which does not contain l, the same must be true of $\mathfrak{M}_4$, from its definition. Hence A cannot lie on l.

Conversely, if A does not belong to $\mathfrak{M}_4$, there passes through it a bounding line of $\mathfrak{M}_4$ which can be determined by the method indicated in the proof of Theorem I. And a bounding line of $\mathfrak{M}_4$ is *a fortiori* a bounding line of $\mathfrak{M}$.

A neat and useful analytic expression can be given to the content of Theorem IV. The linear equation

$$u(x - x_0) + v(y - y_0) = 0$$

with u and v arbitrary, represents the family of lines through the point (x_0, y_0). A line of this family will be a bounding line of a given set of points $\mathfrak{M}$, if and only if for some number pair (u, v) the inequality

$$u(x - x_0) + v(y - y_0) > 0 \tag{5}$$

is satisfied for all points (x, y) of $\mathfrak{M}$. Hence if $\mathfrak{M}$ is bounded and closed, a necessary and sufficient condition that (x_0, y_0) belong to $\mathfrak{M}_4$ is that the system of linear inequalities represented by (5) when (x, y) varies over the set $\mathfrak{M}$ shall admit no simultaneous solution (u, v). This general idea has been used in the study of linear inequalities by Kakeya, Haar, Fujiwara, Stokes, Motzkin, Dines, and McCoy.

THEOREM V. *If $\mathfrak{M}$ is bounded and closed, $\mathfrak{M}_4$ consists precisely of those points each of which can be the center of mass of a distribution of positive masses (of total mass unity)† at suitable points of $\mathfrak{M}$.*

* In a recent paper (Bulletin of the American Mathematical Society, 1936, p. 353), the present writer used the term "convex extension" instead of convex hull. Many persons with whom he has discussed the matter agree that the latter term is inappropriate, but likewise agree that it is firmly intrenched in the mathematical vocabulary.

† The parenthetic phrase has no particular significance here. The possibility of taking the total mass unity is useful in the analytic statement of the theorem (Corollary II, below).

For the proof we need the following lemma, which is however of some importance in itself.

LEMMA. *If $\mathfrak{M}$ is bounded and closed, and s is a supporting line of $\mathfrak{M}_4$, then the set of points in which s intersects $\mathfrak{M}_4$ is precisely the linearly connected extension of the set in which s intersects $\mathfrak{M}$.*

If we denote the set of all points on s by $\mathfrak{S}$, those common to $\mathfrak{S}$ and $\mathfrak{M}$ by $\mathfrak{S}\mathfrak{M}$, and those common to $\mathfrak{S}$ and $\mathfrak{M}_4$ by $\mathfrak{S}\mathfrak{M}_4$, the assertion of the lemma may be written

$$\mathfrak{S}\mathfrak{M}_4 \equiv (\mathfrak{S}\mathfrak{M})_4. \tag{6}$$

That the left side of (6) includes the right follows immediately. For since $\mathfrak{M}_4$ includes $\mathfrak{M}$, $\mathfrak{S}\mathfrak{M}_4$ includes $\mathfrak{S}\mathfrak{M}$. And since $\mathfrak{S}\mathfrak{M}_4$ is linearly connected, it must contain $(\mathfrak{S}\mathfrak{M})_4$ by the definition of the latter set.

To prove that conversely the left side of (6) is included in the right we proceed indirectly, assuming that a point P belongs to $\mathfrak{S}\mathfrak{M}_4$ but not to $(\mathfrak{S}\mathfrak{M})_4$, and obtaining therefrom a contradiction. To facilitate the discussion we introduce a system of coördinate axes with the origin (0, 0) at P and the x-axis along the supporting line s and so directed that for all points of $\mathfrak{M}_4$, and hence of $\mathfrak{M}$, $y \geqq 0$. At least one point of $\mathfrak{M}$ is on the x-axis; for otherwise s would be a bounding line of $\mathfrak{M}$, and (by Theorem IV) P could not belong to $\mathfrak{M}_4$. However, since $(\mathfrak{S}\mathfrak{M})_4$ is linearly connected and is assumed not to contain P, all points of $\mathfrak{S}\mathfrak{M}$ must lie on one side of P; we will suppose for definiteness that, for all points of $\mathfrak{S}\mathfrak{M}$, $x < 0$.

But for some points of $\mathfrak{M}$, $x \geqq 0$; otherwise the y-axis, through P, would be a bounding line of $\mathfrak{M}$. Let us denote the bounded and closed subset of $\mathfrak{M}$ in the first quadrant by $\mathfrak{M}_I$, the minimum distance of $\mathfrak{M}_I$ from the x-axis by h,* and the maximum distance of $\mathfrak{M}_I$ from the y-axis by k. Then the line through $P(0, 0)$ with slope $h/2k$ is certainly a bounding line of $\mathfrak{M}$. Hence (by Theorem IV) P cannot belong to $\mathfrak{M}_4$. *A fortiori* P cannot belong to $\mathfrak{S}\mathfrak{M}_4$ as we assumed, and the lemma is established.

Now we are in a position to prove Theorem V. And we first show that if P belongs to $\mathfrak{M}_4$, it is the center of mass of a suitable distribution of masses, of total mass unity, at points of $\mathfrak{M}$. In the simplest case where $\mathfrak{M}$ is on a single line, say on the x-axis, $\mathfrak{M}_4$ consists, as has been noted, of the points

$$x = ta + (1 - t)b, \qquad (0 \leqq t \leqq 1),$$

of a line segment joining the extreme points $(a, 0)$ and $(b, 0)$ of $\mathfrak{M}$. If P is a point of this segment with abscissa

$$\bar{x} = \bar{t}a + (1 - \bar{t})b,$$

then P is the center of mass of a distribution $\bar{t}$ at $(a, 0)$ and $(1-\bar{t})$ at $(b, 0)$.

* Certainly *positive* in view of our supposition that for all points of $\mathfrak{S}\mathfrak{M}$, $x < 0$.

If $\mathfrak{M}$ does not lie entirely on one line and P is any point of $\mathfrak{M}_4$, we may join any point M of $\mathfrak{M}$ to P and extend (if necessary) the segment MP to intersect the boundary of $\mathfrak{M}_4$ at a point B. If B is a point of $\mathfrak{M}$ we may conclude immediately that P is the center of mass of suitable masses at B and M. If B is not a point of $\mathfrak{M}$ we make use of the supporting line s of $\mathfrak{M}_4$ through B. By the lemma, B belongs to $(\mathfrak{S}\mathfrak{M})_4$ which must be a line segment of s terminated by points Q and R of $\mathfrak{M}$. Hence B is the center of mass of a suitable distribution of masses at Q and R. These masses may be μ_2 and μ_3, such that $\mu_2+\mu_3=1$, or they may be any two masses proportional to μ_2 and μ_3. If the proportionality factor be chosen properly, a complementary mass may be placed at M so that the sum of the three masses is unity and the center of mass is P. The coördinates $(\bar{x}, \bar{y})$ of P may be expressed explicitly in terms of the coördinates of $M(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ as follows. The coördinates of B (on the segment QR) may be expressed in the form $(\mu_2x_2+\mu_3x_3, \mu_2y_2+\mu_3y_3)$; and hence those of P (on the segment MB) may be expressed in the form

$$\bar{x} = \bar{t}x_1 + (1-\bar{t})(\mu_2x_2 + \mu_3x_3), \qquad \bar{y} = \bar{t}y_1 + (1-\bar{t})(\mu_2y_2 + \mu_3y_3)$$

for some value of $\bar{t}$ between 0 and 1. So that, since $\mu_2+\mu_3=1$, the coördinates $(\bar{x}, \bar{y})$ of P may be written

$$\bar{x} = m_1x_1 + m_2x_2 + m_3x_3, \qquad \bar{y} = m_1y_1 + m_2y_2 + m_3y_3,$$

where m_1, m_2, and m_3 are positive and $m_1+m_2+m_3=1$.

Conversely, if P is the center of mass of positive mass distribution at points of $\mathfrak{M}$, then P belongs to $\mathfrak{M}_4$. For otherwise there would exist through P (by Theorem IV) a bounding line of $\mathfrak{M}$. Since all points of $\mathfrak{M}$ would lie on one side of this line, P could not be the center of mass. Analytically, if P were the origin $(0, 0)$, and the bounding line the y-axis, the zero abscissa of P could not possibly be given by the formula

$$\bar{x} = \sum_{i=1}^{r} m_ix_i \Big/ \sum_{i=1}^{r} m_i, \qquad m_i > 0,$$

or by the analogous integral formula for the center of mass of a continuous mass distribution.

The following two corollaries emerge immediately from the preceding proofs:

COROLLARY I. *If $\mathfrak{M}$ is bounded and closed, $\mathfrak{M}_4$ consists of the points of $\mathfrak{M}$, together with the points of all line segments which join points of $\mathfrak{M}$, and the points of all triangular regions with vertices belonging to $\mathfrak{M}$.*

COROLLARY II. *If $\mathfrak{M}$ is bounded and closed, $\mathfrak{M}_4$ consists of those points whose coördinates can be expressed in the form*

$$x = m_1x_1 + m_2x_2 + \cdots + m_rx_r,$$
$$y = m_1y_1 + m_2y_2 + \cdots + m_ry_r,$$

where (x_1, y_1), (x_2, y_2), $\cdots$, (x_r, y_r) *are points of* $\mathfrak{M}$; $m_1, m_2, \cdots, m_r$ *are non-negative; and* $\sum_{i=1}^{r} m_i = 1$. *Furthermore, if such expression is possible for* (x, y), *it is possible with* $r \leqq 3$.

The analytic criteria in Corollary II (and in its generalization to n dimensions) have been useful in many applications. The most obvious is of course the study of non-negative solutions of systems of linear equations.

8. Generalizations. It has already been remarked that the theory presented in the preceding sections generalizes immediately to n-dimensional euclidean space. The two-dimensional space of points $P(x, y)$ is replaced by the space of points $P(x_1, x_2, \cdots, x_n)$, the lines $ax+by+c=0$ by hyperplanes $a_1x_1+a_2x_2 + \cdots +a_nx_n+c=0$, the half-planes by half-spaces, and so on.

It is natural to inquire whether further generalization is possible. In present day mathematics, the term *space* has become almost synonymous with a set of elements of any sort, and adjectives are prefixed to indicate the nature of such special properties as the set is assumed to possess. Thus we speak of metric space, linear or vector space, topological space, function space, Hilbert space, Banach space, and so on.

Without attempting any detailed or comprehensive study we may perhaps inquire as to the properties which a space (set of elements) should have in order that a generalization of the elementary notion of convexity may have significance in the space.

Our property 1 (boundedness) immediately suggests that the space should be a metric space, that it should have associated with it a distance function adequate to determine a definite distance between each pair of elements, with the usual properties of distance. If the metric space be complete (that is contain a limit element associated with each convergent sequence of elements), the properties 2 (closure) and 3 (possession of inner points) can be immediately generalized. Direct generalization of property 4 (linearly connectedness) would seem to require the notion of line in the space, that is, essentially a linear or vector space. A well known space which furnishes all the desiderata mentioned with a minimum of postulates is the so-called Banach space [**4**, p. 53]. And so it is not surprising to find recent papers [**5, 6, 7, 8**] dealing with convexity in Banach space.

But the most obvious is not the only method of generalizing. The property of linearly connectedness, as we have defined it for sets in euclidean space seems to depend upon the presence of lines in the space. Nevertheless Menger [**9**] has defined, relative to sets in a space which is merely assumed to be metric, a property which he calls convexity, and which upon specialization to euclidean space turns out to be entirely equivalent to linearly connectedness.

Generalization of property 5 (complete support at the boundary) calls for generalization of the euclidean hyperplane. This can easily be secured by introduction of a generalized inner product between elements of a Banach space, generalizing the ordinary inner product $a \cdot x \equiv a_1x_1+a_2x_2+ \cdots +a_nx_n$ of two

euclidean vectors. But again the obvious way is not the only way. For Banach, on the last page of his book, sketches very briefly a definition of hyperplane without the use of an inner product, and states a theorem which is at least a partial generalization of Theorem I of the present paper. He attributes this work to Mazur without giving a specific reference; but it is possibly contained in [10], a paper which the present writer has not been able to see, and which was apparently overlooked by Bonnesen and Fenchel as it does not appear in their extensive bibliography.

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NEW ALGEBRAS OF LOGIC

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1. Introduction. In recent years, various new algebras of logic have been proposed which differ in certain ways from the classical Boolean algebra of logic. Three of the most interesting of these systems are the truth value logics of Lukasiewicz and Tarski [1, 2], Heyting's intuitionist logic [3], and the quantum mechanics logic of Garrett Birkhoff and von Neumann [4]. It must not be expected that these new logics will provide us with any new methods of drawing logical conclusions. On the contrary, they all require that we abandon one or more of the rules of classical logic, such as the law of excluded middle, or of double negation or the distributive law. However, these new logics do not agree with each other as to which particular Boolean rules are to be discarded. They are all interesting as purely abstract algebraic systems, apart from any logical interpretation. Since they are new, their algebraic properties have not yet been completely determined. It should be noted that they are algebras of elementary propositions, and are not adequate to represent all types of mathematical reasoning. For example, they contain no symbols meaning "for all x" or "there exists an x such that."

2. What is an algebra of logic? Just what is meant by an "algebra of logic"? This question may be answered by giving a description of Boolean algebra. Like the ordinary algebra of the real number system, a Boolean algebra consists of a set of elements, symbolized by the letters a, b, c, and so on, together with two operations or rules of combination called *addition* and *multiplication*, which assign to each pair of elements a and b of the algebra two further elements $a+b$ and ab called the *sum* and *product* of a and b. There is also an operation called *negation*, which assigns to each element a an element a' called the *negative* of a. These operations satisfy certain postulates given below, which are in part the same as postulates for ordinary algebra. Different algebras of logic, of course, will have different postulate systems.

The interpretation of the Boolean elements and operations, however, is entirely different from that of ordinary algebra. The elements a, b, c, and so on, are to be thought of as representing not numbers, but propositions or statements or sentences. The sum $a+b$ is to be thought of as standing for the proposition that either a or b is true, and the product ab is interpreted to mean the proposition that both a and b are true. The negation a' stands for the proposition that a is false. An algebra of logic may also have other operations, but in Boolean algebra all operations are definable in terms of addition and negation. All the algebras here discussed have the two special elements 0, standing for a proposition which is certainly false, and 1, representing a proposition which is certainly true.

3. The rules of Boolean algebra. We come now to the rules which the Boolean operations are assumed to obey. The notation $a=b$ used in stating these rules can be interpreted to mean that a and b are the same element, or it can be

thought of as standing for an abstract equivalence relation meaning that each of the propositions a and b is a consequence of the other. Some of the Boolean rules are:

$$a + b = b + a \qquad ab = ba$$
$$(a + b) + c = a + (b + c) \qquad (ab)c = a(bc)$$
$$a + 0 = a \qquad a \cdot 1 = a$$
$$a + 1 = 1 \qquad a \cdot 0 = 0$$

These are the well known commutative and associative laws, and the special properties of the elements 0 and 1. The rule $a+1=1$ is the only one of these which fails in ordinary algebra. The rules above hold in all four systems to be discussed. The following Boolean rules fail in one or more of the new logics:

tautology:	$a + a = a$	$aa = a$
distributive:	$a(b + c) = ab + ac$	$a + bc = (a + b)(a + c)$
excluded middle:	$a + a' = 1$	
contradiction:		$aa' = 0$
double negation:	$(a')' = a$	

To understand the significance of these rules for logic, the operations should be interpreted as above. For example, the law of excluded middle reads: For every proposition a, the statement that either a is true or a is false is certainly true.

These laws occur in pairs, in accordance with the principle of duality, which states that from any rule or provable formula another provable formula can be obtained by interchanging sum and product, and interchanging the elements 0 and 1. Sum and product have similar properties, being definable in terms of each other as follows: $ab \equiv (a'+b')'$, $a+b \equiv (a'b')'$. The rules above are not independent; the ones on the right, for example, all follow from those on the left.

There is another important interpretation of Boolean algebra in which the elements $a, b, c, \cdots$ stand for subsets of some "universal" set 1. In this interpretation $a+b$ means logical sum or union, and ab the logical product or common part of the sets a and b, and a' means the complement of a with respect to the set 1.

In Boolean logic, the operation of *implication* $a \supset b$ (read a implies b) is defined to be $a'+b$, that is, "either a is false or b is true." It should be noted that implication is not a *relation* between elements like equality which either holds or does not hold, but a rule of combination like addition, which produces from two elements of the algebra a third element. Many logicians do not like the definition of implication above and consider that it leads to properties that are paradoxical. Some of these properties of implication will be discussed later.

There exist Boolean algebras with 2, 4, 8, $\cdots$ elements, the number of elements, if finite, being a power of 2. The number of elements may also be infinite. All Boolean algebras, however, have the same provable formulas (that

is, algebraic identities) of which the rules above are examples. Since logicians are especially interested in provable formulas, they are apt to prefer the 2-element Boolean algebra as being the simplest. People interested in the relation between logic and the the theory of probability might prefer an infinite number of elements. It may be thought that the interpretation given to the law of excluded middle would rule out of consideration propositions which are neither certainly true nor certainly false, but merely probable. However, let a represent the merely probable proposition "It will rain tomorrow." Then $a+a'=1$ reads "The statement that either it will rain tomorrow or it will not, is certainly true," which seems plausible.

4. The Lukasiewicz-Tarski algebras of logic. In the Lukasiewicz-Tarski algebras of logic each element a has associated with it a real number from 0 to 1 called its truth value, $\tau(a)$. The truth value of a proposition may be interpreted to mean the probability that the proposition is true. In this logic, two propositions with the same truth value are considered to be logically equivalent. In other words, if $\tau(a)=\tau(b)$, then $a=b$. All logical operations are defined in terms of truth values alone.

There are Lukasiewicz-Tarski logics with 2 elements, 3 elements, and in general with any finite number of elements greater than 1. In the two-valued algebra (which is also a Boolean algebra) the truth values are 0 and 1. In the three-valued logic the truth values are 0, 1/2, and 1, and in the $n+1$-valued logic the truth values are $0, 1/n, 2/n, \cdots, 1$.

Lukasiewicz and Tarski take as fundamental operations *negation* and *implication* which they symbolize by *Na* and *Cab*. To make comparison easier we will use instead the notation a' and $a \supset b$, just as in Boolean algebra. Negation is defined in terms of truth values as follows:

$$\tau(a') \equiv 1 - \tau(a).$$

Thus if a has the truth value 1/2, its negative a' also has the truth value 1/2, and a is logically equivalent to its own negative. Such a proposition may be called doubtful, and its negative is also doubtful.

Implication has the following complicated definition in terms of truth values:

$$\tau(a \supset b) \equiv \min\,[1, \tau(a') + \tau(b)].$$

That is, if the sum of the truth values of a' and b is not more than 1, then this sum is taken as the truth value of $a \supset b$. For example, if $\tau(a)=.3$ and $\tau(b)=.6$, then $\tau(a \supset b)=1$, but $\tau(b \supset a)=.7$. In other words, a certainly implies b, but to say that b implies a is to make a statement whose probability is .7. Of any two propositions, the one of lesser truth value certainly implies the other.

In order to compare the Lukasiewicz-Tarski logics with other systems, it is necessary to define addition and multiplication operations in them. This can be done in different ways. Lukasiewicz and Tarski define addition as follows: $a \vee b \equiv (a \supset b) \supset b$. We shall call $a \vee b$ the *logical sum* of a and b. As in Boolean

algebra, the corresponding multiplication operation is defined to be $a \wedge b \equiv (a' \vee b')'$. We shall call $a \wedge b$ the *logical product* of a and b. Logical sum and product satisfy all the laws given for Boolean sum and product except the law of excluded middle and the law of contradiction. It is easily seen that $a \vee b$ is equal to that one of the two elements a and b which has the greater truth value, while $a \wedge b$ is equal to the one of lesser truth value.

Logical sum cannot be taken as a fundamental operation, since implication cannot be defined in terms of it. For this reason it is algebraically convenient to define another addition operation as follows: $a+b \equiv a' \supset b$. We shall call $a+b$ the *arithmetic sum* of a and b. Its interpretation is, "if not a, then b," which is close to the meaning "a or b." In the usual way, the corresponding multiplication operation, called *arithmetic product*, is defined to be $ab \equiv (a'+b')'$. Arithmetic sum and product satisfy all the Boolean laws given above except the law of tautology and the distributive law. Both logical and arithmetic sum and product satisfy the "truth value equation":

$$\tau(a) + \tau(b) = \tau(a \vee b) + \tau(a \wedge b) = \tau(a + b) + \tau(ab),$$

which is familiar from the theory of probability.

Arithmetic sum may be used to replace implication as a fundamental operation, and all operations may be defined in terms of negation and arithmetic sum. The resulting algebra differs from other algebras of logic in that numerical coefficients and exponents occur. Since the law of tautology does not hold, $a+a$ and $a \cdot a \cdot a$ are in general different from a, and may be written $2a$ and a^3.

The two kinds of operations, logical and arithmetic, of the Lukasiewicz-Tarski algebras share between them all the properties of Boolean sum and product. As a consequence, most Boolean formulas have analogs in the Lukasiewicz-Tarski logic. In translating a Boolean formula into this new logic, however, the Boolean sum must sometimes be replaced by logical sum, and sometimes by arithmetic sum. Take for example the Boolean formula:

$$a(a \supset b) = a'.$$

This becomes in the Lukasiewicz-Tarski logic:

$$a(a \supset b) = a \wedge b;$$

that is, arithmetic product occurs on the left and logical on the right. Consider the Boolean formula: $a \supset (b \supset c) = (ab) \supset c$, which may be read "$a$ implies that b implies c" is the same as "a and b together imply c." This continues to hold when ab is taken to be the arithmetic product, but not when ab is replaced by the logical product $a \wedge b$. However, the paradoxical Boolean formula:

$$(a \supset b) + (b \supset a) = 1$$

holds with either logical or arithmetic sum. It is surprising that the formula:

$$[a \wedge (a \supset b)] \supset b = 1$$

does not hold in the Lukasiewicz-Tarski logics. Its interpretation would be: "It

is certainly true that given a, and the fact that a implies b, then b follows." This seems reasonable enough, nevertheless it fails to hold, for example, when a has the truth value 2/3, and b has the truth value 1/3.

5. A postulate system for the Lukasiewicz-Tarski logics. All provable formulas of any Lukasiewicz-Tarski algebra hold also in the two-valued algebra (which is a Boolean algebra) since all the algebras contain the elements 0 and 1. Thus every provable Lukasiewicz-Tarski formula is also a provable Boolean formula. Unlike the case of Boolean algebra, however, different Lukasiewicz-Tarski algebras with different numbers of elements have different provable formulas. For example, the law of tautology $a+a=a$ holds in the two-valued logic but not in the three-valued logic. Similarly, the law $a+a+a=a+a$ holds in the three-valued but not in the four-valued logic and so on. In general the more elements there are, the fewer the provable formulas. This suggests the question of finding what provable formulas hold in all Lukasiewicz-Tarski algebras.

I have attempted to answer this question by constructing the following postulate system which I believe is new. It applies to all Lukasiewicz-Tarski logics. The postulates concern a class of elements $a, b, c, \cdots$, one of which is the special element 0, and an undefined operation called *implication*, which assigns to every pair of elements a and b of the set, distinct or not, a unique element $a \supset b$ of the set.

Definitions: $a \vee b \equiv (a \supset b) \supset b$, $a' \equiv a \supset 0$, $1 \equiv 0'$.

The postulates are, for every a, b, and c:

I. $a \supset (b \supset c) = b \supset (a \supset c)$.

II. $a \vee b = b \vee a$.

III. $a \vee 0 = a$.

IV. Either $a \supset b = 1$, or $b \supset a = 1$.

V. There are exactly $n+1$ elements.

I shall outline the proof that a system satisfying these postulates is a Lukasiewicz-Tarski algebra. It is first shown that either $a \vee b = a$, or $a \vee b = b$. The order relation $a < b$ (read a precedes b) is defined to mean $a \vee b = b$ and $a \neq b$. It is proved that this is a relation of simple or serial order, and that if $a < b$, then $b' < a'$. Arithmetic sum $a+b$ is defined to be $a' \supset b$, and is shown to be commutative and associative.

Let a_1 be the element which immediately follows 0 in the order relation above. Define a_2 to be a_1+a_1, a_3 to be $a_1+a_1+a_1$, and so on. Then it is proved that the elements $0 = a_0, a_1, a_2, \cdots, a_n = 1$ are all distinct and hence comprise all the $n+1$ elements of the algebra. These elements combine according to the rules: $a_r' = a_{n-r}$, and $a_r + a_s = a_{r+s}$, unless $r+s > n$, in which case $a_r + a_s = a_n$. But these are just the rules of combination of the truth values $0, 1/n, 2/n, 3/n, \cdots, 1$ of the $n+1$-valued Lukasiewicz-Tarski logic, when negation and arithmetic addition are taken as the fundamental operations. Hence the postulates define a Lukasiewicz-Tarski logic.

The provable formulas which hold in all Lukasiewicz-Tarski logics are clearly those formulas which follow from the first four of these postulates.

6. Heyting's algebra of logic. Heyting's algebra of logic is a formal postulational development of those rules of logic which the intuitionists of Brouwer's school recognize as valid. Although such a formal treatment of logic is contrary to the spirit of intuitionism, it serves the purpose of defending the intuitionists from the charge made by certain of their critics that a consistent logic which denies the law of excluded middle is impossible [5, 6].

In Heyting's logic the operations of *addition*, *multiplication*, *negation*, and *implication* are taken as fundamental. Since the law of double negation does not hold, it happens that these four operations are independent; no one of them is definable in terms of the other three. For this reason the Heyting postulate system is quite complicated, and it will not be given in full here.

Heyting uses the notation $\neg a$, $a \vee b$, $a \wedge b$, for negation, sum, and product respectively. Instead of this, the Boolean notation will be used here. Heyting, in common with many other logicians, uses the symbol of assertion $\vdash a$, which allows him to dispense with the equality sign. In Heyting's notation $a=b$ would be written $\vdash (a \supset b) \wedge (b \supset a)$.

Heyting's sum, product, and negation satisfy all the Boolean laws listed above except the laws of double negation and excluded middle. In particular, the law of contradiction holds, and since contradiction and excluded middle are duals, it can be seen that the principle of duality must be abandoned.

Since implication is not defined, its properties do not follow from those of the other operations, but must be assumed by postulate. One of the properties of implication is:

$$(a' + b) \supset (a \supset b) = 1,$$

that is, if a is false or b is true, then a implies b. With Heyting as in Boolean algebra, a false proposition implies any proposition, and a true proposition is implied by any proposition.

Unlike Boolean algebra, proof by *reductio ad absurdum* is not valid. For example, in Heyting's logic

$$[a' \supset (bb')] = a'',$$

that is, if to deny a leads to the contradiction bb', that means that a is "not false," but it does not follow that a is true. Other important Heyting formulas are:

$$(a + b)' = a'b',$$

$$(ab)' = a \supset b',$$

$$(a \supset b)' = a''b',$$

that is, the denial of a sum is a product, the denial of a product is an implication, and the denial of an implication is a product.

Four important *Boolean* properties of implication are:

$$a \supset (bc) = (a \supset b)(a \supset c),$$
$$(a + b) \supset c = (a \supset c)(b \supset c),$$
$$(ab) \supset c = (a \supset c) + (b \supset c),$$
$$a \supset (b + c) = (a \supset b) + (a \supset c).$$

Of these, the first two hold in Heyting's logic, but the last two do not, which is just as well, since the last two are usually considered to be paradoxical. The third, for example, states that if two premises a and b imply a conclusion c jointly, then one of them implies the conclusion singly. It is interesting to note what becomes of these formulas in the Lukasiewicz-Tarski logics. If sum and product are taken to mean logical sum and product, all four formulas continue to hold, just as in Boolean algebra. If the sum and product are taken to be the Lukasiewicz-Tarski arithmetic sum and product, in no case does the equality sign hold, but in every case the left side of the formula implies the right side. Common sense tells us that with the last two formulas the reverse should be true, and the right sides should imply the left, and such is indeed the case in Heyting's logic.

Instead of the law of double negation, in Heyting's logic we have a law of triple negation, $a''' = a'$. As a consequence of this law, all negative propositions of the form a' do satisfy the law of double negation. In fact, the negative propositions a' of a Heyting algebra constitute a Boolean subalgebra of the Heyting algebra. In this Boolean subalgebra of negative propositions, product, implication, and negation are the fundamental operations, and addition must be redefined in terms of them. The Heyting sum cannot be used, for the Boolean law of excluded middle $a' + a'' = 1$ (a is false or a is not false) does not hold with the Heyting sum, even for negative propositions.

More generally, it has been shown [7] that no formula of the type $P + Q = 1$ is provable in Heyting's logic unless one of the formulas $P = 1$ or $Q = 1$ is provable. There are no provable alternatives in Heyting's system. This accounts for the absence of some of the paradoxical Boolean formulas, many of which are in the form of alternatives, such as $(a \supset b) + (b \supset a) = 1$, or $(a \supset b) + (a \supset b') = 1$.

The paradoxical Boolean formulas are not avoided entirely, however. It can be shown that if $F = 1$ is a provable Boolean formula, then $F'' = 1$ is a provable Heyting formula. Thus all Boolean formulas have analogs in Heyting's logic, but instead of being true, they may be merely not false. Examples are $(a + a')'' = 1$, and $[(a \supset b) + (b \supset a)]'' = 1$, that is the law of excluded middle is not false, and it is not false that of any two propositions, one of them implies the other.

Another interpretation of the Heyting negation a' is "a is absurd" or "a is impossible," instead of "a is false." Then a'' means "it is impossible that a is impossible" or "a is certainly possible" rather than "a is not false." With this interpretation of negation the law of double negation or of excluded middle would not be expected to hold.

7. The Birkhoff-von Neumann algebra of logic. Garrett Birkhoff and J. von Neumann have written a paper on the logic of quantum mechanics [4] in which they maintain that the rules of Boolean logic do not all apply to the propositions of quantum mechanics. The important part played by linearity and linear spaces in quantum mechanics suggested to them that the system of all linear subspaces of some linear space might be taken as a model for a quantum mechanics logic, just as the system of all subsets of a given set serves as a model for Boolean logic.

In particular, the set of all points, lines, planes, etc. of an n-dimensional projective space is perhaps the simplest model. The *product* of two linear subspaces is defined to be their logical product or intersection. The *sum* of two linear subspaces of the projective space is taken to be, not their logical sum, since that is usually not a linear subspace, but rather the smallest linear subspace containing them both. There is a wide choice of operations to serve as a model for *negation*. Any dual involution of the space which has the property that no two related elements intersect can be used. An example is a polarity in an imaginary quadric.

The formal properties which the operations of this concrete model obey can then be assumed as part of the postulates for a quantum mechanics logic. In the logical interpretation of this abstract system the elements will of course represent propositions, while sum, product, and negation have their usual meanings *or*, *and*, *not*.

It is found that the operations obey all the rules listed above for Boolean algebra except the distributive law. This law is replaced by a weaker law called the *modular identity*:

$$ab + ac = a(b + ac)$$

or, with three terms,

$$ab + ac + ad = a(b + ac + ad).$$

In other words, in taking out a common factor from the sum of several terms, it is removed from one term but must be left in the others.

To insure that the system is different from Boolean algebra, it is further assumed that the special elements 0 and 1 are the only elements having a unique inverse. *Inverse* is defined as follows: the element x is said to be an inverse of a if $a+x=1$ and $ax=0$. It can be seen that the negation a' is always an inverse of a, and in Boolean algebra it is the only inverse. In the projective geometry model, however, elements in general have more than one inverse, since there are many ways of selecting the negation operation.

Up to this point the postulates are satisfied by any n-dimensional projective space. To make the system categorical, a further assumption is made that there exists a truth value or probability function $\tau(a)$ which assigns to each element a of the algebra a real number from 0 to 1, such that

$$\tau(a) + \tau(b) = \tau(a + b) + \tau(ab),$$

$$\tau(0) = 0, \qquad \tau(1) = 1,$$

where τ takes on all values from 0 to 1. The truth value $\tau(a)$ is interpreted to mean the probability that the proposition a is true. Two elements with the same truth value are not necessarily identical. The interpretation of the system as an n-dimensional projective geometry is ruled out by the last assumption, since with that interpretation it can be shown that $\tau(a)$ could take on only a finite number of values, being determined by the dimensionality of the subspace a. With this last postulate the system becomes abstractly equivalent to the continuous geometry which was the subject of the Colloquium Lectures delivered by von Neumann at the Pennsylvania State College in 1937.

There is no implication *operation* in the Birkhoff-von Neumann logic, although there is a *relation* of implication. Just as in Boolean algebra, a implies b if $a+b=b$. The operation $a'+b$, which in Boolean algebra is defined to be implication, has few of the properties in the Birkhoff-von Neumann logic one would expect of an implication operation. For example, if $a'+b=1$, it does not follow that the implication relation $a+b=b$ holds.

The lack of an implication operation is not an important defect. It was one of Boole's objects in constructing an algebra of logic to express all logical ideas in terms of sum, product, and negation, avoiding operations like implication which are neither associative nor commutative. The Birkhoff-von Neumann logic is difficult to work with as an algebra, despite the fact that its postulates are very similar to those of Boolean algebra. Not only is the failure of the distributive law inconvenient, but there is no method of transposing terms from one side of an equation to the other. In the other three algebras, $a=b$ is equivalent to $(a \supset b)(b \supset a)=1$.

8. Some comparisons. In comparing the three new algebras, the following list indicates the systems in which various Boolean laws fail to hold. L stands for the Lukasiewicz-Tarski logical sum and product, A for the arithmetic sum and product, H for the Heyting system, and Q for the Birkhoff-von Neumann quantum mechanics logic.

Tautology A
Distributive A, Q
Excluded middle L, H
Contradiction L
Double negation H

There is no single Boolean formula which we have yet discussed which all three new logics agree in rejecting. There exist such formulas, however. One of them is

$$ab + ab' = a.$$

In the Lukasiewicz-Tarski and Birkhoff-von Neumann logics an analogous formula holds, namely:

$$ab + a(ab)' = a.$$

In the Lukasiewicz-Tarski logics there is the further analogous formula:

$$a \wedge b + ab' = a.$$

In the Heyting logic the dual formula holds:

$$(a + b)(a + b') = a.$$

As for the paradoxical properties of the Boolean implication operation, it can be said that the Lukasiewicz-Tarski logic avoids few of them, while the Heyting logic avoids most of them at the expense of abandoning other properties not usually considered paradoxical. The Birkhoff-von Neumann logic avoids them by not having an implication operation.

In conclusion, it might be mentioned that if one wishes to introduce a truth value function into logic, as is done in the Lukasiewicz-Tarski and Birkhoff-von Neumann logics, in order to establish a connection between logic and the theory of probability, it is not necessary to abandon any of the Boolean rules. It is easy to introduce real-valued functions satisfying the truth value equation into many Boolean algebras, as is done in the theory of measure in point set theory.

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ON THE ABSOLUTE CONVERGENCE OF POLYNOMIAL SERIES

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1. Introduction. By a polynomial series we shall understand an expansion of the form

$$\sum_{n=0}^{\infty} a_n P_n(z),$$

where $\{P_n(z)\}$ is a given sequence of polynomials in the complex variable z and $\{a_n\}$ is a given sequence of complex numbers. With such a series are associated two point-sets C_0 and C_a in the complex z-plane. C_0 is the set of points z for which the series converges and C_a the set for which it converges absolutely. Either C_a or both C_0 and C_a may be vacuous in special cases.

In the present paper we shall assume that $P_n(z)$ is a polynomial of degree n having all its n roots real and we shall be concerned with the corresponding set C_a exclusively. In a number of cases the geometric character of C_a is fairly well known. The following table gives a list of some important cases.* The table gives the interior of C_a rather than C_a itself. The latter may differ from its interior by points on the boundary or, in the third case, also by a finite number of isolated points.

Polynomial	Interior of C_a
z^n	$\lvert z\rvert < r$
$(z-a_1)^n \cdots (z-a_k)^n$	$\lvert z-a_1\rvert \cdots \lvert z-a_k\rvert < r$
$\frac{1}{n!} z(z-1)\cdots(z-n+1)$	$x > x_a$
$\frac{1}{n!} e^z \frac{d^n}{dz^n}(z^n e^{-z})$	$-\eta < \Im(\sqrt{z}) < \eta$
$e^{z^2} \frac{d^n}{dz^n}(e^{-z^2})$	$-\eta < \Im(z) < \eta$
$\frac{1}{2^n n!} \frac{d^n}{dz^n}(z^2-1)^n$	$\lvert z-1\rvert + \lvert z+1\rvert < 2a$

The difference in the character of C_a in the various cases is evidently due to

* The first two cases of the table do not need any explanations. In the third case we are dealing with a binomial series also known as Newton's interpolation series. The remaining cases involve the polynomials of Laguerre, Hermite, and Legendre respectively. That a Legendre series converges in an ellipse with foci at ± 1 is well known, but no proof seems to exist in the literature for the Laguerre and Hermite cases. Cf., however, G. N. Watson, The harmonic functions associated with the parabolic cylinder, Proceedings of the London Mathematical Society, ser. 2, vol. 8, 1910, pages 393–421, and vol. 17, 1918, pages 116–148. These papers contain asymptotic formulas for the Hermitian polynomials and a solution of the problem of expanding an analytic function in an Hermitian series.

the difference in the distribution of the zeros of the polynomials in question. It would seem to be of some interest to study how C_a depends upon known features of this distribution. The present note is devoted to some elementary aspects of this question.

In this paper we shall let $\Im(w)$ denote the *imaginary* part *of* w, $\Re(w)$ the *real* part of w.

2. Induced absolute convergence. The first feature one notices in studying polynomial series is that absolute convergence is contagious. Absolute convergence at one point induces absolute convergence in a set of points unless the first point is a zero of infinitely many of the polynomials in the expansion or a limit point of such zeros. The oldest and best known instance of this phenomenon is given by the classical

THEOREM 1. *If a power series* $\sum_0^\infty a_n z^n$ *is absolutely convergent at* $z=z_0(\neq 0)$, *then it is absolutely convergent in the circle* $|z| \leqq |z_0|$.

It is well known that this theorem is the best of its kind. Another familiar instance is given by

THEOREM 2. *Let* $P_n(z)$ *have real zeros for all* n. *Let** $z_0 \epsilon C_a$, $z_0 = x_0 + iy_0$, $y_0 > 0$. *Then* $z \epsilon C_a$ *if* $z = x_0 + iy$ *and* $-y_0 \leqq y \leqq y_0$.

This follows from the fact that for a real a,

$$|x_0 + iy - a| < |x_0 + iy_0 - a| \quad \text{for} \quad -y_0 < y < y_0,$$

so that $|P_n(z)| < |P_n(z_0)|$ on the line segment in question.

This theorem is the best possible in the following sense. Consider the sets of absolute convergence of all polynomial series absolutely convergent at $z=z_0$. The cross section of all these sets is precisely the line segment $z=x_0+iy$, $-y_0 \leqq y \leqq y_0$, of the theorem. It is clear that the cross section contains this line segment; this is the assertion of Theorem 2. Now consider the following three polynomial series where $a_n = (n \log^2 n)^{-1}$:

$$(1) \qquad \sum_{n=2}^{\infty} a_n (z/z_0)^n,$$

$$(2) \qquad \sum_{n=2}^{\infty} \frac{1}{n!} a_n (z - x_0 - 1)(z - x_0 - 2) \cdots (z - x_0 - n),$$

$$(3) \qquad \sum_{n=2}^{\infty} \frac{1}{n!} a_n (z - x_0 + 1)(z - x_0 + 2) \cdots (z - x_0 + n).$$

The corresponding C_a's are $|z| \leqq |z_0|$, $x \geqq x_0$, and $x \leqq x_0$ respectively. The cross section of these three sets is precisely the line segment in question so that the cross section of all sets C_a cannot be any larger set. This completes the proof of the assertion.

* We recall that the notation $z \in S$ is used to express that the point z is an element of the set S.

We get better results by restricting the position of the zeros. This will be done in the subsequent sections of the paper.

3. Positive zeros. In the present section the zeros of $P_n(z)$ will be supposed to be positive or rather non-negative for every n. Theorem 2 can then be replaced by

THEOREM 3. *Let $P_n(z)$ have real, non-negative zeros for all n. Let $z_0 \epsilon C_a$, $z_0 = x_0 + iy_0$, $y_0 \neq 0$ unless $x_0 < 0$. Then $z \epsilon C_a$ if $|z| \leqq |z_0|$ and $x \geqq x_0$.*

Proof. If $a \geqq 0$ and $z = x + iy$ is so chosen that $|z| \leqq |z_0|$, $x \geqq x_0$, then $|z-a| < |z_0 - a|$. Hence $|P_n(z)| < |P_n(z_0)|$, and the theorem follows.

This result is again the best of its kind in the sense that the cross section of the sets of absolute convergence of all polynomial series, absolutely convergent at $z = z_0$, and corresponding to polynomials with non-negative zeros, is precisely the region stated in the theorem. Indeed, it is the cross section of the C_a's corresponding to the series (1) and (2) above.*

As a consequence of Theorem 3 we shall prove the following well known theorem:

THEOREM 4. *The C_a of the general binomial series*

$$\sum_{n=1}^{\infty} a_n \frac{(z-\lambda_1)(z-\lambda_2)\cdots(z-\lambda_n)}{\lambda_1\lambda_2\cdots\lambda_n}$$

is a half-plane $x > x_a$ (or $\geqq x_a$). Here $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$, $\lambda_n \to \infty$.

Remark. It follows from Theorem 6 below that x_a is always $-\infty$ when $\sum_{n=1}^{\infty} 1/\lambda_n < \infty$, *i.e.*, the half-plane becomes the whole plane.

Proof. We can write the series

$$\sum_{n=1}^{N} a_n \frac{(z-\lambda_1)\cdots(z-\lambda_n)}{\lambda_1\cdots\lambda_n} + \frac{(z-\lambda_1)\cdots(z-\lambda_N)}{\lambda_1\cdots\lambda_N} \sum_{n=N+1}^{\infty} a_n \frac{(z-\lambda_{N+1})\cdots(z-\lambda_n)}{\lambda_{N+1}\cdots\lambda_n}.$$

Suppose that the series is absolutely convergent for $z = z_0 \neq \lambda_n$ for every n, and choose N so large that $\lambda_{N+1} > k$ for a given positive k. Writing $z - \lambda_n = z - k - (\lambda_n - k)$ and applying Theorem 3 we see that the series is absolutely convergent in the region $x \geqq x_0$, $|z-k| \leqq |z_0 - k|$. But k can be taken as large as we please, and it follows that the series is absolutely convergent for $x > x_0$. Hence there exists a line $x = x_a$ to the right of which the series is absolutely convergent and to the left of which it can have this property only at those points λ_n such that $\lambda_n < x_0$.

* If $x_0 < -1$, a slight modification is necessary since some of the zeros will then be negative. It is sufficient to suppress the corresponding factors in the terms.

Theorem 4 is only a special case of a more general theorem. The result is essentially due to the fact that the polynomials involved have only a finite number of zeros in each fixed interval (a, b). Let us denote the number of zeros of $P_n(z)$ in (a, b) by $N(P_n, a, b)$ and put $N(P, a, b)$ equal to the least upper bound of $N(P_n, a, b)$. We have then

THEOREM 5. *Let $N(P, a, b) < \infty$ for every finite interval (a, b) and let $N(P, -\infty, 0) = 0$. Then if a point z_0, not on the positive real axis, belongs to C_a, the half-plane $x > x_0$ also belongs to C_a.*

Proof. Let $z_0 = x_0 + iy_0$, $z_1 = x_1 + iy_1$, $x_1 > x_0$. Determine a real number ξ from the equation $|\xi - z_1| = |\xi - z_0|$. Since the case $\xi \leqq 0$ is covered by Theorem 3, we can suppose $\xi > 0$. The number ξ always exists and is unique. Let us now factor $P_n(z)$ into two factors,

$$P_n(z) = P_{n,1}(z)P_{n,2}(z).$$

Here $P_{n,1}(z) = \Pi(z - a_{n,k})$ where $a_{n,k}$ runs over those zeros of $P_n(z)$ which are $\leqq \xi$. If there are no such zeros, $P_{n,1}(z) \equiv 1$. $P_{n,2}(z)$ contains the remaining factors. The degree of $P_{n,1}(z)$ is at most $N = N(P, 0, \xi)$. We have

$$|P_n(z_1)| = \left|\frac{P_{n,1}(z_1)}{P_{n,1}(z_0)}\right| \left|\frac{P_{n,2}(z_1)}{P_{n,2}(z_0)}\right| |P_n(z_0)|.$$

Here the quotients are products of fractions of the form

$$\left|\frac{z_1 - a_{n,k}}{z_0 - a_{n,k}}\right|.$$

The first quotient contains at most N such fractions in each of which $0 \leqq a_{n,k} \leqq \xi$. The maximum value of such a fraction is a certain quantity M, depending upon z_0 and z_1, the exact value of which is immaterial. In the second quotient all fractions are $\leqq 1$ by virtue of the determination of ξ. Hence

$$|P_n(z_1)| \leqq M^N |P_n(z_0)|.$$

It follows that the series is absolutely convergent at $z = z_1$, and the theorem is proved. That the theorem is the best of its kind follows from the example of series (2).

If we impose somewhat stronger restrictions on the zeros absolute convergence will be ensured in the whole plane.

THEOREM 6. *Let $N(P, -\infty, 0) = 0$ and suppose that $\sum_{k=1}^{n} 1/a_{n,k} \leqq M < \infty$ for every n. Then if the polynomial series converges absolutely for a z_0, not on the positive real axis, it converges absolutely in the whole plane.*

Proof. In the following argument we shall need a couple of well known inequalities from the theory of the exponential function. We have $1 + u \leqq e^u$ for all real u, and $1 - u > e^{-2u}$ for $0 < u < \frac{1}{2}$.

Let us define $P_n(z)=\prod_{k=1}^{n}[1-z/a_{n,k}]$. We begin by finding an upper limit of $|P_n(z)|$. We have clearly

$$|P_n(z)| \leqq \prod_{k=1}^{n}[1+|z|/a_{n,k}] < \exp\left\{|z|\sum_{k=1}^{n}(1/a_{n,k})\right\} \leqq \exp[M|z|],$$

where we have used the first inequality for e^u. Next we need a lower bound for $|P_n(z_0)|$. The inequality $\sum_1^n(1/a_{n,k}) \leqq M$ implies that $a_{n,k} \geqq k/M$, supposing $a_{n,k} \leqq a_{n,k+1}$. Since z_0 is not a real positive number we can find a finite positive quantity $B(z_0)$ such that

$$\prod'|1-z_0/a_{n,k}| > B(z_0),$$

where the product extends over those values of k for which $a_{n,k} \leqq 2|z_0|$. By virtue of the second inequality for e^u we see that the remaining factors, if any, satisfy the inequality

$$\prod''|1-z_0/a_{n,k}| \geqq \prod''[1-|z_0|/a_{n,k}] > \exp[-2|z_0|\sum''(1/a_{n,k})] > \exp[-2M|z_0|].$$

Hence

$$|P_n(z)| \leqq \frac{1}{B(z_0)}\exp[M(|z|+2|z_0|)]|P_n(z_0)|.$$

This estimate suffices to prove the theorem.

We cannot replace the condition

$$\limsup_{n\to\infty}\sum_{k=1}^{n}1/a_{n,k} < \infty$$

by the weaker assumption

$$\liminf_{n\to\infty}\sum_{k=1}^{n}1/a_{n,k} < \infty.$$

In order to prove this it is sufficient to take

$$a_{2n+1}=1, \qquad P_{2n+1}(z)=\binom{z-1}{2n+1},$$

$$a_{2n}=1, \qquad P_{2n}(z)=\binom{z-1}{n}\binom{1-z}{n}.$$

This choice satisfies the weaker condition, but not the stronger one. The resulting series is absolutely convergent for $x>1$ and diverges for $x<0$.

We have seen that the domain of absolute convergence of a polynomial series is a half-plane if $N(P_n, a, b)$ is always bounded for finite intervals and $N(P_n, -\infty, 0)=0$. Here the first condition is merely sufficient and far from necessary. This is illustrated by the following rather special theorem:

THEOREM 7. *Let* $N(P_n, -\infty, 0) = 0$ *and*

$$\min(An^\alpha\omega, n) \leqq N(P_n, 0, \omega) \leqq \min(Bn^\alpha\omega, n),$$

where $0 < A < B$ *and* $0 < \alpha < 1$. *If a point* z_0, *not on the positive real axis, belongs to* C_α, *so does the half-plane* $\Re(z) > \Re(z_0)$.

Proof. Let $z_0 = x_0 + iy_0$, $z_1 = x_1 + iy_1$, $x_1 > x_0$. Let t be a real variable $\geqq 1$. An elementary calculation shows that

$$t^2\left\{\left|\frac{z_1 - t}{z_0 - t}\right| + \frac{1}{t}(x_1 - x_0) - 1\right\} \equiv \eta$$

is bounded for $t \geqq 1$. The upper bound of $|\eta|$, M_0 say, is a function of z_1 and z_0, the exact value of which does not concern us. Hence

$$\left|\frac{z_1 - t}{z_0 - t}\right| = 1 - \frac{1}{t}(x_1 - x_0) + \frac{\eta}{t^2}, \qquad |\eta| \leqq M_0.$$

In the rest of the proof we use the same notation as in the proof of Theorem 5. The case $\xi \leqq 0$ being covered by Theorem 3, we suppose $\xi > 0$. For sufficiently large values of n we have

$$|P_{n,1}(z_1)/P_{n,1}(z_0)| \leqq \exp[n^\alpha B\xi \log M].$$

To every large n we can find a least integer m such that

$$a_{n,k} \geqq \rho \equiv \max[1, \xi, \tfrac{1}{2}(x_1 - x_0), 2M_0/(x_1 - x_0)], \qquad k \geqq m,$$

and for every such n, we have $m < Bn^\alpha\rho + 1$. For such values of k and n

$$\left|\frac{z_1 - a_{n,k}}{z_0 - a_{n,k}}\right| \leqq 1 - \tfrac{1}{2}(x_1 - x_0)\frac{1}{a_{n,k}}.$$

On the other hand, if $\omega \leqq n^{1-\alpha}/A$ then

$$a_{n,k} \leqq \omega \quad \text{for} \quad k \leqq N(P_n, 0, \omega),$$

and

$$An^\alpha\omega \leqq N(P_n, 0, \omega).$$

Combining these inequalities we get

$$1/a_{n,k} \geqq An^\alpha/k,$$

and

$$\left|\frac{z_1 - a_{n,k}}{z_0 - a_{n,k}}\right| \leqq 1 - \frac{A}{2}(x_1 - x_0)n^\alpha\frac{1}{k}, \qquad m \leqq k \leqq n.$$

Hence

$$|P_{n,2}(z_1)/P_{n,2}(z_0)| \leqq \exp\left[-\frac{A}{2}(x_1 - x_0)n^\alpha \log(n/m)\right]$$

$$\leqq \exp\left\{-\frac{A}{2}(x_1 - x_0)n^\alpha[(1-\alpha)\log n - \log B\rho]\right\},$$

and finally

$$|P_n(z_1)| \leqq \exp\left\{-\frac{A}{2}(x_1 - x_0)n^\alpha[(1-\alpha)\log n - \log B\rho] + n^\alpha B\xi \log M\right\} \cdot |P_n(z_0)|.$$

This estimate shows that $z_1 \epsilon C_a$ and completes the proof of the theorem.

The proof breaks down if $\alpha \geqq 1$ and the theorem is actually false for such an α. This is illustrated by the weighted binomial series

$$\sum_{n=1}^{\infty} a_n \binom{n^\alpha z}{n}.$$

If $\alpha > 1$, the series converges like a power series, *i.e.*, C_a is a circle $|z| < R$ or $|z| \leqq R$. If $\alpha = 1$, then C_a is bounded by a transcendental curve

$$\Re[z \log z + (z-1)\log(z-1)] = \text{const.}$$

These results are easily obtained from Stirling's formula in the theory of the Gamma function.

4. Symmetric zeros. The results derived for positive zeros are easily extended to symmetric zeros, *i.e.*, to polynomial series such that $P_n(-z) = \pm P_n(z)$. We have merely to separate the terms of even degree from those of odd degree and factor out a term z from the latter. The resulting series are polynomial series in z^2 to which the preceding analysis applies. It is sufficient to quote one single theorem as an example of the results so obtained. I take the analog of Theorem 5.

THEOREM 8. *Let the zeros of $P_n(z)$ be real and symmetric with respect to the origin and let $N(P, a, b) < \infty$ for every finite interval (a, b). Then if a non-real point z_0 belongs to C_a, the hyperbolic domain $\Re(z^2) > \Re(z_0^2)$ also belongs to C_a.*

EUCLID'S ALGORITHM FOR LARGE NUMBERS

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Euclid's algorithm which is so fundamental to the theory of regular continued fractions and elementary theory of numbers, is also of immense practical value in such well known problems as discovering the partial quotients in the regular continued fraction expansion of a given real number, and the solution of linear diophantine equations, to mention only two applications. In these problems, which occur frequently in experimental research, it is often necessary to carry the algorithm to a great many stages, as for example when one needs the greatest common divisor of two numbers each having, let us say, 30 digits, or when one wishes an extremely accurate rational approximation to a given irrational. In such cases, in which one would naturally use a computing machine, the algorithm involves numerous trivial operations with extremely large numbers. The purpose of this note is to show how more than 90% of these operations with large numbers may be eliminated. If one wishes historical evidence of the difficulties or rather the tedium of Euclid's algorithm in such cases, one may take the problem of expanding π in a regular continued fraction. In 1685 Wallis* computed the first 34 partial quotients of π. This calculation, made nearly a century before π was proved irrational, was verified as far as the 26th partial quotient by Lambert in 1770. But since then no one has extended the calculation of Lambert, and the fact that the 34th partial quotient given by Wallis should be 99 instead of unity has remained unnoticed until today.

1. Notation and general formulas. Let x_0 and x_1 be a pair of positive real numbers. Then Euclid's algorithm generates from x_0 and x_1 a set $\{x_\nu\}$ of real numbers and a set $\{q_\nu\}$ of integers by means of the equations

$$
\begin{aligned}
x_0 &= q_0 x_1 + x_2, \\
x_1 &= q_1 x_2 + x_3, \\
x_2 &= q_2 x_3 + x_4, \\
&\cdots\cdots\cdots \\
x_\nu &= q_\nu x_{\nu+1} + x_{\nu+2}, \\
&\cdots\cdots\cdots,
\end{aligned} \tag{1}
$$

in which q_ν is the greatest integer not exceeding $x_\nu/x_{\nu+1}=\xi_\nu$. Eliminating $x_2, x_3, x_4, \cdots$ from (1) we obtain the regular continued fraction for ξ_0,

$$
\xi_0 = \frac{x_0}{x_1} = q_0 + \frac{1}{q_1 + q_2 + \cdots} = [q_0, q_1, q_2, \cdots].
$$

Similarly

* John Wallis, A Treatise of Algebra, London, 1685, pp. 46–55.

$$\xi_\nu = \frac{x_\nu}{x_{\nu+1}} = [q_\nu, q_{\nu+1}, q_{\nu+2} \cdots]. \tag{2}$$

The nth convergent of ξ_0, namely

$$[q_0, q_1, q_2, \cdots, q_n],$$

is usually denoted by A_n/B_n, but to avoid the use of too many sub-subscripts in what follows we shall also write $A(n)$ and $B(n)$ for the numerator and denominator of the nth convergent of ξ_0. More generally, the nth convergent of ξ_ν,

$$[q_\nu, q_{\nu+1}, \cdots, q_{\nu+n}], \tag{3}$$

will be denoted by $A(n, \nu)/B(n, \nu)$ so that

$$A_n = A(n) = A(n, 0) \qquad \text{and} \qquad B_n = B(n) = B(n, 0).$$

The A's and B's satisfy the following recursion formulas

$$\begin{aligned} A(m, \nu) &= A(m-1, \nu)q_{m+\nu} + A(m-2, \nu), \\ B(m, \nu) &= B(m-1, \nu)q_{m+\nu} + B(m-2, \nu), \end{aligned} \tag{4}$$

with the initial conditions

$$\begin{aligned} A(-1, \nu) &= 1, \qquad A(0, \nu) = q_\nu, \\ B(-1, \nu) &= 0, \qquad B(0, \nu) = 1. \end{aligned} \tag{5}$$

If we eliminate $x_{\nu+2}, x_{\nu+3}, \cdots, x_{\nu+n-1}$ between the $(\nu+1)$st, $(\nu+2)$nd, $\cdots$ $(\nu+n-1)$st equations of the system (1) we obtain

$$x_{\nu+n} = (-1)^n\{B(n-2, \nu)x_\nu - A(n-2, \nu)x_{\nu+1}\}, \tag{6}$$

a formula which is readily proved by induction using (1), (4), and (5), and which lies at the root of our modification of the Euclid algorithm. The following formulas will also be of use and are easily established by induction from (4) and (5).

$$\begin{aligned} A_{\nu+n} &= A_\nu A(n-1, \nu+1) + A_{\nu-1}B(n-1, \nu+1), \\ B_{\nu+n} &= B_\nu A(n-1, \nu+1) + B_{\nu-1}B(n-1, \nu+1). \end{aligned} \tag{7}$$

Given the two numbers x_0 and x_1 having a very large number of significant figures, the application of Euclid's algorithm begins with the computation of the partial quotients q_ν $(\nu = 0, 1, 2, \cdots)$. We note first that since q_0 is merely the greatest integer in x_0/x_1, crude approximations to x_0 and x_1 will in general be sufficient to determine q_0. More generally, if (x_0, x_1) are replaced by approximate values (y_0, y_1), then the first few partial quotients of x_0/x_1 and y_0/y_1 will be the same. If y_0 and y_1 have only a few significant figures, the first few partial quotients are easily obtained. Of course, if the Eulcid's algorithm for y_0/y_1 is carried too far, the q's will begin to disagree with those for x_0/x_1. There are two ways to find out how far the q's may be trusted. Perhaps the safest way is to choose a second pair (z_0, z_1) such that x_0/x_1 lies between y_0/y_1 and z_0/z_1. Then the partial

quotients of y_0/y_1 and x_0/x_1 will agree at least as far as those of y_0/y_1 and z_0/z_1 agree. Since the whole calculation is based on the first few q's, the fact that they are obtained in two ways is a welcome check on the work, rather than a waste of effort. The other method of telling how far we may trust the q's will be discussed presently.

Having obtained the first few partial quotients, up to say q_{k_1}, we next compute the numbers

$$A(k_1, 0), \quad B(k_1, 0), \quad A(k_1 - 1, 0), \quad B(k_1 - 1, 0) \tag{8}$$

by the recurrence formulas (4) with $\nu = 0$. The relation

$$A(k_1, 0)B(k_1 - 1, 0) - A(k_1 - 1, 0)B(k_1, 0) = (-1)^{k_1 - 1}$$

affords an almost infallible check on the work.

Thus far the calculation has involved only small numbers. In fact we have not used x_0 and x_1, but merely approximations to them. By setting $\nu = 0$, and $n = k_1 + 1$ and $k_1 + 2$ in (6) we get

$$x_{k_1+1} = (-1)^{k_1+1}\{B(k_1 - 1, 0)x_0 - A(k_1 - 1, 0)x_1\}, \tag{9}$$

$$x_{k_1+2} = (-1)^{k_1+2}\{B(k_1, 0)x_0 - A(k_1, 0)x_1\}. \tag{10}$$

Here we encounter for the first time operations with really large numbers. To be quite certain that we are on the right track we may apply the following well known* theorem.

THEOREM A. *Let A_k/B_k, A_{k-1}/B_{k-1}, be two consecutive convergents to a number η. Then these fractions are consecutive convergents of ξ if and only if*

$$\left| \xi - \frac{A_k}{B_k} \right| < \frac{1}{B_k(B_k + B_{k-1})}.$$

Setting $\xi_0 = x_0/x_1$ and $k = k_1$ we obtain from (10) the condition

$$x_{k_1+2} < \frac{x_1}{B(k_1, 0) + B(k_1 - 1, 0)} \tag{11}$$

which is necessary for the correctness of the q's so far. We now replace x_{k_1+1} and x_{k_1+2} by approximations y_{k_1+1} and y_{k_1+2} and compute as before the first few partial quotients of y_{k_1+1}/y_{k_1+2}. By (2) these partial quotients will be

$$q_{k_1+1}, q_{k_1+2}, \cdots, q_{k_1+1+k_2}. \tag{12}$$

Using these q's we next compute the numbers

$$A(k_2, k_1 + 1), \quad B(k_2, k_1 + 1), \quad A(k_2 - 1, k_1 + 1), \quad B(k_2 - 1, k_1 + 1) \tag{13}$$

by (4) and check them by means of the relation

$$A(k_2, k_1 + 1)B(k_2 - 1, k_1 + 1) - A(k_2 - 1, k_1 + 1)B(k_2, k_1 + 1) = (-1)^{k_2 - 1}.$$

Using (6) with $\nu = k_1 + 1$ and $n = k_2 + 1$ and $k_2 + 2$ we get the equations

* See for instance Lucas, Théorie des Nombres, p. 449.

$$x_{k_1+k_2+2} = (-1)^{k_2+1}\{B(k_2-1,\ k_1+1)x_{k_1+1} - A(k_2-1,\ k_1+1)k_{k_1+2}\}, \tag{14}$$

$$x_{k_1+k_2+3} = (-1)^{k_2}\{B(k_2,\ k_1+1)x_{k_1+1} - A(k_2,\ k_1+1)x_{k_1+2}\}, \tag{15}$$

which involve for the second time operations with large numbers. A test of the correctness of the set (12) of new q's is this time

$$x_{k_1+k_2+3} < \frac{x_{k_1+2}}{B(k_2,\ k_1+1) + B(k_2-1,\ k_1+1)}. \tag{16}$$

The process may now be continued using approximate values of $x_{k_1+k_2+2}$ and $x_{k_1+k_2+3}$ and obtaining a new set of q's, a new pair of A's and B's, and a new pair of x's. In this way the partial quotients may be extended as far as desirable.

In some problems the convergents A_n/B_n are of no interest. In these cases the above process is adequate. For example if we have a given number ξ_0 expressed let us say in decimals, and we wish to examine its partial quotients to see if they terminate, become periodic, or obey some law or other, we have only to choose $x_0=\xi_0$, and $x_1=1$ and apply the above process. As a second example we may wish to find the greatest common divisor $\delta=(x_0,\ x_1)$ of two large integers x_0 and x_1. From (9) and (10) we see that any divisor common to x_0 and x_1 will divide x_{k_1+1} and x_{k_1+2} and conversely. Hence from (14) and (15) and all further similar equations we have

$$\delta = (x_0,\ x_1) = (x_{k_1+1},\ x_{k_1+2}) = (x_{k_1+k_2+2},\ x_{k_1+k_2+3}) = \cdots.$$

Since these x's decrease rapidly (by (11), (16), etc.) we soon come to a pair whose G.C.D. is easily found.

Of those problems in which the convergents are of importance a large majority require merely one convergent. This is the case for example when one wishes to get a rational approximation to a real number, or in the solution of linear diophantine equations, in which case the penultimate convergent of a rational number is needed. Sometimes one needs a sequence of convergents (or even intermediate convergents) beginning with A_n/B_n, the earlier convergents being of no use. This happens for example when one is looking for a rational approximation to a given real number which not only is sufficiently accurate but whose numerator or denominator has some further property.

In all these cases one may use formulas (7) to advantage as follows. We have already found the numbers (8), (13), etc. Substituting them in (7) with $\nu=k_1$ and first $n=k_2$ and then $n=k_2+1$, we get at once

$$\begin{aligned}
A_{k_1+k_2} &= A_{k_1}A(k_2-1,\ k_1+1) + A_{k_1-1}B(k_2-1,\ k_1+1),\\
B_{k_1+k_2} &= B_{k_1}A(k_2-1,\ k_1+1) + B_{k_1-1}B(k_2-1,\ k_1+1),\\
A_{k_1+k_2+1} &= A_{k_1}A(k_2,\ k_1+1) + A_{k_1-1}B(k_2,\ k_1+1),\\
B_{k_1+k_2+1} &= B_{k_1}A(k_2,\ k_1+1) + B_{k_1-1}B(k_2,\ k_1+1).
\end{aligned} \tag{17}$$

Thus we proceed from the numerators and denominators of one pair of consecutive convergents (8) to those of another isolated pair (17). Repeating the

process as many times as is necessary one obtains a consecutive pair of convergents A_n/B_n, A_{n-1}/B_{n-1} for whatever value of n one may wish. The relation

$$A_nB_{n-1} - A_{n-1}B_n = (-1)^{n-1}$$

serves as a final check. One may also examine consecutive convergents (or intermediate convergents) in the neighborhood of A_n/B_n using (4) with $\nu=0$.

2. An application. To illustrate the foregoing method we consider the regular continued fraction for π. Taking a value of π correct to 100 decimal places* we shall first find the partial quotients

$$\pi = [3, 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, \cdots], \tag{18}$$

from which

$$\begin{aligned} A_{17} &= 2549491779, & B_{17} &= 811528438, \\ A_{18} &= 6167950454, & B_{18} &= 1963319607. \end{aligned} \tag{19}$$

Using now our 100 figure accuracy we find† (by (6) with $\nu=0$, $n=19, 20$)

$$\begin{aligned} x_{19} &= A_{17} - B_{17}\pi = 4.474\cdot 10^{-10}, \\ x_{20} &= -A_{18} + B_{18}\pi = 1.497\cdot 10^{-10}. \end{aligned} \tag{20}$$

As a check (11) gives

$$x_{20} < 1/(B_{18} + B_{17}) = 3.603\cdot 10^{-10}$$

which is in accord with (20).

Next we find that

$$\begin{aligned} \xi_{19} &= x_{19}/x_{20} = [q_{19}, q_{20}, \cdots, q_{32}, \cdots] \\ &= [2, 1, 84, 2, 1, 1, 15, 3, 13, 1, 4, 2, 6, 6, \cdots]. \end{aligned} \tag{21}$$

Hence

$$\begin{aligned} A(12, 19) &= 61245426, & B(12, 19) &= 20495141, \\ A(13, 19) &= 376962143, & B(13, 19) &= 126146437. \end{aligned} \tag{22}$$

Next we compute

$$\begin{aligned} x_{33} &= B(12, 19)x_{19} - A(12, 19)x_{20} = 1.185\cdot 10^{-18}, \\ x_{34} &= A(13, 19)x_{20} - B(13, 19)x_{19} = 1.188\cdot 10^{-20}. \end{aligned}$$

As a check (16) gives

$$x_{34} < x_{20}/(B(13, 19) + B(12, 19)) = 1.021\cdot 10^{-18}.$$

---

* π=3.14159 26535 89793 23846 26433 83279 50288 41971 69399 37510 58209 74944 59230 78164 06286 20899 86280 34825 34211 70680.

† The actual values of the x's have been suppressed to save space.

Next we find

$$\xi_{33} = x_{33}/x_{34} = [q_{33}, q_{34}, \cdots, q_{50}, \cdots] \tag{23}$$
$$= [99, 1, 2, 2, 6, 3, 5, 1, 1, 6, 8, 1, 7, 1, 2, 3, 7, 1, \cdots],$$

from which

$$A(16, 33) = 6185428223, \qquad B(16, 33) = 62033398, \tag{24}$$
$$A(17, 33) = 7032882291, \qquad B(17, 33) = 70532479,$$

and

$$x_{51} = B(16, 33)x_{33} - A(16, 33)x_{34} = 1.249 \cdot 10^{-28},$$
$$x_{52} = A(17, 33)x_{34} - B(16, 33)x_{33} = 4.958 \cdot 10^{-29}.$$

As a check

$$x_{52} < x_{34}/(B(17, 33) + B(16, 33)) = 8.965 \cdot 10^{-29}.$$

Next we find

$$\xi_{51} = x_{51}/x_{52} = [q_{51}, q_{52}, \cdots, q_{76}, \cdots] \tag{25}$$
$$= [2, 1, 1, 12, 1, 1, 1, 3, 1, 1, 8, 1, 1, 2, 1, 6, 1, 1, 5, 2, 2, 3, 1, 2, 4, 4, \cdots],$$

and

$$A(24, 51) = 2414289141, \qquad B(24, 51) = 958421828, \tag{26}$$
$$A(25, 51) = 10210815077, \qquad B(25, 51) = 4053478055,$$

from which

$$x_{77} = B(24, 51)x_{51} - A(24, 51)x_{52} = 1.206 \cdot 10^{-38},$$
$$x_{78} = A(25, 51)x_{52} - B(25, 51)x_{51} = 7.099 \cdot 10^{-40}.$$

As a check

$$x_{78} < x_{52}/(B(25, 51) + B(24, 51)) = 9.893 \cdot 10^{-39}.$$

Finally

$$\xi_{77} = x_{77}/x_{78} = [q_{77}, q_{78}, \cdots, q_{90}, \cdots] \tag{27}$$
$$= [16, 1, 161, 45, 1, 22, 1, 2, 2, 1, 4, 1, 2, 24, \cdots],$$

$$A(12, 77) = 482872247, \qquad B(12, 77) = 28414566, \tag{28}$$
$$A(13\ 77) = 11759887912, \qquad B(13, 77) = 692009353;$$

hence

$$x_{92} = A(13, 77)x_{78} - B(13, 77)x_{77} = 7.328 \cdot 10^{-49},$$

whereas

$$x_{92} < x_{78}/(B(13, 77) + B(12, 77)) = 9.854 \cdot 10^{-49}.$$

Combining (18), (21), (23), (25), and (27) we have all the partial quotients of π from $q_0=3$ to $q_{90}=24$.

If we wish the 90th convergent A_{90}/B_{90} we proceed as follows. Setting $\nu=18$, $n=13, 14$ in (7) we have from (19) and (23)

$$
\begin{aligned}
A_{31} &= A_{18}A(12, 19) + A_{17}B(12, 19) = \ 430010946\ 591069243,\\
B_{31} &= B_{18}A(12, 19) + B_{17}B(12, 19) = \ 136876735\ 467187340,\\
A_{32} &= A_{18}A(13, 19) + A_{17}B(13, 19) = 2646693125\ 139304345,\\
B_{32} &= B_{18}A(13, 19) + B_{17}B(13, 19) = \ 842468587\ 426513207.
\end{aligned} \tag{29}
$$

As a check

$$A_{32}B_{31} - A_{31}B_{32} = -1.$$

Next setting $\nu=32$, $n=17, 18$ in (7) we obtain from (29) and (24),

$$
\begin{aligned}
A_{49} &= A_{32}A(16, 33) + A_{31}B(16, 33),\\
B_{49} &= B_{32}A(16, 33) + B_{31}B(16, 33),\\
A_{50} &= A_{32}A(17, 33) + A_{31}B(17, 33),\\
B_{50} &= B_{32}A(17, 33) + B_{31}B(17, 33).
\end{aligned} \tag{30}
$$

Next we set $\nu=50$, $n=25, 26$ in (7) and obtain from (30) and (26),

$$
\begin{aligned}
A_{75} &= A_{50}A(24, 51) + A_{49}B(24, 51),\\
B_{75} &= B_{50}A(24, 51) + B_{49}B(24, 51),\\
A_{76} &= A_{50}A(25, 51) + A_{49}B(25, 51),\\
B_{76} &= B_{50}A(25, 51) + B_{49}B(25, 51).
\end{aligned} \tag{31}
$$

Finally we set $\nu=76$, $n=14$ in (7) and obtain from (31) and (28),

$$
\begin{aligned}
A_{90} &= A_{76}A(13, 77) + A_{75}B(13, 77),\\
B_{90} &= B_{76}A(13, 77) + B_{75}B(13, 77).
\end{aligned}
$$

The actual values of A_{90} and B_{90} are

$$
\begin{aligned}
A_{90} &= 3062\ 43329\ 44969\ 82257\ 53216\ 23878\ 54374\ 05787\ 90366\ 50780,\\
B_{90} &= \ 974\ 80279\ 34167\ 85521\ 47430\ 34062\ 01616\ 85335\ 37806\ 95273.
\end{aligned}
$$

By Theorem A, $\pi - A_{90}/B_{90}$ should be less than $1.01 \cdot 10^{-96}$. By actual computation we find this difference to be less than $8 \cdot 10^{-97}$.

QUESTIONS, DISCUSSIONS, AND NOTES

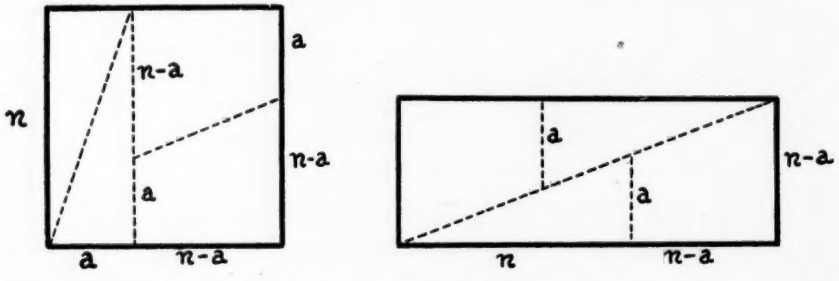
EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

LEWIS CARROLL AND A GEOMETRICAL PARADOX

WARREN WEAVER, Rockefeller Foundation

In a familiar geometrical paradox* a square of area $8\times 8=64$ square units is cut into four parts which may be refitted to form a rectangle of apparent area $5\times 13=65$ square units. In connection with editing unpublished mathematical material† left by Charles L. Dodgson (Lewis Carroll), the present author learned that Lewis Carroll generalized this paradox by characterizing the dimensions of all possible squares which may be cut in this same way and refitted into a rectangle with an apparent gain of one unit of area. Carroll's notes are incomplete and vague so that it is not possible accurately to reconstruct all details of his argument. Using the figure



he wrote the condition

$$(2n^2 - 3na + a^2) - n^2 = 1,$$

or

$$n^2 - 3na + a^2 - 1 = 0. \tag{1}$$

Assuming (or arguing in a way not stated) that a and n are the first two terms of a sequence of integers U_n any two successive terms of which produce an instance of the paradox, Carroll obtained (with details again uncertain) the difference equation

* This paradox is described in W. W. R. Ball's Mathematical Recreations and Essays, 5th edition, London, Macmillan, 1911, p. 53. It apparently was first published in 1868 by Schlömilch. See Zeitschrift für Mathematik und Physik, vol. 13, p. 162.

† This material belongs to Mr. M. L. Parrish of Philadelphia, the owner of the finest Carroll collection in existence.

$$U_n + \Delta U_n - \Delta^2 U_n = 0,$$

for the determination of the sequence. Solving this equation by standard methods, he obtained the solution*

$$U_n = \frac{(3+\sqrt{5})^n - (3-\sqrt{5})^n}{2^n\sqrt{5}},$$

and drew up the table

n	3	8	21	55	etc.
a	1	3	8	21	

The calculation is not dated, but Dodgson undoubtedly wrote this portion of the note book in question between 1890 and 1893. It seems clear that Dodgson was unaware of the similar generalization which was carried out much earlier by V. Schlegel.† A third (independent?) generalization was published by E. B. Escott.‡

The present note is written partly to call attention to the connection of the author of the immortal Alice in Wonderland with this well known geometrical puzzle; and partly to indicate that this problem may be solved by such elementary methods as to bring it within the range of a class in elementary algebra. Perhaps the argument, ballasted by a little heavier material from the Schlegel article, would be of some interest to junior mathematical clubs.

One seeks pairs of positive integers a, n satisfying the relation (1). Solving this quadratic for n, we obtain

$$n = \frac{3a \pm \sqrt{5a^2+4}}{2}. \tag{2}$$

Suppose one has a positive integral value of a which, when substituted into this formula, results in an integral value of n when one choice of sign is made. First one notes that the radicand must be a perfect square. If a be odd, then this radicand is odd and the positive square root is odd. But then both the sum and difference of the odd number $3a$ and the odd square root will be even, and not only one but both values of n are integral. The value, say n_-, which corresponds to the negative sign is non-negative and less than a, while the value n_+ is larger

* This formula was produced by J. P. M. Binet in Comptes Rendus de l'Académie des Sciences de Paris, tome 17, 1843, p. 563 in connection with a treatment of Fibonacci series. R. C. Archibald has pointed out the relationship between this geometrical puzzle and Fibonacci series. See this MONTHLY, vol. 25, 1918, p. 236.

† Zeitschrift für Mathematik und Physik, vol. 24, 1874, p. 123. This paper discusses various number theoretic properties of the sequence U_n, and includes the loss as well as the gain of one unit of area.

‡ Open Court Magazine, vol. 21, 1907, p. 502.

than a. The same end result holds if a be even. Thus we have three positive* integers n_-, a, and n_+ which we will call U_0, U_1, and U_2.

Moreover (1) is symmetrical in n and a. Thus the equation

$$a = \frac{3n \pm \sqrt{5n^2 + 4}}{2} \tag{3}$$

must be satisfied by the pair of values $n = U_2$, $a = U_1$. But by exactly the argument above, if the integral value $n = U_2$ leads to a smaller value $a = U_1$, the choice of the plus sign will yield another integral value $a = U_3$ larger than U_2.

Thus it is clear that there exists an infinite sequence of integers $U_0, U_1, U_2 \cdots$ such that any successive pair U_r, U_{r+1}, constitute a pair of values of a and n which meet the condition of the problem. These numbers could be determined by successive use of (2). Beginning with any one known value of a and using the plus sign, one finds a value of n which is then used again in the formula as the next value of a, and so on.

But a somewhat simpler actual determination can be made, once the above facts are known. For we are now justified in writing

$$U_r^2 - 3U_rU_{r-1} + U_{r-1}^2 - 1 = 0,$$

$$U_{r+1}^2 - 3U_{r+1}U_r + U_r^2 - 1 = 0.$$

Subtracting and factoring, we obtain

$$(U_{r-1} - U_{r+1})(U_{r-1} + U_{r+1} - 3U_r) = 0,$$

so that

$$U_{r+1} = 3U_r - U_{r-1}. \tag{4}$$

One notices at once that $a = 0$ leads, in (2), to the integral value $n = 1$. With this beginning, as many terms as are desired are readily computed from the recurrence relation (4). The argument that one thus obtains all the possible pairs of integral solutions of (1) can well be left to the elementary algebra classes.

Editorial Note. In a letter to the Editor, Dr. Weaver gives a further interesting side light on the source material for the present article. He writes: "I have been engaged some little time in studying and editing the mathematical manuscript material which was left by C. L. Dodgson (Lewis Carroll). A large box of mathematical manuscripts all carefully catalogued as to subject, was willed by Lewis Carroll to one of his friends and subsequently sold to Mr. M. L. Parrish of Philadelphia." One of these manuscripts is the inspiration for the present paper. R. E. G.

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* The smallest may be zero.

COLLINEATION GROUPS WHICH ARE TRANSITIVE ON THE RATIONAL POINTS OF A CONIC

G. B. HUFF, Southern Methodist University

In a note in the MONTHLY, William Sell* showed that there exists a linear homogeneous transformation group generated by four involutions which would send any integral solution of $A^2+B^2=C^2$ into any other integral solution. The editor noted (*l. c.*) that this had interesting geometric implications when A, B, C are interpreted as homogeneous coördinates in the plane. In that case the involutions were harmonic perspectivities, each one defined by a point and its polar line with respect to the conic. From this point of view the generators were not independent and it was noted that three involutions were sufficient to generate the whole set of rational points on the conic from any one of them. The set, however, was not symmetric and the question was raised as to the possibility of the existence of a symmetric set of generators.

To examine this question we will show that the study of the ternary group leaving a proper conic invariant may be reduced to the study of a binary group. From this point of view it will be clear that many sets of generators may be devised. None of those shown are symmetric, and it is shown that the entire group is not generated if (a) the center of each involution is on the axes of the other two, or if (b) the centers of the involutions lie by pairs on tangent lines of the conic.

1. We denote any proper conic with rational coefficients by $(xx)=0$ and let (ax) represent the first polar of a point a: (a_0, a_1, a_2) with respect to (xx). We seek to obtain a parametric representation of $(xx)=0$ and find the binary involution induced on the parameters by the harmonic perspectivity in a and its polar line $(ax)=0$.

If z is any point on $(xx)=0$ and if a is a point different from z, then the further intersection of $\lambda z+\mu a=0$ with $(xx)=0$ is given by $2\lambda(az)+\mu(aa)=0$. Thus that point, z', is determined by the equations:

$$z' = (aa)z - 2(az)a. \tag{1}$$

Equations (1) may be interpreted in another way. If a is not on the conic, it can be the center of an involution. The image z' of a conic point z is found by joining it to a and finding the further point of intersection. Thus the equations of the plane involution in a are:

$$x' = (aa)x - 2(ax)a. \tag{1'}$$

Let w, y be two distinct rational points of the conic different from z. Then $\alpha w+\beta y$, where α and β are rational gives all rational points of the line joining w, y. The line joining any one of these to z meets the conic in a rational point. A particular point on the conic is determined uniquely by α, β only after the point 1, 1 is chosen on the line $\overline{wy}$. This can be done conveniently by letting

* William Sell: On integral solutions of $A^2+B^2=C^2$, this MONTHLY, vol. 43, 1936, pp. 481–3.

$(zx)=0$ meet $\overline{wy}$ where $\alpha=\beta$. This will be true if we write $a_{\alpha,\beta}\equiv\alpha(yz)w-\beta(wz)y$. Thus, $z_{\alpha,\beta}$, the point on the conic obtained by joining z to $a_{\alpha,\beta}$ is given by:

$$z_{\alpha,\beta} = (a_{\alpha,\beta}a_{\alpha,\beta})z - 2(a_{\alpha,\beta}z)a_{\alpha,\beta}. \tag{2}$$

Evaluating the symbols we have:

LEMMA I. *If w, y, z are three distinct rational points on the proper conic with rational coefficients $(xx)=0$, that conic has the parametric representation*

$$z_{\alpha,\beta} = (\alpha^2-\alpha\beta)(yz)w + (\beta^2-\alpha\beta)(wz)y + \alpha\beta(wy)z, \tag{3}$$

where w, y, z are the points $(1, 0)$, $(0, 1)$, $(1, 1)$, respectively. As α, β run over all relatively prime integral values, this gives all the rational points on the conic.

The plane involution T_a with center at a and equations (1′) permutes the points of the conic and hence induces a transformation on the parameters α, β. To obtain the equations of this involution we note that it is completely defined by its fixed points which are the points in which the line $(ax)=0$ meets the conic. Substituting $z_{\alpha,\beta}$ in $(ax)=0$, we obtain

$$(\alpha^2-\alpha\beta)(yz)(aw) + (\beta^2-\alpha\beta)(wz)(ay) + \alpha\beta(wy)(az) = 0,$$

or

$$(yz)(aw)\alpha^2 + [(wy)(az) - (aw)(yz) - (ay)(wz)]\alpha\beta + (wz)(ay)\beta^2 = 0.$$

From the classical theory, we know that

$$t: \quad \begin{aligned} \alpha' &= -b\alpha - 2c\beta, \\ \beta' &= 2a\alpha + b\beta, \end{aligned}$$

is the involution whose fixed points are determined by

$$a\alpha^2 + b\alpha\beta + c\beta^2 = 0.$$

Hence T_a corresponds to the binary involution t_a:

$$t_a: \quad \begin{aligned} \alpha' &= -[(wy)(az) - (aw)(yz) - (ay)(wz)]\alpha - 2(wz)(ay)\beta, \\ \beta' &= 2(aw)(yz)\alpha + [(wy)(az) - (aw)(yz) - (ay)(wz)]\beta. \end{aligned} \tag{4}$$

If $a=\lambda(yz)w+\mu(wz)y+\nu(wy)z$, this takes the form:

$$t_a: \quad \begin{aligned} \alpha' &= -[-2\nu(wz)(wy)(yz)]\alpha - 2(wz)[\lambda(yz)(wy) + \nu(wy)(yz)]\beta, \\ \beta' &= 2(yz)[\mu(wz)(wy) + \nu(wy)(wz)]\alpha + [-2\nu(wz)(wy)(yz)]\beta. \end{aligned}$$

LEMMA II. *If $(xx)=0$ is given the parametric form (3), and if a is written*

$$a = \lambda(yz)w + \mu(wz)y + \nu(wy)z$$

then the binary involution corresponding to T_a is:

$$t_a: \quad \begin{aligned} \alpha' &= \nu\alpha - (\lambda+\nu)\beta, \\ \beta' &= (\mu+\nu)\alpha - \nu\beta, \end{aligned} \qquad \Delta = \begin{vmatrix} \nu & -(\lambda+\nu) \\ (\mu+\nu) & -\nu \end{vmatrix}. \tag{5}$$

The determinant $\Delta=\lambda\mu+\lambda\nu+\mu\nu$ cannot vanish since this is the condition that a be on $(xx)=0$.

2. We may now obtain a set of generators for the ternary group by devising a set of binary involutions which will generate all the pairs of integers α, β from any given pair. Consider the set of binary involutions:

$$(6)\qquad t_1:\ \begin{matrix}\alpha'=\alpha-2\beta,\\ \beta'=-\beta;\end{matrix}\qquad t_2:\ \begin{matrix}\alpha'=\alpha+\beta,\\ \beta'=\alpha-\beta;\end{matrix}\qquad t_3:\ \begin{matrix}\alpha'=\beta,\\ \beta'=\alpha.\end{matrix}$$

The group generated by these contains $t_4=t_2t_3t_2$ which has the equations:

$$t_4:\ \begin{matrix}\alpha'=-\alpha,\\ \beta'=\beta.\end{matrix}$$

It is evident that (0,1), (1, 1), (1, −1), (1, 0) are conjugate under these elements. Moreover t_3 and t_4 can reduce any other point to one such that $\alpha>\beta>0$. Under t_1 this goes into a point such that $|\alpha'|<\alpha$, $|\beta'|=\beta$. This new point is either one of the first four or it may be reduced again. Thus the set of generators (6) will generate a group transitive on the rational points of the conic. The involution (5) takes the form of t_1, t_2, t_3 for $(\lambda, \mu, \nu)=(1, -1, 1)$, $(-2, 0, 1)$, $(1, -1, 0)$, respectively. Hence,

THEOREM I. *If $(xx)=0$ is any proper conic with rational coefficients, and if w, y, z are any rational points on the conic, then all the rational points of the conic are generated from any one by the plane involutions with centers at $a_i=\lambda_i(yz)w+\mu_i(wz)y+\nu_i(wy)z$, $(i=1, 2, 3)$, where $\lambda_i, \mu_i, \nu_i=1, -1, 1; -2, 0, 1; 1, -1, 0$.*

Other sets of generators may be devised in a similar manner. This particular set is used because it is simple and because the ternary generators reduce to those given by Sell for $(xx)=x_1^2+x_2^2-x_0^2$, and $w:(1, 0, 1)$, $y:(1, 0, -1)$, $z:(0, 1, 1)$.

3. The generators obtained in **2** are not symmetric. The center of T_1 is always on the axis of T_2 and the line joining the centers of T_2 and T_3 is tangent to the conic. Let us consider the cases where the two types of symmetry hold throughout.

When the center of each involution is on the axes of the other two, then any one is the product of the other two and the group generated is the "four group." Let a_1, a_2, a_3 be at the vertices of a self polar triangle and choose w, y as the points of tangency from a_1; choose z as either point of tangency from a_2. Then the values of λ, μ, ν determining a_1, a_2, a_3 may be computed and turn out to be:

$$a_1:\ (1, 1, -1);\quad a_2:\ (1, -1, 0);\quad a_3:\ (1, 1, 0).$$

Substituting these in (5) yields the binary involutions:

$$t_1:\ \begin{matrix}\alpha'=-\alpha,\\ \beta'=\beta;\end{matrix}\qquad t_2:\ \begin{matrix}\alpha'=\beta,\\ \beta'=\alpha;\end{matrix}\qquad t_3:\ \begin{matrix}\alpha'=-\beta,\\ \beta'=\alpha.\end{matrix}$$

Hence, we have the following:

THEOREM II. *If the centers of three plane involutions are at the vertices of a self-polar triangle of the conic* $(xx)=0$ *the group generated is the four group.*

4. Consider the case where the generating involutions are symmetric in that the centers lie in pairs on tangent lines of the conic. In this case the axes form an inscribed triangle with vertices at the points of tangency.

Choose w, y, z as the points of tangency. Then the points a_1, a_2, a_3 are given by the values of λ, μ, ν as follows:

$$a_1\colon\ (-1,1,1); \qquad a_2\colon\ (1,-1,1); \qquad a_3\colon\ (1,1,-1).$$

Substituting these in (5) yields:

$$t_1\colon\ \begin{aligned}\alpha' &= \alpha,\\ \beta' &= 2\alpha-\beta;\end{aligned} \qquad t_2\colon\ \begin{aligned}\alpha' &= \alpha-2\beta,\\ \beta' &= -\beta;\end{aligned} \qquad t_3\colon\ \begin{aligned}\alpha' &= -\alpha,\\ \beta' &= \beta.\end{aligned}$$

Evidently these can not generate the entire group since the parity of both α, β is invariant under each of these. By an argument similar to that in **2,** it may be shown that any characteristic other than (0, 1), (1, 1), (1, 0) may always be reduced to one of these. Since these are the points w, y, z (Lemma I), we may state the result:

THEOREM III. *If the centers of three involutions lie by pairs on tangent lines of the conic* $(xx)=0$, *the group generated is an infinite group which divides the rational points of the conic into three distinct conjugate sets. Each conjugate set contains one point of tangency and no point in one set is conjugate to a point in another set.*

The structure of this group is very interesting. For instance, any pair of the involutions generates an infinite dihedral group of the sort studied by Barber.*

5. The interesting point in this is that binary transformations are used to obtain results concerning plane collineations. The method may be used to answer other questions that may be raised in connection with the group of all collineations leaving a conic unaltered. The generating involutions given in **2** are unsymmetric exactly like those given by Sell and indeed are projective to them. By following the method of **2,** other sets may be devised which are not projective under rational transformations to these. No set, however, can exist which possesses either type of symmetry throughout.

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* S. F. Barber, Planar Cremona transformations, American Journal of Mathematics, vol. 56, 1933, p. 120.

RECENT PUBLICATIONS

EDITED BY TOMLINSON FORT, Lehigh University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Actualités Scientifiques et Industrielles, Paris, Hermann, 1937. The following additional titles have been received. See this MONTHLY, vol. 42, 1935, pp. 40–41, and vol. 44, 1937, pages. 373–374, for earlier lists of titles in this series.

(Économique Rationnelle, exposés publiés sous la direction de G. et E. Guillaume, Nos. 1–5.)

504. *Méthode. L'économique élevée au rang d'une science exacte fondée axiomatiquement. Les conditions de prévisions pratiques.* 44 pages. 5f.

505. *L'économie pure. Le domaine des mécanismes élémentaires. Formation des prix et des salaries.* 76 pages. 8f.

506. *L'économie pure en interférence avec le domaine juridique. Mécanisme de la formation des bénéfices, des déficiences du pouvoir d'achat et des crises.* 55 pages. 6f.

507. *Le domaine économique-juridique national en interférence avec des mondes étrangers.* 59 pages. 6f.

508. *Modèles mathématiques des mondes économiques.* 136 pages. 15f.

Statistical Methods. Applied to Experiments in Agriculture and Biology. By G. W. Snedecor. Ames, Iowa, Collegiate Press, 1937. 13+341 pages. $3.75.

REVIEWS

Mathematical Snack Bar. By Norman Alliston. Cambridge, England, W. Heffer and Sons. Ltd., Distributed by Chemical Publishing Co. of N. Y. Inc., 1936. 7+155 pages. $3.00.

The book, as the author indicates in the preface, is a collection of notes and results obtained over a period of some twenty-five years of roving investigation. Modestly no claim is made either to great originality or to new results. The range of topics treated is wide including chapters devoted to the discussion of the triangle, quadrilateral, polygons in general, pyramids, ways of making a biquadratic expression square, residues, primes, Fermat problems, problems in Diophantine analysis, etc. Of these the ones dealing with triangles, quadrilaterals, and biquadratics I found the most interesting. One new point associated with the triangle is introduced and called the Tips Centre. The book is full of results and scattered bits of information; truly it is a mathematical snack bar.

But the most unusual thing about this volume is its striking language. If the mathematical investigator of today will consider the problem for just a moment he will realize that scientific language in general has now reached the point where it contains much more rigor than rhythm, more clarity than color,

more verity than vigor, is much more propositional than persuasive, indeed is more logical than linguistical. Not so with the language of this book. Its structure is so curious and so condensed as to be very awkard so far as mathematical thought is concerned; occasionally, when read aloud, it sounds almost like a translation. At times it is quite non-understandable.

Although it will be somewhat difficult to use because of its language and also because of its utter lack of formality with regard to statements concerning the setting and background of problems and with regard to statements of definitions, theorems, proofs, etc., yet this little book may well prove to be an interesting and useful reference. It may be thought of as a companion volume to such books as Johnson's *Modern Geometry* and Carmichael's *Theory of Numbers*.

C. O. OAKLEY

Introduction to College Algebra. By William L. Hart. New York, D. C. Heath and Company, 1937. 5+246 pages. $1.84.

This text is intended for use as a three-hour semester course for college freshman who did not take advanced algebra in high school. It omits the following topics which are included in the author's *College Algebra*: mathematical induction, inequalities, complex numbers, permutations and combinations, probability, partial fractions, and infinite series. Determinants of the second and third order are introduced in connection with the solution of linear equations, and there is a paragraph on compound interest in the chapter on logarithms. The chapter on the theory of equations does not include Horner's Method or the algebraic solutions of the cubic and the quartic. The appendix contains a four place table of logarithms, a table of powers and roots, and a compound interest table. Answers are given for the odd exercises.

L. T. MOORE

Plane Trigonometry. By Mills, Atkin and Flagg. Chicago, Scott, Foresman and Company, 1937. 12+170 pages. $1.60.

The authors of this text have listed in the introduction an imposing array of seventeen features, among which may be mentioned: easy transition to new ideas, geometrical approach, emphasis on the use of natural functions, interesting applications, and historical notes.

The arrangement of the material differs from that of most texts in this subject as will be seen by noting the order of the nine chapter headings. They are: introduction, indirect measurement, ratios involving the sides of a right triangle, functions of any angle, solving right triangles, solving the general triangle, fundamental trigonometric relations, trigonometric functions of any angle in terms of the functions of an acute angle, logarithms and their use, solving triangles by the use of logarithms, complex numbers, and DeMoivre's theorem. Since the chapter on solving the general triangle precedes the chapters on trigonometric analysis, the formulas for the tangents of the half angles are derived from the two formulas for the area, *viz.*, $K=\sqrt{s(s-a)(s-b)(s-c)}$ and

$K=rs$. For the former of these the student is referred to any plane geometry.

There are some ingenious geometric proofs, but it is doubtful if the geometric proof of the Law of Tangents is as valuable to the student and at the same time as easy as the analytic derivation from the Law of Sines. The numerical problems are designed for use of the four-place tables included in the text in which the natural and the logarithmic functions of the angle are given at intervals of one degree.

At the beginning of Chapter V the authors very properly call attention to the two-fold nature of trigonometry, *i.e.*, the solution of triangles and trigonometric analysis, and they point out the importance of the latter phase of the subject for further study in mathematics. However, in the opinion of the reviewer, too much emphasis is placed on the former which may be called the arithmetic of trigonometry and too little on the latter, the algebra of trigonometry. The problems in identities and equations in both the direct functions and the inverse functions are too few and too easy to give the student necessary power for later work in calculus, the physical sciences, and engineering. This criticism is made in view of the fact that present-day instruction in elementary algebra includes the sine and tangent of acute angles, the use of tables, and the solution of right triangles by means of logarithms.

The book has an attractive appearance, the figures are especially well drawn, and the applications are interesting and wisely chosen, due importance being given to vector diagrams. The historical notes are very appropriate and the chapter on complex numbers deserves mention. It is unfortunate that many classes in trigonometry do not have sufficient time to include this topic.

J. I. TRACEY

Plane Trigonometry. By W. K. Morrill. New York, Farrar and Rinehart, 1938. 9+167 pages. $1.60.

This book was written primarily to provide a brief course enabling a student "after six or eight weeks of study, to proceed with analytic geometry and the calculus." However, starred sections are added for use in a more ambitious course. Thus most of the usual topics are included, except complex numbers.

The purpose of the author has been well carried out. Especially commendable are the exercises of which there are more than a thousand, with more difficult problems at the ends of most lists. The answers to the odd-numbered exercises are provided. There are no tables, but the explanations cover tables of different types. In this connection there is a misprint in the fourth line of the footnote on page 36.

The author uses the $\equiv$ symbol for identities. Example 3 on page 66 is open to criticism. Surely, in proving an identity, the left- and right-hand sides should not be stated to be identical until after it has been proved that each is equal to the same expression. The formulas for sine and cosine of a sum are proved only for acute angles, but the exercises suggest some extensions, and the formulas are stated to be true for all positive and negative angles. (Zero ?)

The book is attractively printed and bound and will be found to be well adapted for use in many courses.

J. W. CLAWSON

Plane Trigonometry. With Tables. By F. W. Sparks and P. K. Rees. New York, Prentice-Hall, 1937. 10+161+76 pages. $2.50.

The authors present the subject in a terse, direct style with clear but not elaborate explanations. The functions are defined at the start for angles of any magnitude in standard position. All of the usual topics are discussed, including inverse functions and complex numbers. The formulas for the cosine and sine of the sum of two angles are proved only for the case when each angle and the sum is acute; but it is stated that the validity for all angles may be established by similar arguments. The derivation of the dependent formulas is especially good. The reason why the double sign is omitted in the formula $\tan(\theta/2) = (1-\cos\theta)/\sin\theta$, is given on page 71, a matter often slurred over.

The reviewer does not always like the treatment of identities. In a worked-out example on page 21, the authors start with an identity to be proved, and, after several manipulations on both sides, end the treatment with the statement "Therefore $\sin^2 x = \sin^2 x$." Under inverse functions on pages 125 and 126, the same criticism applies.

The problems are adequate, except that many teachers would prefer to have more numerical problems with the parts of triangles given to four or five significant figures instead of two or three. Five-place logarithm tables and the answers to the even-numbered problems are provided. Unimportant misprints were discovered on pages 51 and 87; but the typography throughout is admirable.

J. W. CLAWSON

Analytic Geometry and Calculus. By M. Morris and O. E. Brown. New York, McGraw-Hill Book Company, 1937. 10+507 pages. $3.75.

The authors have adopted the point of view that a first course in analytical geometry should be a preparation for the calculus. To this end their choice of material is excellent. In the 104 pages of plane and 35 pages of solid analytical geometry there are only to be found those topics which the student will need in his work in calculus and elementary differential equations. There is nothing novel in the presentation, but the exposition is clear and the style attractive.

The first chapter of the part of the book dealing with the calculus is headed: Variables, Functions, Limits, and Continuity. The discussion of these topics is adequate and comprehensive. It would be difficult to improve on the method and illustrations used in developing the concept of limit. Detailed proofs are given of such theorems on limits as: *The limit of the product of two functions of a common variable is equal to the product of the limits.* In the reviewer's opinion the student has not, at the beginning of a first course in the calculus, the maturity to appreciate a careful proof of these theorems. But if at this stage proofs are

given, those worked out by the authors of the book under review are as readable and convincing as one could hope for.

In dealing with derivatives the notation $f'(x)$ is used until the introduction of the differential. This procedure is none other than common sense dictates, but at this point far too few authors use common sense. There is a short and adequate section on infinitesimals concluding with a proof of Duhamel's theorem. This theorem involves two sets of infinitesimals, u_n and v_n, and an essential condition for the theorem to hold is that for a given $\epsilon>0$, $\left|v_n/u_n-1\right|<\epsilon$ for all n sufficiently great. The authors choose to state this in the equivalent form $\left|(u_n-v_n)/u_n\right|<\epsilon$ for all n sufficiently great. To the immature student the second form will seem artificial and mystifying. It is better to state the condition in the first form and change it over into the second as the proof progresses, especially since the theorem is later applied (p. 311) in terms of the first form.

Considered from the point of view of a first course, the introduction of the definite integral and the fundamental theorem of the integral calculus is satisfactory. The chapter on infinite series is especially well done. Mention should be made of an interesting section on curves, derived curves, and second derived curves, the relations among them, and their geometric significance. In addition to the material usually included in a first course, the book contains such topics as Rolle's Theorem, Theorem of the Mean, L' Hospital's Rule, Taylor's Theorem for a function of two variables, and some of the more delicate tests for the convergence of a series.

The atmosphere of the book is mature; but the style is appealing, and the whole set-up gives evidence of considerable class room seasoning and disillusionment, which makes it safe for a first course.

R. L. JEFFERY

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 462 East Foster Ave., State College, Pa.

BOOKS FOR CLUBS

This list is a continuation from previous numbers of the MONTHLY; see December 1937 issue, and January and March of the present year.

17. *The Frozen Fountain.* By Claude F. Bragdon. New York, Knopf, 1932. 125 pages. $3.75. One of the best in a series by this author, dealing with the art of design in space and its application in architecture.
18. *A Primer of Higher Space.* By Claude F. Bragdon. New York, Knopf, 1923. 2nd revised edition. $2.00. Shows the use of projections of four dimensional regular solids in designs in architecture.
19. *The New Image.* By Claude F. Bragdon. New York, Knopf, 1928. 190 pages. $3.00.

20. *Brain Teasers.* By Philip J. Rulon. Boston, L. C. Page and Co., 1932. 250 pages. $1.50. Contains many mathematical amusement problems.
21. *Numbergrams.* By Norman F. Sparhawk. Boston, Van Press, 1932. 50 pages. $1.25. Arithmetic problems to be solved by substituting numbers for letters.
22. *Puzzles in Wood.* By Edwin M. Wyatt. Milwaukee, Bruce Publ. Co., 1928. 64 pages. $0.75. Directions for making puzzles, most of them mathematical, out of wood or cardboard.
23. *Historical Introduction to Mathematical Literature.* By G. A. Miller. New York, Macmillan, 1916. A guide book to the mathematical world before 1916. Includes the status of living mathematicians, mathematical societies and journals, international congresses, encyclopedias, a history of tables from the time of Ahmes, errors, and a bibliography.
24. *The Foundations of Science.* By Henri Poincaré. New York, The Science Press, 1921. Introduction by Josiah Royce. The chapters include: On the Nature of Mathematical Reasoning, Mathematical Magnitude, and Experience, The Non-euclidean Geometries, Space and Geometry, Relative Motion and Absolute Motion, Energy and Thermodynamics, The Calculus of Probabilities, Intuition and Logic in Mathematics, Mathematical Creation, and others.
25. *Science and the Human Temperament.* By Edwin Schroedinger. Norton, 1935. A collection of popular lectures, including an unusually illuminating and dignified discussion of the place of the scientist in a civilized society.
26. *Queen of the Sciences.* By E. T. Bell. Baltimore, Williams and Wilkins, 1931, $1.00. A somewhat conversational discussion of some of the foundations, frontiers, and fundamentals of modern mathematics.

CLUB REPORTS

1936–37

Junior Mathematical Club, University of Chicago

President, M. F. Smiley; Program Chairman, L. Green; Treasurer, H. Chatland; Social Chairmen, Miss M. B. Haberzetle and I. M. Niven. The following talks were given by graduate students: The hypothesis of the continuum; Generalized projective geometry; Positive ternary quadratic forms; Modular invariants; Order relations; Matrix approximations and generalized least square lines; Euclid's parallel postulate; Functions of matrices; Structure theorems of algebras; Hermitian series; A generalization of groups; Volterra's mathematical treatment of the struggle for existence. Besides these, the following lectures were also delivered: "A test for the convergence or divergence of a series of positive terms," by Professor T. Vijayaraghavan of the University of Dacca; "Some modern results in the theory of transcendental numbers," by Dr. G. C. Webber of the Armour Institute of Technology; "Some new mathematical problems in biophysics," by Professor N. Rashevsky of the University of Chicago. Besides the customary afternoon teas held for a half-hour prior to each meeting, the club also sponsored a number of bridge parties during the year.

Mathematics Club, Smith College

Secretary, Patricia A. Taggart. Topics discussed included: Origin of our numerals; Magic squares; Location of complex roots of polynomials; The Japanese bead-tray computator; A review of Stephen Leacock's article, *Humor thought in mathematics.* Four members of the Club were elected to Sigma Xi and of these, three were also elected to Phi Beta Kappa.

Pi Mu Epsilon, Lehigh University

President, C. H. Hoffman; Secretary-Treasurer, D. B. Wheeler; Director, Professor D. H. Lehmer. At the regular monthly meetings student members led the mathematical discussions which were frequently followed by talks by faculty members. The chapter sponsored a lecture by Professor F. W. Owens, Pennsylvania State College, on "Unusual coordinate systems" and a talk by Dr. W. A. Menisen of Bell Telephone Laboratories, inventor of the crystal clock, on "Precise time measurement." One evening the club members were guests at the home of Professor Tomlinson Fort. The year closed with a formal initiation banquet at which Dr. P. M. Palmer, Dean of Arts and Science College, was the principal speaker.

Mathematics Club, Oshkosh State Teachers College

President, A. Womaski; Vice-President, Dorothy Lindgren; Secretary, M. Batterman; Treasurer, K. Dornstreich; Faculty Advisers, Dr. M. M. Beenken, Dr. Irene Price. Monthly meetings were held with papers by student members and enlivened by mathematical recreations. Topics presented included: The International Congress; Egypt and the Egyptian pyramids; History of analytic geometry; Chinese contributions to mathematics; Finger printing; Educational value of mathematics; The San Francisco bridge; Mathematical pitfalls. The graduating members of the club were entertained at a mathematical party by the faculty advisers and the year closed with a successful picnic for all.

Mathematics Club, George Washington University

President, A. M. DuPrez, Jr.; Secretary-Treasurer, Marion M. Sandomire; Faculty Adviser, Dr. F. E. Johnston. Topics discussed included: Theoretical economics of consumer behavior; Some assumptions of geometry; Polygonal numbers; History of Fermat's last theorem; Application of mathematics in atmospheric optics; Star spectra; A new method of enumeration; Equi-long transformations and their geometry; Bessel's functions; Spheres related to the tetrahedron; Nomographs. Dr. Johnston gave an exhibition of rare books on mathematics together with some modern expositions of Galois's theory of groups and of non-euclidean geometry. The social events included a Christmas party and a spring picnic as guests of Dr. and Mrs. J. H. Taylor at Long Beach, Maryland.

Kappa Mu Epsilon, Alabama State Teachers College, Florence

President, B. Simmons; Vice-President, F. Riley; Secretaries, Martha Howland, Mary A. Richeson; Treasurer, Virginia Little; Faculty Adviser, Professor Orpha A. Culmer. This chapter in its second year's work concentrated on astronomy, sponsoring a lecture by Professor A. M. Harding, University of Arkansas. A picnic and an anniversary banquet were the chief social events.

Pi Mu Epsilon, University of Oklahoma

Director, Dr. J. C. Brixey; Vice-Director, J. Boler; Secretary, W. Kitchens; Treasurer, J. Long; Corresponding Secretary, Dr. Dora McFarland. The chapter sponsored the meetings of the University Mathematics Club. Topics included: Art and mathematics; Mathematical amusements; Numerology; Review of *Men of Mathematics*; Logarithms. The programs were designed to be of interest to the general public. At the initiation dinner talks were given on "Number systems" by Dr. S. B. Townes and on "Problems of a mathematics teacher" by L. Dwight. The chapter held its third annual mathematics prize contest which is restricted to students who have not entered their senior year.

Mathematics Club, Butler University

President, J. St. Helens; Vice-President, Mildred Rugenstein; Secretary, Margaret Stemp; Treasurer, J. Thurston. At the monthly meetings topics discussed included: Mathematical fallacies; Number systems; Mathematics in the United States; The calendar; Astronomy. Dr. B. C. Getchell gave a talk on "Personal experiences of the solar eclipse expedition in Maine." The Christmas party and a picnic were enjoyable social events.

Pi Mu Epsilon, University of Pennsylvania

President, D. L. Herr. This chapter divides its activities into three parts: First, monthly meetings at which brief papers are read by student members on some mathematical problems, either pure mathematics or some of its applications to other scientific fields; second, bi-monthly prize contests, open to the entire university body, which include solution of original problems chosen with reference to second and third year students; third, prizes awarded to the best student paper, presented each semester at a chapter meeting.

Pi Mu Epsilon, University of Missouri

This chapter, one of the oldest in the organization, offers an annual prize of $10.00 to the student, not yet a senior, who receives the highest mark in a special examination covering algebra, trigonometry, analytic geometry, and differential calculus. The meetings are devoted to subjects of pure and applied mathematics with one social meeting in addition to the annual initiation banquet.

Mathematics Club, University of Iowa

President, R. Little; Secretary, Margaret Kampmeier; Faculty Adviser, Professor N. B. Conkwright. Monthly meetings were held throughout the year and the following topics discussed: Conormal points on a parabola; A special one to one correspondence; Centro-surface of the ellipsoid; Some topics in the theory of numbers: Algebra and calculus of matrices; Mathematical recreations. The annual mathematics dinner for the senior class, graduate students, and faculty was the principal social event of the year.

Pi Mu Epsilon, Syracuse University

This is the original chapter of Pi Mu Epsilon. The record of their activities has been continuous since 1908. During the last year they have had various lectures by faculty members, a Christmas party, and the annual initiation banquet. They are laying plans for a more general program with students presenting most of the papers.

Mathematics Club, Connecticut College

Monthly meetings were held. Topics included: Building and loan associations; Theory of investment; Short cuts in mathematical computation; Curves; Works of Sir Isaac Newton. Marian Bills, Aetna Life Insurance Company, led a panel discussion on "Opportunities for women in life insurance work."

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 327. *Proposed by Philip Franklin, Massachusetts Institute of Technology.*

It was shown by Fermat that, for any number k, it is possible to find k distinct right triangles, each with integral sides, having the same area. Find sets for $k=1, 2, 3, 4, 5$, having as common areas $6, 210, 840, 341880 = 2^3 \cdot 3 \cdot 5 \cdot 7 \cdot 11 \cdot 37$,

and $37383746400 = 2^5 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13^2 \cdot 19$ respectively. Are these the smallest possible values?

E 328. *Proposed by C. A. Murray, West Texas State Teachers College.*

In the equation, $y = x^3 + px^2 + qx$, show how to determine all pairs of integral values of p and q, for which the equation $y = 0$ will have distinct integral roots, and the two bend points integral coördinates.

E 329. *Proposed by V. Thébault, Le Mans, France.*

How may we locate two points on the sides of a triangle so that the segment joining them may divide the area in a given ratio and have a maximum or minimum length?

E 330. *Proposed by Fred Discepoli, New York.*

Find a triangular number whose digit pattern in the decimal system is *aabb*, and determine whether or not the solution is unique.

E 331. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

Determine a triangle whose six angle bisectors, three altitudes, five radii, and area are rational numbers.

E 332. *Proposed by J. Rosenbaum, Bloomfield, Connecticut.*

Prove that $1^p + 2^p - 3^p + 4^p - 5^p - 6^p + 7^p + 8^p - \cdots + (2^n - 1)^p = 0$ for every positive integer $p < n$, where the sign of a term of the form m^p is negative or positive according as m is or is not one of the 2^k integers following 2^{k+1}, where k is any integer.

SOLUTIONS

E 253 [1937, 49; 1937, 480]. *Proposed by V. Thébault, Le Mans, France.*

Find two positive integers differing by five, with the sum of their squares a perfect cube, and show that the solution is unique.

Comment by C. W. Trigg, Eagle Rock High School, Los Angeles.

On page 24, and in the subsequent theory, of his *Diophantine Analysis* Carmichael shows that the only solutions of $x^2 + y^2 = z^3$ are given by

(1) $\quad x = m^3 + mn^2, \quad y = m^2n + n^3, \quad z = m^2 + n^2$

(2) $\quad x = m^3 - 3mn^2, \quad y = 3m^2n - n^3, \quad z = m^2 + n^2.$

If in (1) we set $x + 5 = y$, we get $m(m^2 + n^2) + 5 = n(m^2 + n^2)$, whence $(m^2 + n^2)(n - m) = 5$. Now $m \neq n$, so $n = 2$, $m = 1$, $x = 5$, $y = 10$, and $z = 5$. If we use $x - 5 = y$ in (2), we get $m(m^2 - 3n^2) - 5 = n(3m^2 - n^2)$, so that $(m + n)(m^2 + n^2 - 4mn) = 5$. Now, $m + n \neq 1$, so $m = 5 - n$. Hence $(5 - n)^2 + n^2 - 4n(5 - n) = 1$. That is, $6(n - 4)(n - 1) = 0$. Since $n < m$, $n = 1$, $m = 4$, $x = 52$, $y = 47$, and $z = 17$.

If we set $x + 5 = y$ in (2), the roots of the resulting equation in n are irrational. If we set $x - 5 = y$ in (2) we get, as in a previous case, $n = 1$, $m = 4$, $x = 52$, $y = 47$,

$z=17$. Therefore the two solutions published in the August-September 1937 number of this MONTHLY, page 480, are the only ones.

E 294 [1937, 539]. *Proposed by V. Thébault, Le Mans, France.*

Show that the difference between the cubes of two successive positive integers can never be either the double or the triple of a perfect square.

Solution by "Dumb," Dartmouth College.

Since $(x+1)^3-x^3=3x(x+1)+1$, it is divisible by neither 2 nor 3, and hence not the double or the triple of any integer. The restricting of the statement of the problem to the double or triple of a perfect square seems superfluous.

Editorial Note. In his solution, Joel Brenner proved that the difference of two consecutive cubes could not be divisible by multiples of 2, 3, 5, 11, or 17, and could not be 13 times a perfect square. Fred Discepoli proved that if the difference of two consecutive cubes were a prime times a perfect square, then the prime had to be one more than a multiple of 3.

Also solved by N. A. Brigham, J. H. Butchart, J. H. Cross, Daniel Finkel, M. J. Gottlieb, V. W. Graham, Joseph Milkman, F. C. W. Olson, J. Rosenbaum, E. P. Starke, C. W. Trigg, Z. W. Wilchinsky, and the proposer.

E 295 [1937, 539]. *Proposed by P. W. Lonergan, DeWitt Clinton High School, New York.*

In preparing a board for Chinese checkers, a conventional six-pointed star is formed by placing an equilateral triangle upon another of equal size, so that the trisection points of the sides of one coincide with those of the other. Each side of each "point" is divided into four parts, and the complete figure is cut into small equilateral triangles by drawing through each marked point, lines parallel to the other sides of the figure. Find the number of small equilateral triangles, and the total number of equilateral triangles of all sizes, in the complete figure.

Solution by W. E. Buker, Pittsburgh, Pa.

Orient the star so that, as on a map, two of the points would be due north and south. We count only the triangles with vertex north, which will be one-half the total. The following figures represent the number of such triangles, whose sides are progressively 1, 2, 3, $\cdots$, 12 times the sides of the smallest triangle, with northward vertices:

$$96 + 75 + 58 + 45 + 36 + 28 + 21 + 15 + 10 + 6 + 3 + 1 = 394.$$

The total number of the smallest sized triangles is $2 \cdot 96 = 192$, and of triangles of all sizes is $2 \cdot 394 = 788$.

Also solved by C. W. Trigg.

E 296 [1937, 539]. *Proposed by D. L. MacKay, Evander Childs High School, New York.*

D, E, and F are the centers of the semicircles constructed on the sides BC, CA, and AB as diameters, and exterior to the triangle ABC. If d and e are the

lengths of the common external tangents between the points of contact for the semicircles D and E, and D and F, construct triangle ABC, given d, e, and angle A.

Solution by V. W. Graham, The Harcourt Street High School, Dublin, Ireland.

Let the incircle touch BC, CA, AB at L, M, and N respectively, so that $AN=s-a$, etc., with the usual notation. If we know the radius of the incircle the solution is obvious, for drawing the incircle we obtain N and M, and since $AN\cdot NB=d^2$, this gives B, and similarly for C.

We are given $(s-a)(s-b)=d^2$ and $(s-a)(s-c)=e^2$, so that $a(s-a)=d^2+e^2$. Therefore $s(s-a)=(s-a)^2+a(s-a)=(s-a)^2+d^2+e^2=r^2\cot^2\frac{1}{2}A+d^2+e^2$.

Now $rs=[s(s-a)(s-b)(s-c)]^{1/2}$, so that

$$r=de/[s(s-a)]^{1/2}=de/[r^2\cot^2\tfrac{1}{2}A+d^2+e^2]^{1/2}$$
$$=de\cdot\tan\tfrac{1}{2}A/[r^2+(d^2+e^2)\tan^2\tfrac{1}{2}A]^{1/2}.$$

Now let

$$de\cdot\tan\tfrac{1}{2}A=q^2 \text{ and } (d^2+e)\tan^2\tfrac{1}{2}A=p^2,$$

so that p and q are known. Then if $r^2+p^2=t^2$, we have $rt=q^2$, and we see that our problem has reduced to another standard one (see Dilworth, *A new Sequel to Euclid*, page 84; the Ed. Company of Ireland) namely—Given the difference of the squares of two segments and their rectangle or product, to find their lengths.

Also solved by J. H. Butchart, Joseph Milkman, E. P. Starke, and the proposer.

E 297 [1937, 539]. *Proposed by W. F. Cheney, Jr., Connecticut State College.*

When the proper fraction, WPA/USA, is reduced to lowest terms, it becomes UV/VU. Here each letter represents a digit in the decimal system. Determine the original fraction and show that the solution is unique.

Solution by C. W. Trigg, Eagle Rock High School, Los Angeles.

In the first place, neither W, U, nor V is zero. V and U are not both even, and $U<V$. Furthermore, $U+V$ is not a multiple of 3.

In the second place, A is not zero, for if it were, we would have WP/US reducing to UV/VU, but $US<VU$.

In the third place, $WPA=k(UV)$ and $USA=k(VU)$, so k can not end in 1, 3, 7 or 9. If k is even, the above restrictions limit the values possible for UV to 16, 38, and 49. If k ends with 5, neither U nor V is 5, and the possible values of UV are 13, 17, 19, 37, and 79.

In the fourth place, the hundred's digit of USA is the same as the units digit of VU. This final restriction readily eliminates all of the remaining possibilities above with the unique exception of $185/365=37/73$.

Also solved by W. E. Buker, Fred Discepoli, William Douglas, V. W. Graham, C. E. Snoke, Jr., E. P. Starke, B. C. Zimmerman, and the proposer.

E 298 [1937, 539]. *Proposed by C. T. Holmes, Brunswick, Me.*

As a first step in making a string model of the hyperboloid, $x^2/a^2+y^2/b^2-z^2/c^2=1$, we drill holes through two pieces of board along the curve in which the surface is cut by the plane $z=Z$, that is, the ellipse

$$(1) \qquad x^2/(ka)^2 + y^2/(kb)^2 = 1, \ z = Z, \quad \text{where} \quad k = \sqrt{(c^2+Z^2)}/c > 1.$$

The rulings must cut the plane $z=0$ in the ellipse

$$(2) \qquad x^2/a^2 + y^2/b^2 = 1, \ z = 0,$$

which is similar to (1). Hence the projections of these rulings on the plane $z=0$ are chords of (1) which are tangent to (2) at their midpoints.

If through a point P_1 on the ellipse (1) we draw the chord P_1P_2 tangent to (2), then through P_2 draw the chord P_2P_3 tangent to (2), then through P_3 the chord P_3P_4 tangent to (2), and so on, what is the condition that we return ultimately to the point P_1, so that we shall not be under the tiresome necessity of drilling infinitely many holes?

Solution by W. B. Campbell, Drexel Institute of Technology.

If, in the coplanar ellipses (1) and (2) (with Z replaced by 0), we multiply the ordinates by a/b, we obtain the circles

$$(3) \qquad x^2 + y^2 = (ka)^2,$$

$$(4) \qquad x^2 + y^2 = a^2,$$

and any chords of (1) tangent to (2) become chords of (3) tangent to (4), all of equal length. Each chord then subtends a central angle $2\phi<\pi$, where $\sec\phi=k$, $\tan\phi=Z/c$. A continuous broken line of such chords will return exactly to its initial point on (3) if and only if $2\phi/2\pi$ is rational. If the ratio is expressed as the quotient p/q of two relatively prime integers, the qth chord will be the first so to return, and similarly for a set of chords connected with the given coplanar ellipses, for any choice of initial point on (1). [The same conclusion is reached by writing (1) and (2) in the parametric form $x=a\cos\theta$, etc., and finding the constant difference in eccentric angles of the two points in which (1) is cut by a tangent at any given point on (2).]

In the space model, alternate chords end on ellipse (1) in the upper plane $z=+Z$, and the string returns to the initial point thereon after q trips if q is even, or after $2q$ trips if q is odd, each trip contributing a ruled line to the desired locus. If $2n$ such lines are used, there will be n holes on ellipse (1), the eccentric angles of an adjacent pair of holes differing by $2\pi/n$. The distance between holes will vary approximately from $kb\cdot\sin(2\pi/n)$ to $ka\cdot\sin(2\pi/n)$. The minimum value of q is 3, for which $p=1$, $\phi=\pi/3$, $Z=c\sqrt{3}$, but this requires 6 trips. For $p/q=1/4$, $\phi=\pi/4$, and the fourth trip returns to P_0. Values of p/q meeting the requirements include (1/3), (1/4), (1/5), (2/5), (1/6), (1/7), (2/7), (3/7), (1/8), (3/8), (1/9), (2/9), (4/9), (1/10), (3/10), (1/11), (2/11), (3/11), (4/11), (5/11), (1/12), (5/12), (1/13), $\cdots$.

Also solved by E. P. Starke, Maud Willey, and the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3868. *Proposed by Arnold Dresden, Swarthmore College.*

Prove that, if $0 \leqq \alpha \leqq x_1 \leqq x_2$ and n is a positive integer, then

$$x_2^{1/n} - x_1^{1/n} \leqq (x_2 - \alpha)^{1/n} - (x_1 - \alpha)^{1/n}.$$

3869. *Proposed by Otto Dunkel, Washington University.*

Two perpendicular planes are tangent to a paraboloid and their intersection has a given direction. Determine the form of the surface generated by the intersection as the pair of planes varies. This supplements problem 3773 [1936, 111] solved 1938, page 124.

3870. *Proposed by A. Moessner, Nürnberg-N, Germany.*

The equations

$$A_1^3 + A_2^3 + A_3^3 + A_4^3 = B + C + D,$$
$$A_1^6 + A_2^6 + A_3^6 + A_4^6 = B^2 + C^2 + D^2,$$
$$A_1^3 + A_2^3 + A_3^3 = A_4^3,$$

are satisfied by $A_1=3$, $A_2=4$, $A_3=5$, $A_4=6$, $B=91$, $C=152$, $D=189$. What is the general solution in integers?

3871. *Proposed by V. Thébault, Le Mans, France.*

Let O, I, I_a, I_b, I_c be the centers of the circumcircle, inscribed circle, and the escribed circles of a given triangle ABC; and let Q_a, Q_b, Q_c, Q_a', Q_b', Q_c' be the intersections of the sides of the triangle with the interior and exterior bisectors of its angles. The parallels to the Euler line through the orthogonal projections on $Q_a'Q_b'Q_c'$, Q_bQ_c, Q_cQ_a, Q_aQ_b of an arbitrarily chosen point of that line meet respectively OI, OI_a, OI_b, OI_c in M, M_a, M_b, M_c. Show that the sum of the powers of O with respect to the circles with diameters IM, I_aM_a, I_bM_b, I_cM_c is zero.

SOLUTIONS

3730 [1935, 177]. *Proposed by William Chisholm, Quincy, Mass.*

There is said to be a short and simple method of finding one solution x of each of the equations

$$x^5 - (m+n)(5x^2+5x+1) = (m+n)^2, \tag{1}$$

$$x^2+m^2+n^2-2(mn+mx+nx) = 8\sqrt{2mnx(x+m+n)}. \tag{2}$$

What is the method?

Solution of (2) *by V. W. Graham, The High School, Dublin, Ireland.*

The solution of (1) by F. Underwood was given in this MONTHLY, vol. 44, 1937, p. 111. For the solution of (2) set $x=y+m+n$, and there results

$$y^2 - 4mn = 8[2mn(y+m+n)(y+2m+2n)]^{1/2}. \tag{1}$$

Squaring each side and adding $16mny^2$ to each side of the resulting equation, we obtain

$$\begin{aligned}(y^2+4mn)^2 &= 16mn[3y+4(m+n)]^2,\\ y^2+4mn &= \pm 4\sqrt{mn}[3y+4(m+n)].\end{aligned} \tag{2}$$

The solution of these two quadratic equations gives

$$\begin{aligned}y &= 6\sqrt{mn} \pm 4\sqrt[4]{mn}(\sqrt{m}+\sqrt{n}),\\ y &= -6\sqrt{mn} \pm 4i\sqrt[4]{mn}(\sqrt{m}-\sqrt{n}).\end{aligned} \tag{3}$$

The values of x are then easily obtained.

3778 [1936, 245]. *Proposed by L. J. Adams, High School, Beverly Hills, Calif.*

Solve

$$\frac{dy}{dx} = \frac{1}{x} + \frac{1}{y}.$$

Solution by A. K. Waltz, Clarkson College, Potsdam, N. Y.

Set $x=e^u$, and the given equation becomes

$$y\left(\frac{dy}{du} - 1\right) = e^u. \tag{1}$$

Assume a solution of (1) in the form

$$y = \sum_{n=0}^{\infty} a_n u^n = \sum_{n=0}^{\infty} a_n (\log x)^n. \tag{2}$$

After substitution in (1) we have

$$\sum_{n=0}^{\infty} a_n u^n \left[\sum_{n=1}^{\infty} n a_n u^{n-1} - 1\right] = \sum_{n=0}^{\infty} u^n/n!.$$

Equating coefficients of the same powers of u on each side, we find

$$\begin{aligned}(n+1)\sum_{i=0}^{n/2} a_i a_{n+1-i} - a_n &= \frac{1}{n!}, \quad n \text{ even};\\ (n+1)\sum_{i=0}^{(n-1)/2} a_i a_{n+1-i} - a_n + \frac{n+1}{2}a_{(n+1)/2}^2 &= \frac{1}{n!}. \quad n \text{ odd}.\end{aligned} \tag{3}$$

The coefficients $a_1, a_2, \cdots$ in terms of a_0 may then be computed in succession. A general expression for a_n in terms of a_0 is not easily determined. The first three a's are

$$(4)\qquad \begin{aligned} &a_1 = 1 + 1/a_0 = 1 + k, \qquad a_2 = k(1 - k - k^2)/2!, \\ &a_3 = [k - k^2(1 - k - k^2)(2 + 3k)]/3!, \qquad a_0 \neq 0. \end{aligned}$$

The equation (1) may be written

$$\frac{dy}{du} = 1 + \frac{e^u}{y}.$$

The right side of this equation is analytic in u and y in the neighborhood of $u=0$ and $y=a_0\neq 0$. Hence there exists an integral $y(u)$ analytic in a neighborhood of $u=0$ which reduces to a_0 for $u=0$; and this integral may be obtained by the above method. (See Goursat-Hedrick's *Mathematical Analysis*, vol. 2, part II, page 45.)

The radius of convergence of the integral is not easily found, but it is greater than or equal to the radius given in (9) page 47 of the same reference.

3781 [1936, 245]. *Proposed by Otto Dunkel, Washington University.*

Evaluate

$$\sum_{k=1}^{n} \frac{\cos m(2k - 1)\pi/n}{\sin^2 (2k - 1)\pi/2n}$$

in terms of the integers m and n, $0 \leqq m \leqq n$; also the sum where the sine is in the numerator instead of the cosine.

Solution by the Proposer.

By the method of partial fractions we have

$$(1)\qquad \frac{x^m}{x^n + 1} = -\frac{1}{n}\sum_{k=1}^{n} \frac{x_k^{m+1}}{x - x_k}, \qquad 0 \leqq m < n,$$

where m and n are positive integers and x_k is a root of $x^n+1=0$. If $m=n$, the equation (1) is true after adding unity to the right side. After taking the derivative of each member of (1), and the equation for $m=n$, and then setting $x=1$, we have

$$(2)\qquad \frac{(2m - n)n}{4} = \sum_{k=1}^{n} \frac{x_k^{m+1}}{(1 - x_k)^2}, \qquad 0 \leqq m \leqq n.$$

We now transform (2) by means of the following equations:

$$(3)\qquad \begin{aligned} x_k &= e^{(2k-1)\pi i/n}, \qquad x_k^m + x_k^{-m} = 2\cos m(2k - 1)\pi/n, \\ &\qquad\qquad x_k^m - x_k^{-m} = 2i \sin m(2k - 1)\pi/n, \\ (1 - x_k)^2 &= -(1 - x_k)(1 - x_k^{-1})x_k = -[2 - (x_k + x_k^{-1})]x_k \\ &= -2[1 - \cos (2k - 1)\pi/n]x_k. \end{aligned}$$

We have first

$$\sum_{k=1}^{n} \frac{x_k^m}{1-\cos(2k-1)\pi/n} = \frac{n(n-2m)}{2}. \tag{4}$$

Since x_k^{-1} is also a root, we have another equation by replacing x_k in (4) by x_k^{-1}. The addition of these two equations gives

$$\begin{aligned} &\sum_{k=1}^{n} \frac{\cos m(2k-1)\pi/n}{1-\cos(2k-1)\pi/n} = \frac{n(n-2m)}{2}, \qquad 0 \leqq m \leqq n, \\ &\sum_{k=1}^{n} \frac{\cos m(2k-1)\pi/n}{\sin^2(2k-1)\pi/2n} = n(n-2m). \end{aligned} \tag{5}$$

We may combine the terms for k and $n-k+1$ and write

$$\sum_{k=1}^{[n/2]} \frac{\cos m(2k-1)\pi/n}{\sin^2(2k-1)\pi/2n} = \frac{n(n-2m)}{2} + \left\{\left[\frac{n}{2}\right] - \frac{n}{2}\right\}(-1)^m. \tag{6}$$

In a similar manner we get

$$\sum_{k=1}^{n} \frac{\sin m(2k-1)\pi/n}{1-\cos(2k-1)\pi/n} = 0. \tag{7}$$

The corresponding terms in (7) for k and $n-k+1$ cancel, and if there is a middle term it is zero.

We may also write

$$m = \frac{n}{2} - \frac{1}{n}\sum_{k=1}^{n} c_k^{(m)}, \qquad 0 = \sum_{k=1}^{n} s_k^{(m)}, \tag{8}$$

where $c_k^{(m)}$ is the summand in the first equation of (5), while $s_k^{(m)}$ is that in (7).

We shall now use the above results to prove the interpolation formula of Szegö mentioned in solution II of 3705 [1936, 246]. Let

$$\begin{aligned} f(\phi) &= A_0 + \sum_{m=1}^{n} (A_m \cos m\phi + B_m \sin m\phi), \\ g(\phi) &= \sum_{m=1}^{n} (A_m \sin m\phi - B_m \cos m\phi), \end{aligned} \tag{9}$$

where A_m, B_m are given constants. The theorem states that

$$g'(\phi) = \frac{n}{2} f(\phi) - \frac{1}{n}\sum_{k=1}^{n} \frac{f[\phi + (2k-1)\pi/n]}{1-\cos(2k-1)\pi/n}. \tag{10}$$

We have first by (8)

$$m\cos m\phi = \cos m\phi\left\{\frac{n}{2} - \frac{1}{n}\sum_{k=1}^{n} c_k^{(m)}\right\} + \sin m\phi\left\{\frac{1}{n}\sum_{k=1}^{n} s_k^{(m)}\right\}$$

$$= \frac{n}{2}\cos m\phi - \frac{1}{n}\sum_{k=1}^{n}\frac{\cos m[\phi + (2k-1)\pi/n]}{1-\cos(2k-1)\pi/n},$$

(11)

$$m\sin m\phi = \frac{n}{2}\sin m\phi - \frac{1}{n}\sum_{k=1}^{n}\frac{\sin m[\phi + (2k-1)\pi/n]}{1-\cos(2k-1)\pi/n},$$

$$0 = \frac{n}{2} - \frac{1}{n}\sum_{k=1}^{n}\frac{1}{1-\cos(2k-1)\pi/n}.$$

Then from

$$g'(\phi) = \sum_{m=1}^{n}(A_m m\cos m\phi + B_m m\sin m\phi)$$

and the three equations in (11) the desired result easily follows. We leave this final step to the reader since the equations are not easy to print.

3783 [1936, 309]. *Proposed by J. Barinaga, Madrid University.*

Show that

$$\begin{vmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2^2 & 3^2 & \cdots & n^2 \\ 1 & 2^3 & 3^3 & \cdots & n^3 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1 & 2^n & 3^n & \cdots & n^n \end{vmatrix} = 1!2!3!\cdots n!\left[\binom{n}{1} - \frac{1}{2}\binom{n}{2} + \frac{1}{3}\binom{n}{3} - \cdots + (-1)^{n+1}\frac{1}{n}\binom{n}{n}\right].$$

I. *Solution by J. A. Greenwood, Duke University.*

Consider the determinant of the nth order differing from the one in the problem only in the exponent, which is $l+i-2$ for the ith row, where l is a positive integer and $i=2, 3, \cdots, n$. Subtract the first column from each of the following. This results in a determinant of order $n-1$ which may be expanded in the form

$$\sum_{r=1}^{n}(-1)^{r+1}\frac{(n!)^l}{r^l}V(1, 2, \cdots, r-1, r+1, \cdots, n), \tag{1}$$

where V is the Vandermonde determinant

$$V(1, 2, \cdots, r-1, r+1, \cdots, n) = \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & r-1 & r+1 & \cdots & n \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot & \cdot & \cdots & \cdot \\ 1 & 2^{n-2} & \cdots & (r-1)^{n-2} & (r+1)^{n-2} & \cdots & n^{n-2} \end{vmatrix}$$

$$= \frac{(n-1)!(n-2)!\cdots 2!1!}{(r-1)!(n-r)!}.$$

For $r=1$ we have $V(2, 3, \cdots, n)$. The determinant considered then reduces to

$$1!2!\cdots(n-1)!(n!)^{l-1}\sum_{r=1}^{n}(-1)^{r+1}\binom{n}{r}\frac{1}{r^{l-1}}.$$

For $l=2$ we obtain the special case of the problem.

II. *Solution by Morgan Ward, California Institute of Technology.*

Let $F(x)$ denote the polynomial which is obtained from the determinant in the problem by replacing its first column by $1, x^2, x^3, \cdots, x^n$. The given determinant is then $F(1)$. We have first

$$F(0) = 1!2!\cdots n!n \tag{1}$$

by reductions similar to those used for Vandermonde determinants. Since $F(x)$ vanishes for $x=2, 3, \cdots, n$, we have

$$F(x) = A(x-2)(x-3)\cdots(x-n)(x-B), \tag{2}$$

where the constants A and B will now be determined. Taking the logarithmic derivative of the two members of (2), then setting $x=0$, and observing that $F'(0)=0$, we have

$$\frac{F'(0)}{F(0)} = -\sum_{i=2}^{n}\frac{1}{i} - \frac{1}{B} = 0. \tag{3}$$

Set $x=0$ in (2) and we have in turn

$$AB = (-1)^n 1!2!\cdots(n-1)!n,$$

$$A = (-1)^{n+1}1!2!\cdots(n-1)!n\sum_{i=2}^{n}\frac{1}{i},$$

$$F(x) = (-1)^{n+1}1!2!\cdots(n-1)!n(x-2)(x-3)\cdots(x-n)\left[x\sum_{i=2}^{n}\frac{1}{i}+1\right],$$

$$F(1) = 1!2!\cdots(n-1)!n!\sum_{i=1}^{n}\frac{1}{i}.$$

Since

$$\sum_{i=1}^{n}\frac{1}{i} = \sum_{i=1}^{n}(-1)^{i+1}\binom{n}{i}\frac{1}{i},$$

this completes the proof.

Solved also by Frank Ayres, Jr., and J. F. Locke.

Editorial Note. The proposer stated that the equality in the problem is a simple consequence of a formula given by L. Theisinger in the *Monatshefte für Mathematik und Physik*, vol. 26, 1915, p. 134. It will be observed that the second as well as the first solution generalizes the problem. If we set $x=n+1$ in $F(x)$ in the second solution, and make slight simplifications on each side, we obtain the evaluation of a determinant related to the one of the problem.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

The National Council of Teachers of Mathematics will have its fourth summer meeting with the N. E. A. in New York City on June 27, 28, 29. The general theme of the three day session will be "Mathematics that Functions." The general plan is as follows:

(1) "Progressive" Arithmetic Teaching, June 27, 2 p.m.
(2) "The Forgotten Pupil," a joint meeting with the Department of Secondary Education, June 27, 3 p.m.
(3) Discussion Luncheon, June 28, 12:30 p.m.
(4) "Progressive" High School Mathematics, June 28, 2:30 p.m.
(5) Mathematics as it Functions in Business and Industry, June 29, 2 p.m.

Details including speakers, places of meetings, reservations for luncheon will appear in the May issue of the *Mathematics Teacher*.

The symposium on the algebra of geometry and related subjects at the University of Notre Dame, announced in the January number of this MONTHLY, was held February 11 and 12, in five sessions. At the first session, on Friday morning, the following papers were presented: Boolean algebra and topology, by Professor M. H. Stone; Partially-ordered function spaces, by Mr. Garrett Birkhoff. At the session on Friday afternoon the following papers were presented: On structures and their applications, by Professor Oystein Ore; Applications of the lattice theory to field structure, by Dr. Saunders MacLane; Some problems concerning groups, by Dr. S. K. Senior. On Friday evening Professor E. V. Huntington gave a general lecture on the method of postulates, entitled: The duplicity of logic. On Saturday morning the following papers were presented: Continuous geometry, by Professor John von Neumann; Alt's algebra of affine geometry and related problems, by Professor Karl Menger; A theorem on the algebra of geometry, by Dr. A. N. Milgram. At the last session, on Saturday afternoon, the following papers were presented: An application of the theory of division algebras to geometry, by Professor A. A. Albert; An algebraic topic with geometric aspects, by Professor Emil Artin.

The Walker Ames Professorship of Mathematics represents an attempt on the part of the University of Washington to take advantage of the fine summer climate of Seattle to establish the latter as a center of mathematical activity in the northwest. Professor Weyl of the Institute for Advanced Study follows Professor Veblen, who initiated the Ames Professorship in the summer of 1937.

Assistant Professor R. G. Archibald of Columbia University has been appointed to an assistant professorship at Queens College, Flushing, N. Y.

Assistant Professor A. E. Brandt of Iowa State College is on leave of absence

this year. He is serving as senior statistician at the United States Bureau of Soils, Department of Agriculture.

Assistant Professor A. B. Brown of Columbia University has been appointed to an assistant professorship at Queens College, Flushing, N. Y.

Assistant Professor S. S. Cairns of Lehigh University has been appointed to an assistant professorship at Queens College, Flushing, N. Y.

Assistant Professor A. T. Craig of the University of Iowa has been promoted to an associate professorship.

Professor Peter Field of the University of Michigan has leave of absence for the second semester of 1937–38.

Associate professor C. A. Garabedian of Wheaton College, Massachusetts, has been promoted to a professorship.

Dr. Evan Johnson of the Uniontown Extension Center of the Pennsylvania State College has been promoted to an assistant professorship.

Professor Karl Menger of the University of Notre Dame has been made chairman of the department of mathematics.

Professors R. E. Moritz and G. I. Gavett of the mathematics department of the University of Washington will retire to half time teaching beginning with the fall quarter of the academic year 1938–39. Professor Moritz has served from 1904 to 1936 as executive secretary of the mathematics department. Professor Gavett has been a member of the department from 1907 to the present time.

Dr. D. C. Murdoch of the University of Toronto has been awarded a Sterling Fellowship at Yale University.

Dr. E. R. Ott has been promoted to an assistant professorship at the University of Buffalo.

Dr. E. L. Post of the College of the City of New York has been promoted to an assistant professorship.

Professor E. L. Rees of the University of Kentucky is on leave of absence the second semester 1937–38 and is traveling in South America.

Dr. W. O. Rogers of Pennsylvania State College has been promoted to an assistant professorship.

Professor J. A. Shohat of the University of Pennsylvania has been elected Fellow of the American Institute of Statistics.

Mrs. O. H. Stecker of the Pennsylvania State College has been promoted to an assistant professorship.

Assistant Professor J. M. West of the Pennsylvania State College has been promoted to an associate professorship.

Dr. R. B. Allen, professor emeritus of mathematics at Kenyon College, died March 4, 1938, at the age of sixty-six. He had taught at Kenyon College since 1906, retiring in June 1937. He was a charter member of the Mathematical Association.

Professor J. B. Meyer of the State Teachers College, Valley City, North Dakota, died on February 25, 1938. He had been a member of the Association since 1920.

SUMMER COURSES

The following courses in mathematics are announced for the summer of 1938.

Catholic University of America. In addition to the usual elementary courses the following advanced courses will be offered: By Dr. E. J. Finan: Theory of equations; Theory of groups. By Professor J. N. Rice: Analytic geometry of three dimensions; Calculus of observations. By Professor O. J. Ramler: College geometry; Differential equations; Synthetic projective geometry.

University of California at Los Angeles. In addition to the usual elementary courses the following advanced courses will be offered: By Dr. A. L. Foster: Advanced calculus. By Professor P. H. Daus: Elementary mathematics from an advanced standpoint. By Professor W. M. Whyburn: Calculus of variations. By Vice-President E. R. Hedrick: The teaching of mathematics.

University of Chicago. First term, June 17 to July 22; second term, July 23 to August 28. During the Summer Quarter of 1938 the department of mathematics at the University of Chicago will offer a well balanced program, but will place especial emphasis on algebra and the theory of numbers. A conference on algebra is to be held June 28 to July 1 at which a number of eminent mathematicians will present papers. Courses emphasizing some phase of algebra are: A seminar on topics in algebra, by Professor A. A. Albert and his associates; Modern higher algebra, by Professor Albert; Theory of algebraic numbers, by Dr. D. M. Dribin; Algebraic functions, by Professor G. A. Bliss; Continuous groups, by Dr. Nathan Jacobson; and Elementary theory of numbers, by Professor L. E. Dickson. In addition to the above courses and those in calculus, differential equations, and solid analytical geometry, the following courses will be given: By Professor W. T. Reid: Introduction to the theory of functions; Integral equations. By Professor M. R. Hestenes: Theory of functions of real variables. By Professor E. P. Lane: Metric differential geometry; Projective differential geometry.

University of Colorado. First term, June 20 to July 23; second term, July 25 to August 27. In addition to the usual elementary courses, the following advanced courses will be offered: By Professor A. J. Kempner: Advanced calculus (both terms); Theory of numbers (both terms); Mathematical recreations (first term). By Professor C. A. Hutchinson: Differential geometry (both terms). By Professor Claribel Kendall: Theory of equations (second term).

Columbia University. The graduate courses in mathematics offered in the summer session of 1938 are as follows: By Professor Kasner: Survey of modern mathematics; Curves and families of curves. By Professor Koopman: Differential equations. By Professor P. A. Smith: Introduction to higher algebra. By Professor Fite: Theory of functions of a complex variable.

Duke University. First term, June 13 to July 23. By Professor Elliott: Differential equations; Infinite series. By Professor Gergen: Elementary potential theory; Complex variable. By Professor Rankin: Integral calculus; Teaching of mathematics. By Professor Thomas: Modern algebra; Thesis seminar. Second term, July 26 to September 3. By Professor Carlitz: Number theory; Complex variable. By Professor Thomas: Solid analytic geometry; Modern algebra.

University of Illinois. In addition to the usual elementary courses the following advanced courses will be offered: By Professor H. W. Bailey: Advanced aspects of euclidean geometry; Advanced calculus. By Professor D. G. Bourgin: Applications of mathematics. By Dr. O. K. Bower: Elementary statistics—a course for beginners. By Professor A. R. Crathorne: Theory of statistics; Calculus of variations. By Professor Arnold Emch: Theory of equations; Non-euclidean geometry. By Professor H. Levy: Projective geometry; Theory of matrices. By Dr. C. W. Mendel: Complex variable.

University of Kentucky. The following advanced courses will be offered: First term, Projective geometry, by Professor P. P. Boyd, and Theory of equations, by Professor H. H. Downing. Second term, Higher algebra by Professor C. G. Latimer, and Infinite series, by Professor F. Elizabeth LeStourgeon.

University of Iowa. June 11th to August 5th. In addition to courses in college algebra, trigonometry, analytic geometry and calculus, the following advanced courses will be offered: By Dr. Ruth Lane: The teaching of mathematics. By Dr. Edwin Oberg: Differential equations; Fourier and allied series. By Professor L. E. Ward: Advanced calculus, infinite series; Second course in ordinary differential equations. By Professor C. C. Wylie: Mathematics of finance; Astronomy; Readings in astronomy. By Professor J. F. Reilly: Second course in college algebra; Solid analytical geometry; Seminar in finite differences. By Professor E. W. Chittenden: Topics for teachers; Foundations of mathematics; Seminar in analysis. By the staff: Reading and research.

University of Michigan. June 27 to August 19. In addition to elementary courses and the standard courses in differential equations, theory of equations, solid analytic geometery, and advanced calculus, the following courses will be offered: By Professor Carver: Social statistics; Mathematical theory of statistics, II. By Professor Coe: Analytic mechanics; Vector analysis. By Professor A. H. Copeland: Theory of probability. By Professor A. T. Craig: Mathematical theory of statistics, I; Advanced theory of statistics. I. By Professor Dushnik: Empirical formulas. By Dr. Elder: Theory of numbers. By Dr. Friedrichs: Fourier series; Applied mathematics, engineering problems. By Dr. T. N. E. Greville: Finite differences. By Professor T. H. Hildebrandt: Infinite series;

Integral equations. By Professor L. C. Karpinski: Teachers' seminar in algebra; History of geometry and trigonometry. By Mr. Kazarinoff: Synthetic projective geometry. By Professor Norman Miller: Theory of functions of a complex variable. By Dr. Myers: Differential geometry. By Professor J. A. Nyswander: Algebraic theory. By Professor R. L. Wilder: Introduction to the foundations of mathematics; General spaces. In addition, there will be a seminar in pure mathematics conducted by Professors Hildebrandt and Wilder, and one in statistics by Professor Craig.

University of Minnesota. In addition to the usual undergraduate courses, the following courses will be offered: First term, June 15 to July 23. By Professor Dunham Jackson: Advanced algebraic theory; Fourier series. By Professor A. L. Underhill: Intermediate calculus; Differential equations. By Professors Jackson and Underhill: Reading and research in advanced mathematics. Second term, July 25 to August 27. By Professor Gladys Gibbens: Selected topics in advanced mathematics.

University of North Carolina. First term, June 13 to July 23. By Professor Archibald Henderson: Theory of equations; Non-euclidean geometry. By Professor Mackie: Theory of functions of a real variable. By Professor Hill: History of mathematics. By Professor E. T. Browne: Algebraic invariants. Second term, July 25 to August 31. By Professor J. W. Lasley: Synthetic projective geometry. By Professor Hoyle: Advanced calculus; Lie theory of differential equations. By Professor E. A. Cameron: Advanced algebra.

Northwestern University. In addition to courses in analytic geometry and elementary calculus, the following courses will be offered: By Professor F. E. Wood: Series and differential equations; College geometry. By Professor D. R. Curtiss: Determinants and theory of equations; Higher algebra. By Professor H. E. Buchanan: Functions of a complex variable. By Professor A. W. Tucker: Introduction to topology.

Ohio State University. The following courses are announced for the summer of 1938. By Professor C. C. Morris: Finite differences. By Professor Tibor Radó: Projective geometry; Fourier's series. By Professor S. E. Rasor: Introduction to the theory of functions of a complex variable. By Dr. Suckau: Elementary theory of equations; Infinite series and products.

University of Pennsylvania. In addition to the usual undergraduate courses, the following advanced courses will be given: By Professor H. H. Mitchell: Infinite series and products. By Professor Beal: Differential equations; Differential geometry. By Professor P. A. Caris: Diophantine analysis. By Professor J. A. Shohat: Finite differences.

Teachers College, Columbia University. July 5 to August 13. By Professor W. D. Reeve: Mathematics education field course in Germany and England, sailing from New York June 30 with return sailing from Southampton August 19. By Professor C. B. Upton: Reorganization of senior high school mathematics; Teaching arithmetic in primary grades; Teaching arithmetic in intermediate grades. By Professor J. R. Clark: Teaching geometry in secondary

schools; Teaching algebra in junior high schools. By Professor C. N. Shuster: Modern business arithmetic; Field work in mathematics. By Dr. J. A. Swenson: Professionalized subject matter in senior high school mathematics; Demonstration class in eleventh year mathematics. By Mr. R. R. Smith: Teaching algebra in secondary schools; Professionalized subject matter in intermediate algebra. By Dr. N. Lazar: Demonstration class in demonstrative geometry; By Dr. A. D. Bradley: Navigation and map projections; Professionalized subject matter in junior high school mathematics.

University of Texas. First term, June 7 to July 18; second term, July 19 to August 29. In addition to the usual elementary courses the following courses will be offered: First term. By Professor A. E. Cooper: Differential equations. By Professor H. V. Craig: Advanced calculus; Vector and tensor analysis. By Professor R. N. Haskell: Potential theory. By Professor P. M. Batchelder: Advanced calculus; Teaching problems. By Professor E. L. Dodd: Probability; Theory of means. By Professor H. J. Ettlinger: Advanced calculus, partial differential equations. By Professor R. L. Moore: Foundations of geometry; Theory of sets. Second term. By Professor C. M. Cleveland: Differential equations. By Professor E. G. Keller: Non-linearity in engineering, physics, and astronomy; Celestial mechanics. By Dr. F. B. Jones: Advanced calculus. By Professor R. G. Lubben: Non-euclidean geometry. By Professor H. S. Vandiver: Foundations of algebra; Modern algebra.

University of Virginia. The following courses in advanced mathematics will be offered: First term, June 20 to July 30. By Professor E. J. McShane: Higher algebra; Calculus of variations. Second term, August 1 to September 3. By Professor G. T. Whyburn: Foundations of geometry; Advanced calculus.

University of Washington. June 20 to August 19. By Professor McFarlan: Integral and differential calculus; Calculus of variations. By Professor Carpenter: Differential equations (with Professor Jerbert); Introduction to applied mathematics. By Professor Cramlet: Vectors and tensors. By Professor Weyl: Selected mathematical problems.

University of Wisconsin. In addition to the usual elementary courses the following courses will be offered: By Dr. Kershner: Theory of equations. By Professor Langer: Theory of analytic functions (six or nine weeks); Fourier's series (six or nine weeks). By Professor March: Theoretical mechanics; Harmonic analysis; Dimensional analysis. By Professor Sokolnikoff; Advanced calculus; Vector and tensor analysis; Definite integrals. By Dr. Wagner: Advanced geometry.

University of Wyoming. In addition to elementary courses, the following will be offered: By Professor O. H. Rechard: Theory of functions of a complex variable (first and second term). By Professor C. F. Barr: College geometry (first term).

DEFINITIONS OF STIELTJES INTEGRALS OF THE RIEMANN TYPE*

T. H. HILDEBRANDT, University of Michigan

The Stieltjes integral defined by Stieltjes [1] in 1894 and brought into the limelight by the theorem of F. Riesz [2] that the most general linear functional operation on the space of continuous functions is best expressed by such an integral, has received so much attention within the last two decades that it is impossible to cover the subject in a paper of reasonable length. We therefore propose to limit ourselves to the case where the range of integration is a finite interval $J: a \leqq x \leqq b$, the functions involved are real valued and bounded, and the definitions are of the Riemann type. After a brief treatment of functions of intervals, we discuss two fundamental types of Stieltjes integrals, and then present a number of modifications of these definitions, introduced to overcome certain deficiencies of the ordinary integrals, particularly the limitations on the simultaneous discontinuities of the two functions involved.

1. The Riemann integral definitions. It seems desirable to recall briefly the two standard methods used for defining the Riemann integrals. Suppose σ denotes any subdivision $a=x_0<x_1<x_2< \cdots <x_{n-1}<x_n=b$, of $J=(a, b)$ and let $x_{i-1} \leqq \xi_i \leqq x_i$. Then if $|\sigma|$ is the maximum of x_i-x_{i-1}, $i=1, \cdots, n$, which we shall call the *norm* of the subdivision, we define:

$$\int_a^b f(x)dx = \lim_{|\sigma|\to 0} \sum_{i=1}^{n} f(\xi_i)(x_i - x_{i-1}).$$

The other method of defining an integral is via the upper and lower integral. Let M_i and m_i be respectively the least upper bound (LUB) and the greatest lower bound (GLB) of $f(x)$ on (x_{i-1}, x_i). Then $\bar{\int}_a^b f(x)dx$ is the greatest lower bound of $\sum_{i=1}^{n} M_i(x_i-x_{i-1})$ and $\underline{\int}_a^b f(x)dx$ is the least upper bound of $\sum_{i=1}^{n} m_i(x_i-x_{i-1})$ for all subdivisions σ of (a, b). When $\bar{J}=\underline{J}$ then f is said to be integrable and the common value is the integral of f. Now $\sum M_i(x_i-x_{i-1})$ is the LUB of $\sum_{i=1}^{n} f(\xi_i)(x_i-x_{i-1})$ when the ξ_i are allowed to vary on (x_{i-1}, x_i). So that the upper integral is a GLB(LUB) combination, which one associates with a superior or greatest of the limits. Similarly the lower integral is an LUB(GLB) combination or a smallest of the limits. But the obvious limit is not in the sense $|\sigma| \to 0$. Rather we note that if additional points are added to σ, then $\sum_{i=1}^{n} M_i(x_i-x_{i-1})$ decreases if anything, and so $\bar{\int}_a^b f(x)dx$ is the limit of this sum as additional points are added to the subdivisions. Defining $\sigma_1 \geqq \sigma_2$ to mean that every point of σ_2 is a point of σ_1 (or every interval of σ_1 is a subinterval of σ_2), then a function of subdivisions $\phi(\sigma)$ has a *limit in the σ sense* if for every $e>0$ there exists a σ_e such that for all $\sigma \geqq \sigma_e$, we have $|\phi(\sigma)-A| \leqq e$ [3], [6]. Then $\bar{J}$ and $\underline{J}$ are greatest and least limits in this σ-sense, and their equality is equivalent to the existence of a limit of $\sum_{i=1}^{n} f(\xi_i)(x_i-x_{i-1})$ in the σ-sense. If we call a

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limit of a function of subdivisions $\phi(\sigma)$ as $|\sigma| \to 0$, a *norm limit*, then the Darboux theorem stating the equivalence of the integral by the Riemann and by the upper and lower integral processes is in effect a statement of the fact that the σ-limit in this case is also a norm limit.

2. Integrals of functions of intervals [4] [5] [6]. For the sake of brevity it seems best to take a leap at this point and consider briefly the integrals of functions of intervals, of which the Riemann integral is a special case. Considering the subintervals $I=(c, d)$ of the fundamental interval $J=(a, b)$ as a range, a function of intervals $F(I)$ assigns to each I of J one or more real values. Examples:

$f(\xi)(d-c)$ with $c \leqq \xi \leqq d$, used in the preceding paragraph;

$f(d)-f(c)$; $\quad |f(d)-f(c)|$;

$\omega(f; I)=\omega(f; c, d)=\text{LUB}[|f(x_1)-f(x_2)|; c \leqq x_1, x_2 \leqq d]$;

$f(\xi)(g(d)-g(c))$, $c \leqq \xi \leqq d$;

and so on. Any function of intervals $F(I)$ gives rise to a function of subdivisions $\phi(\sigma)$ by the process of dividing the fundamental interval J via the subdivision σ into a finite number of nonoverlapping intervals: $I_1, I_2, \cdots, I_n$, and evaluating $\sum_{i=1}^{n} F(I_i)$. If this $\phi(\sigma)$ has a limit then $F(I)$ is said to have an integral, *i.e.*, we define *the norm integral* (NJ)

$$N\int_a^b F(dI)=\lim_{|\sigma|\to 0}\phi(\sigma)=\lim_{|\sigma|\to 0}\sum_{i=1}^{n} F(I_i),$$

and *the σ-integral* (σJ)

$$\sigma\int_a^b F(dI)=\lim_{\sigma}\phi(\sigma)=\lim_{\sigma}\sum_{i=1}^{n} F(I_i).$$

We note the following theorems for these integrals:

(a) EXISTENCE THEOREMS. 2.11. *A necessary and sufficient condition that $\int F(dI)$ exist is that a corresponding Cauchy condition of convergence be satisfied,* where by a Cauchy condition of convergence for a $\phi(\sigma)$ we mean $\lim(\phi(\sigma_1)-\phi(\sigma_2))=0$, where the limit is either as $|\sigma_1|$ and $|\sigma_2| \to 0$ or involves a condition of the form $\sigma_1, \sigma_2 \geqq \sigma_e$, depending upon the integral. If we define the interval function

$$\omega(SF, I)=\text{LUB}\left[\left|\sum_{\sigma_1} F(\Delta I)-\sum_{\sigma_2} F(\Delta I)\right|; \text{ for all } \sigma_1 \text{ and } \sigma_2 \text{ of } I\right],$$

then this theorem is equivalent to:

2.12. *A necessary and sufficient condition that $\int F(dI)$ exist is that $\int \omega(SF, dI)=0$, both integrals being either norm or σ-integrals.*

Since $\sum\omega(SF, \Delta I)$ is a monotone nondecreasing function of σ in the sense $\sigma_1 \geqq \sigma_2$, this theorem for the case of the σ-integral is replaceable by:

2.13. *A necessary and sufficient condition that $\sigma\int F(dI)$ exist is that*

$$\mathrm{GLB}\,[\textstyle\sum_\sigma \omega(SF, \Delta I); \sigma] = 0.$$

An important necessary condition for the existence of these integrals follows immediately from 2.12. To state the condition, we consider the intervals

$$I = (x_1, x_3), \quad I_1 = (x_1, x_2), \quad I_2 = (x_2, x_3)$$

so that $I = I_1 + I_2$, and define $F(I)$ *pseudo-additive at the point* x_2 [7], if

$$\lim_{x_1 \to x_2, x_3 \to x_2} [F(I) - (F(I_1) + F(I_2))] = 0;$$

$F(I)$ is *pseudo-additive on the right at* x_1 if

$$\lim_{x_3, x_2 \to x_1} [F(I) - (F(I_1) + F(I_2))] = 0.$$

Obviously a function can be both pseudo-additive on the right and left at a point without being pseudo-additive at the point. In terms of these notions we have:

2.14. *A necessary condition for the existence of a σ-integral is that $F(I)$ be pseudo-additive on the left and right at all points of J; for the existence of the norm integral, that $F(I)$ be pseudo-additive at all points of J.*

The pseudo-additivity plays a rôle in the equivalence of the norm and σ definitions of integrals, *i.e.*, we have:

2.15. *A necessary and sufficient condition that the $N\int F(dI)$ exist is that the $\sigma\int F(dI)$ exist and F be pseudo-additive at all points of J. The values of the integrals will agree.*

(b) Interval Properties. With respect to the integral as a function of the limits, we have the usual results:

2.21. *If $\int_a^b F(dI)$ exists, and $a \leqq c < d \leqq b$, then $\int_c^d F(dI)$ exists and $\int_a^c F(dI) + \int_c^b F(dI) = \int_a^b F(dI)$.*

The integrals then define interval functions, which have the additive property: If $I = I_1 + I_2$, and I_1 and I_2 have only an endpoint in common, then $F(I) = F(I_1) + F(I_2)$. For the converse theorem, there is a difference between the norm and σ-integrals, *i.e.*, we have:

2.22σ. *If $\sigma\int_a^c F(dI)$ and $\sigma\int_c^b F(dI)$ exist then $\sigma\int_a^b F(dI)$ exists as their sum.*

2.22N. *If $N\int_a^c F(dI)$ and $N\int_c^b F(dI)$ exist and F is pseudo-additive at the point c, then $N\int_a^b F(dI)$ exists as their sum.*

By combining the additivity of the integral with Existence Theorem 2.12, we obtain the following *approximation theorem*:

2.23. *If $\int F(dI)$ exists then*

$$\left| \int F(dI) - \sum_{\sigma} F(\Delta I) \right| \leqq \sum_{\sigma} \omega(SF, \Delta I),$$

the subdivision σ being the same on both sides of the inequality.

This theorem in turn gives rise to theorems which are similar to the result that the Riemann integral is a continuous function of its upper limit:

2.24N. *If $N\int F(dI)$ exists, then for every $e>0$, there exists a d_e such that if $|I| \leqq d_e$, where $|I|$ is the length of I, then $|F(I) - N\int_I F(dI)| \leqq e$.*

2.24σ. *If $\sigma\int F(dI)$ exists, then for every $e>0$, and x, there exists a d_{ex} such that if I has x as an endpoint, and $|I| \leqq d_{ex}$, then $|F(I) - \sigma\int_I F(dI)| \leqq e$.*

(c) LINEARITY. Since the norm and σ-limits have the usual properties of limits [3], we have:

2.31. *If $\int F(dI)$ exists and c is any constant, then $\int cF(dI)$ exists equal to $c\int F(dI)$.*

2.32. *If $\int F_1(dI)$ and $\int F_2(dI)$ exist, then $\int (F_1+F_2)(dI)$ exists and is equal to their sum.*

It is assumed in the latter theorem, that the set of values $(F_1+F_2)(I)$ is contained in the algebraic sum of the sets of values $F_1(I)$ and $F_2(I)$. These theorems state that the integral is a linear functional operation.

(d) CONVERGENCE. With respect to a sequence of interval functions $F_n(I)$ converging to an interval function $F(I)$ we have:

2.41. *If $\int F_n(dI)$ exist uniformly in the sense that $\lim \sum_{\sigma} \omega(SF_n, \Delta I) = 0$ uniformly in n, and $\lim_n F_n(I) = F(I)$ for each I in the sense that every value of $F(I)$ is the limit of some sequence of values $F_n(I)$, then $\int F(dI)$ exists and $\lim \int F_n(dI) = \int F(dI)$, limits and integrals being in the same sense throughout.*

3. Functions of bounded variation. As an instance of the above theory, we note the functions of bounded variation. For this purpose we define relative to a point function $f(x)$ the interval function $F(I) = |f(d) - f(c)|$, with $I = (c, d)$. This gives rise to the single valued function of subdivisions $\phi(\sigma) = \sum_{i=1}^{n} |f(x_i) - f(x_{i-1})|$, which is monotonic nondecreasing in the σ-sense. Hence the σ-integral exists when these sums are bounded. In this case $f(x)$ is of bounded variation, and the total variation of $f(x)$ on (a, b) is the least upper bound of this $\phi(\sigma)$, *i.e.*, the σ-integral of $F(I)$. Then if $f(x)$ is of bounded variation, $F(I)$ must be pseudo-additive on the right and left, at each point. The former condition requires that if $x < x_1 < x_2$, then $\lim_{x_1, x_2 \to x} [|f(x) - f(x_2)| - (|f(x) - f(x_1)| + |f(x_1) - f(x_2)|)] = 0$, and it is possible to show that this is equivalent to the fact that $f(x+0)$ exists. Similarly $f(x-0)$ must exist. The total variation does not in general exist as a norm limit (*i.e.*, expressible as a norm integral). The latter necessitates the

addition of the pseudo-additive condition to the existence of the σ-integral, which can be shown to require that at every point $f(x)$ lies between $f(x+0)$ and $f(x-0)$, *i.e.*, that $f(x)$ be a regular function of bounded variation.

From Theorem 2.21 it follows at once that the total variation of a function is an additive function of intervals. If $V_a^x f$ denote the total variation of f in the interval (a, x) then the monotoneity of $V_a^x f \pm f(x)$ follows, and this gives the decomposition of a function of bounded variation into the difference of two monotone functions.

Of importance is also the decomposition which throws the discontinuities into the limelight. If for a function f of bounded variation, we define the function $f_d(x) = \sum_{\xi<x}[f(\xi+0)-f(\xi-0)]+f(x)-f(x-0)$, where the sum obviously includes only a denumerable number of non-zero terms, and is absolutely convergent, then $f_c(x) = f(x) - f_d(x)$ is continuous and of bounded variation.

Theorem 2.24 implies that $f(x)$ and $V_a^x f$ have the same continuity properties at all points.

4. The ordinary Stieltjes integral. By taking for the interval function $F(I)$ the many valued expression $f(\xi)(g(d)-g(c))$, where $I=(c, d)$ and $c \leqq \xi \leqq d$, and forming the corresponding norm and σ-integrals, we get the Riemann-Stieltjes [8], and the Pollard-Moore-Stieltjes [9, 10] integrals, respectively, *i.e.*,

$$\mathrm{RS}\int_a^b f dg = N\int_a^b f dg = \lim_{|\sigma|\to 0} \sum_{i=1}^{n} f(\xi_i)(g(x_i) - g(x_{i-1}))$$

and

$$\mathrm{PMS}\int_a^b f dg = \sigma\int_a^b f dg = \lim_{\sigma} \sum_{i=1}^{n} f(\xi_i)(g(x_i) - g(x_{i-1})).$$

We shall call these integrals the *ordinary norm* and *σ-integrals* respectively.

Existence Theorems 2.11 and 2.12 apply at once. If we denote $\omega(SF, I) = \omega(Sf\Delta g, I) = \mathrm{LUB}[|\sum_{\sigma_1} f\Delta g - \sum_{\sigma_2} f\Delta g|\,; \sigma_1, \sigma_2 \text{ of } I]$, we have results due to H. L. Smith [10]:

4.11. *A necessary and sufficient condition that $\int f dg$ exist is that $\int \omega(Sf\Delta g, dI) = 0$.*

The pseudo-additive property becomes:

4.12. *A necessary and sufficient condition that $F(I) = f(\xi)(g(d)-g(c))$ be pseudo-additive (on the right) (on the left) is that f and g have no common discontinuities (on the right) (on the left).*

Applied to the existence of integrals we find:

4.13. *A necessary condition for the existence of $N\int f dg$ ($\sigma\int f dg$) is that f and g have no common discontinuities (no common discontinuities on the same side).*

4.14. *The $N\int f dg$ exists if and only if the $\sigma\int f dg$ exists, and f and g have no common discontinuities.*

This theorem is the generalization of the Darboux theorem mentioned in §1. The Linearity Theorems 2.31 and 2.32 give:

4.31. *The integral $\int f dg$ is a bilinear functional in f and g, i.e.,*

$$\int \left(\sum_{i=1}^{n} a_i f_i \right) d \left(\sum_{j=1}^{m} b_j g_j \right) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \int f_i dg_j,$$

it being assumed that the integrals on the right exist.

Further theorems may be obtained by making the proper substitutions in the theorems of §2. On the other hand, the following two theorems depend upon the special form of the interval function.

4.5. INTEGRATION BY PARTS THEOREM [11]. *If $\int_a^b f dg$ exists then $\int_a^b g df$ exists also and $\int_a^b f dg + \int_a^b g df = f(b)g(b) - f(a)g(a)$.*

The existence of $\int g df$ follows directly from the fact that for any interval I: $\omega(Sf\Delta g, I) = \omega(Sg\Delta f, I)$ which in turn is deducible from the identity:

$$\sum_{i=1}^{n} f(\xi_i)(g(x_i) - g(x_{i-1})) + \sum_{i=0}^{n} g(x_i)(f(\xi_{i+1}) - f(\xi_i)) = f(x_n)g(x_n) - f(x_0)g(x_0)$$

with $\xi_0 = x_0$ and $\xi_{n+1} = x_n$. The same identity then yields the theorem by proper choice of the x_i and ξ_i.

4.6. SUBSTITUTION THEOREM [12]. *If $G(x) = \int_a^x f(x) dg(x)$, and $h(x)$ is bounded, then $\int_a^b h(x) dG(x)$ and $\int_a^b h(x) f(x) dg(x)$ exist simultaneously and are equal.*

This is an immediate consequence of the observation that

$$\left| \sum_{\sigma} h(\xi_i)(G(x_i) - G(x_{i-1})) - \sum_{\sigma} h(\xi_i) f(\xi_i)(g(x_i) - g(x_{i-1})) \right|$$

$$\leqq \sum_{\sigma} \left| h(\xi_i) \right| \left| \int_{x_{i-1}}^{x_i} f dg - f(\xi_i)(g(x_i) - g(x_{i-1})) \right| \leqq M_{|h|} \sum_{\sigma} \omega(Sf\Delta g; \Delta I),$$

where $M_{|h|}$ is the LUB of $|h(x)|$ on (a, b), and the approximation theorem corresponding to 2.23 is used. We might note that a similar substitution theorem holds when $\int f dg$ is replaced by $\int F(dI)$.

5. Ordinary Stieltjes integrals with respect to functions of bounded variation. In the Stieltjes integrals as defined above we have required in a general way only the boundedness of the functions f and g. Even this might be dropped if sufficient conditions are added to make the associated function of intervals bounded. If we fix the function g, then the existence of the integral determines an associated class of functions f, integrable with respect to g. For instance if $g(x) = x$, then we obtain the class of Riemann integrable functions. If we wish the integral to exist for every g (or f) of a certain class this determines an associated or conjugate class of functions f (or g). For example, one may consider the class of continuous functions as the least common subclass of integrable functions. We have the result [9]:

5.1. *If $\int f dg$ exists for every $f(x)$ continuous on $a \leqq x \leqq b$, then $g(x)$ must be of bounded variation.*

The proof is of the contrapositive type, *i.e.*, the assumption that $g(x)$ be not of bounded variation leads to the construction of a continuous function f so that $\int f dg$ does not exist. Since f is continuous the integral is both a norm and σ-integral. The conjugate theorem holds also *viz.*:

5.2. *If $\int f dg$ exists for all functions of bounded variation g, then f is continuous.*

This follows immediately from the fact that f and g cannot have simultaneous discontinuities. On the other hand, $\int f dg$ exists if f is continuous and g is of bounded variation, the situation for which Stieltjes originally defined his integral. So that with respect to the Stieltjes integral the class of functions of bounded variation and continuous functions might be said to be conjugate. These interrelations have no doubt been responsible for the fact that the consideration of Stieltjes integrals has been restricted largely to integration with respect to functions of bounded variation. (But see [13].) Obviously if we limit the functions g to be considered as of bounded variation, we can obtain additional properties of our integrals.

In the line of modifications of existence theorems we note that for any interval $I=(c, d)$, we have:

$$\omega(f, I) \mid g(d) - g(c) \mid \leqq \omega(Sf\Delta g, I) \leqq \omega(f, I)(v(d) - v(c)),$$

where $\omega(f, I)$ is the oscillation of f on I, and $v(x)$ the variation function $V_a^x g$ corresponding to g. As a consequence $\lim \sum_\sigma \omega(f, I)|\Delta g| = 0$ is a necessary condition and $\lim \sum_\sigma \omega(f, I)\Delta v = 0$ is a sufficient condition for the existence of the Stieltjes integrals. However it is possible to prove the equivalence of the limits just mentioned, both being taken either in the norm, or in the σ-sense, so that we obtain:

5.31. *If g is of bounded variation, and $v(x)$ the variation of g in the interval (a, x) then a necessary and sufficient condition for the existence of $\int f dg$ is that of $\int f dv$.*

This shows that for existence theorems in this setting it is sufficient to assume that g be monotonic, justifying the building up of the Stieltjes integral relative to functions of bounded variation from the monotone function angle. The condition $\lim \sum_\sigma \omega(f, I)\Delta v = 0$ can now be used in the usual way, to prove the following analog to Riemann integrals [14], [15]:

5.32N. *A necessary and sufficient condition that the norm or Riemann-Stieltjes integral $\int f dg$ exist, g of bounded variation, is that the v-measure of the points of discontinuity of f be zero.*

In this theorem we assume that corresponding to any point x there correspond the points $v(x-0) \leqq v \leqq v(x+0)$ on the v-axis, and the v-measure of a set

of points on the x-axis is the measure of the corresponding points on the v-axis. For the σ- or Pollard-Moore-Stieltjes integral, a modification is necessary. We assume that each point x on the x-axis is bi-faced. To x^+ corresponds the points $v(x) \leqq v \leqq v(x+0)$, to x^- corresponds the points $v(x-0) \leqq v \leqq v(x)$. If x is a point of discontinuity of f on the right we label it x^+ and correspondingly on the left. Then if we call the $v^{\pm}$ measure of a set of bi-faced points on the x-axis, the measure of the corresponding points on the v-axis, we have

5.32. *A necessary and sufficient condition that the σ- or Pollard-Moore-Stieltjes integral $\int f dg$ exist, g of bounded variation, is that the points of discontinuity of f form a set of zero $v^{\pm}$ measure.*

We might remark that if $g(x)$ is monotonic then upper and lower integrals can be defined as for the Riemann integral, and their equality produces the σ-integral [16].

If we decompose the function g into the sum of its continuous part and its function of discontinuities, and if $\int f dg$ exists, then $\int f dg_c$ and $\int f dg_d$ both exist. Moreover the $\int f dg_d$ is easily evaluated and we find that it has the form: $\sum_x f(x)(g(x+0)-g(x-0))$, so that

$$5.33. \qquad \int f dg = \int f dg_c + \sum_x f(x)(g(x+0) - g(x-0)).$$

The sum involves at most a denumerable number of terms and is absolutely convergent. This form is identical for both norm and σ-integrals. One of the advantages of the Stieltjes integral especially in the applications derives from this property of combining a term of continuous type with an infinite series, and making possible the simultaneous treatment of these two cases [17], [18].

The application of the Convergence Theorem 2.41 for integrals of functions of intervals, yields in this setting:

5.41. *If $f_n(x)$ converges uniformly to $f(x)$ on $a \leqq x \leqq b$, and $g(x)$ is of bounded variation, and if $\int f_n(x) dg(x)$ exists for every n, then $\int f(x) dg(x)$ exists and $\lim_n \int f_n(x) dg(x) = \int f(x) dg(x)$.*

5.42. *If $g_n(x)$ is uniformly of bounded variation, i.e., $V_a^b g_n \leqq M$ for all n, and if $g_n(x)$ approaches $g(x)$ for every x, then $\lim_n \int f dg_n = \int f dg$, for every continuous function $f(x)$.*

Bray [19] has proved this theorem for the weaker assumption that $g_n(x)$ converge to $g(x)$ at a dense set of points on (a, b) including a and b.

We note finally the following theorem, closely related to the fundamental lemma of the calculus of variations and demonstrable in the same way, (*e.g.*, adapting the proof of Bliss, *Calculus of Variations* p. 20) [20].

5.5. *If $\int f dg = 0$ for every continuous function $f(x)$, then $g(x)$ is constant on (a, b) excepting at most at a denumerable set of points interior to (a, b).*

6. The modified Stieltjes integrals. The existence of the ordinary Stieltjes integrals restricts the simultaneous occurrence of discontinuities in the functions f and g. On the other hand in the applications it is often desired to integrate any function of bounded variation with respect to any other. This has led to various modifications of the definitions.

If we base our definition of Stieltjes integral on interval functions of the type $f(\xi)(g(d)-g(c))$, then modification centers in the multiplier $f(\xi)$. Restricting ξ to being an endpoint of the interval (c, d) leads to an analog of the Cauchy definition anticipating the Riemann integral. For the case $g(x)=x$, Gillespie [21] has shown that the class of functions integrable by the Cauchy and Riemann method of definition is the same. It seems probable that his proof can be carried over to the Stieltjes integral for a continuous $g(x)$ of bounded variation. Except for properties derivable from corresponding theorems on functions of intervals, this definition does not seem to have been discussed.

The restriction on discontinuities of the functions f and g derives to some extent from the presence of the endpoints of the interval (c, d) in the range of ξ. This suggests that one try to restrict ξ to the interior of the interval, *i.e.*, we consider a norm and σ-integral based on $F(I)=f(\xi)(g(d)-g(c))$, with $c<\xi<d$. We shall call such integrals when they exist, modified Stieltjes integrals [11], and denote them mod$\int$. The theorems of §2 can then be stated for this class of integrals. In particular the inequality $\omega(Sf\Delta g, I)\geqq\omega^*(f, I)|\Delta g|$, where $\omega^*(f, I)$ is the oscillation of f on the open interval I, leads at once to the result:

6.11σ. *A necessary condition that* mod $\sigma\int fdg$ *exist is that for every point either* g *be continuous on the right or* $f(x+0)$ *exist and either* g *be continuous on the left or* $f(x-0)$ *exist.*

6.11N. *A necessary condition that* mod $N\int fdg$ *exist, is that for every point either* f *be continuous or* $g(x+0)$ *and* $g(x-0)$ *exist and be equal.*

If f and g have no common discontinuities, then these integrals reduce to the ordinary ones.

The integration by parts theorem is not in general true. It is demonstrable that the theorem holds for the modified norm integral if and only if f and g be not both discontinuous at a and b. This is not sufficient for the σ-integral however.

In case the function g is assumed to be of bounded variation, we can demonstrate that the modified integral exists if and only if the corresponding modified integrals with respect to the total variation v of g exist. For the σ-integral we have:

6.12. *The* mod $\sigma\int fdg$ *exists if* f *have only discontinuities of the first kind, and* g *be of bounded variation,* so that the mod $\sigma\int$ gives a value for any function of bounded variation with respect to any other function of bounded variation.

If the mod $\int fdg$ exist, then $\int fdg_c$, where g_c is the continuous part of the func-

tion of bounded variation g, exists as an ordinary integral. Moreover we have:

$$\text{mod } \sigma \int f dg = \int f dg_c + \text{mod } \sigma \int f dg_d$$

$$= \int f dg_c + \sum_x [f(x+0)(g(x+0)-g(x))+f(x-0)(g(x)-g(x-0))].$$

For this σ-integral, the integration by parts theorem is not valid in the usual form. We can show:

6.2. *If f and g are both functions of bounded variation then*

$$\text{mod } \sigma \int_a^b f dg + \text{mod } \sigma \int_a^b g df = f(b)g(b) - f(a)g(a)$$

$$+ \sum_x [f(x-0)(g(x)-g(x-0)) - f(x)(g(x+0)-g(x-0)) + f(x+0)(g(x+0)-g(x))],$$

where as usual $\sum_x$ need be extended only to the discontinuities of f and g, as a matter of fact only to their common discontinuities. Obviously if the $\sum_x$ term vanishes. the integration by parts theorem holds in its usual form.

7. The Stieltjes mean integrals. Another method of modification is to replace the multiplier $f(\xi)$ by the mean at a specified set of points on the interval. The simplest of these suggested by H. L. Smith [**10**], is where only the endpoints are used, *i.e.*, we define $F(I) = \frac{1}{2}(f(d)+f(c))(g(d)-g(c))$, and according to Steffensen [**22**] has its uses in actuarial mathematics (see also [**18**] and [**25**]). If the integrals of $F(I)$ exist according to this definition, we shall call them with H. S. Kaltenborn [**23**], the Stieltjes mean integrals $SM\int f$. Obviously if the ordinary integral exists, the corresponding mean integral also exists, and gives the same value, but not conversely. For instance $SM\int g dg$ exists for any function g and has the value $\frac{1}{2}((g(b))^2-(g(a))^2)$. Obviously the theorems of §2 can be adapted to this situation.

Interesting is the fact that the integration by parts theorem is valid and very easily demonstrated since

$$\tfrac{1}{2}(f(d) + f(c))(g(d) - g(c)) + \tfrac{1}{2}(g(d) + g(c))(f(d) - f(c)) = f(d)g(d) - f(c)g(c).$$

But the substitution theorem: *if $F(x) = SM\int f dg$ exists, then the simultaneous existence and equality of $SM\int h dF$ and $SM\int f dg$* is valid if and only if $\lim \sum_\sigma \Delta h \Delta f \Delta g = 0$. On the other hand if in the $SM\int f dg$ we think of hf as a function $h(x)f(y)$ and define $G(I) = \frac{1}{4}(h(d)+h(c))(f(d)+f(c))(g(d)-g(c))$, then the theorem is valid without a supplementary condition.

When g is assumed to be of bounded variation, it is not possible to deduce the existence of $SM\int f dv$, v the variation function of g, from that of $\int f dg$ in the norm integral case. This is also probably true for the σ-integral. On the other hand:

7.12. *If g is of bounded variation and f has only discontinuities of the first kind, then $SM\sigma\int f dg$ exists.*

The Stieltjes mean σ-integral then offers a Stieltjes integral where every function of bounded variation is integrable with respect to every other function of bounded variation and where the integration by parts theorem holds. If we assume f to have only discontinuities of the first kind and g to be of bounded variation then

$$SM\sigma \int f dg = \sigma \int f dg_c$$
$$+ \tfrac{1}{2} \sum_x [f(x-0)(g(x)-g(x-0))+f(x)(g(x+0)-g(x-0))$$
$$+f(x+0)(g(x+0)-g(x))].$$

The $\sum_x$ terms involve the values $g(x-0)$, $g(x)$, $g(x+0)$, but if $f(x)$ is continuous the corresponding term reduces to $f(x)(g(x+0)-g(x-0))$, so that the value $g(x)$ enters only where $f(x)$ is discontinuous also.

8. A Young-Stieltjes integral. The remaining modifications in the interval function $F(I)$ basic to a Stieltjes integral depend on special properties of the functions involved. In particular if we assume that $g(x)$ has only discontinuities of the first kind, *i.e.*, assume that $g(x+0)$ and $g(x-0)$ exist at all points, then we can utilize this fact in our $F(I)$, and segregate out the endpoints of the interval I. As a first attempt, we set with W. H. Young [**24**]:

8.01. $F(I)=f(c)(g(c+0)-g(c))+f(\xi)(g(d-0)-g(c+0))+f(d)(g(d)-g(d-0))$,

where $c<\xi<d$. It is to be noted that with the exception of the endpoints a and b of the interval of integration, the value of $g(x)$ at the points of division plays only a disappearing rôle, so that it is justifiable to assume that $g(x)$ lies between $g(x-0)$ and $g(x+0)$ for all x. Then the norm integral exists if and only if f and g have no common discontinuities. Since it is then possible to take a sequence of subdivisions avoiding the points of discontinuity of g, for which the sums approach the integral, we have the following theorem at once [**5**]:

8.11N. *The norm integral based on the function $F(I)$ of* 8.01, *exists if and only if the ordinary norm integral exists and is equal to it.*

On the other hand the σ-integral of this interval function may exist when f and g have common discontinuities. As a matter of fact it can be shown that

8.11σ. *If g is of bounded variation then*

$$Y\sigma \int f dg = \int f dg_c + \sum_x f(x)(g(x+0) - g(x-0)).$$

This agrees in form with that of the ordinary integrals in this case, and as a matter of fact, also with that given by the Lebesgue-Stieltjes integral process.

In a sense our result is reversible. If we should define a Stieltjes integral with respect to a function of bounded variation g by the condition that $\int f dg_c$ exists, in the ordinary sense, and that the formula of 8.11σ is valid, then this integral is

also obtainable as a σ-integral of an interval function defined by 8.01, so that the properties of §2 apply at once. We note that the integration by parts theorem in its usual form is not generally valid for the Young σ-integral. It can be shown that it takes the form given in 6.2 [**25**].

9. The de Finetti-Jacob-Stieltjes integral. The form of the Stieltjes integral obtained in 8.11σ suggests another modification. If $f(x)$ has at most discontinuities of the first kind at a point, where $g(x)$ of bounded variation is discontinuous, it seems natural that the values $f(x-0)$, $f(x)$, and $f(x+0)$ should make themselves felt in the integral, *i.e.*, we might set down as a definition:

$$9.01.\quad \int f dg = \int f dg_c + \sum_x \left[a(x)f(x-0)+b(x)f(x)+c(x)f(x-0)\right],$$

where $a(x)$, $b(x)$, and $c(x)$ are zero excepting at points of discontinuity of $g(x)$. de Finetti and Jacob [**25**] make a suggestion of this type. They assume that to each x, there corresponds two values of $g(x)$, a right handed one $g_r(x)$ and a left-handed one $g_l(x)$, in addition to $g(x+0)$ and $g(x-0)$, and then define for $g(x)$ monotonic:

$$9.02.\quad \int f dg = \int f dg_c + \sum_x \left[f(x-0)(g_l(x)-g(x-0))+f(x)(g_r(x)-g_l(x)) + f(x+0)(g(x+0)-g_r(x))\right]$$

which form can obviously be applied to functions g of bounded variation in an extended sense. The form 9 01 can be shown to be equivalent if $a(x)+b(x)+c(x) = g(x+0)-g(x-0)$, a condition which is desirable if at points of continuity of $f(x)$ the terms reduce to the form of 5.33 or 8.11σ. Our previous considerations pose the question whether such a form can be obtained as an integral of an interval function. Paralleling the Young expression 8.01, we find:

$$9.03.\quad \text{If}\quad F(I) = \tfrac{1}{2}f(c)(g_r(c)-g_l(c))+f(\xi)(g_l(d)-g_r(c))+\tfrac{1}{2}f(d)(g_r(d)-g_l(d)),$$

with a slight modification at the endpoints a and b of J, and if $\sigma\int F(dI)$ exists, and g is of bounded variation, then this integral can be expressed in the form 9.02.

It may be of interest to note that the most general linear continuous functional operation on the space of bounded functions having at most discontinuities of the first kind leads to an integral of the type 9.02 [**26**].

10. Other definitions. It seems desirable to mention finally some modifications of the limits processes. One is suggested by Lebesgue [**27,** p. 272]. He assumes g of bounded variation, and considers a limit with $|\sigma| \to 0$, but with the additional condition that σ shall ultimately contain all points of discontinuity of g. A moment's reflection will show that the integral thus defined is equivalent to the σ-integral. Another suggestion is by Ridder [**28**]. He assumes $g(x)$ monotonic, and suggests that $|\sigma|$ be measured not on the x axis, but on the $g(x)$ axis. As a consequence if $|\sigma| \leqq e$, σ would include all points of discontinuity of g where

the jump exceeds e. The relationship of this procedure to that of Lebesgue and consequently to the σ-integral is obvious. Finally, Copeland [29], wishing to apply a Stieltjes integral process to probability questions, uses a method of definition based on probability considerations. For details, it is necessary to refer to his paper. He obtains a process which gives an integral for any function of bounded variation with respect to any function of bounded variation, but assumes that for each function of bounded variation $g(x)=\frac{1}{2}(g(x+0)+g(x-0))$. Also the integration by parts theorem is valid. The values obtained agree with those of the Stieltjes mean σ-integrals for this special combination of functions.

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ANALYTIC TREATMENT OF PERSPECTION WITH BEARING ON PICTURIZATION

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Introductory remarks. The present paper was motivated by Mr. Neil Little's interesting and well illustrated article published in this MONTHLY, vol. 44, May, 1937, under the title *An analytic study of the non-perspective picturization of quadric surfaces.*

In view of the fact that I have given much thought to the subject of "Picturization" and also because Mr. Little does not derive all of the equations (though correct) which he uses, I have felt impelled to write the present paper as well as one recently published in this MONTHLY, vol. 44, October, 1937, entitled, *Meaning and function of a picture.*

Furthermore, one of the things that is generally overlooked, and also not mentioned in Mr. Little's article, is that a single projection is not sufficient to adequately represent an object of space upon a plane. In the present paper, this fact is emphasized, and in Remarks 1 and 3 it is shown analytically that two projections are, in general, sufficient for the purpose of adequately representing a space object on a plane surface. It might be interesting to mention, in passing, that on account of ignorance concerning the dimensions of a battleship it becomes necessary for a gunner on an enemy battleship to make use of a range finder (which involves double projection).

With the statement in the last paragraph of Mr. Little's article, in which he speaks of the inadequacy of pictures in our textbooks on analytic geometry and calculus, I fully concur. In recognition of this fact, I was requested many years ago by Professors Woods and Bailey of the Massachusetts Institute of Technology to draw pictures of the quadric surfaces for volume 2 of their work entitled, *A Course in Mathematics* and published by Ginn and Company in 1909. These pictures were also used in their later one-volume work entitled, *Analytic Geometry and Calculus*, published by Ginn and Company in 1917. These pictures were properly drawn according to the rules of Cavalier perspective, which is the type of parallel projection used by Mr. Little in the first part of his article.

1. Statement of two problems in perspection. By the term perspection in projective geometry is meant the process of obtaining a so-called conical figure by connecting the various points of a figure of space with a fixed point V by means of straight lines. In the treatment here considered we will think of the point V as lying at infinity in a given direction (δ).

If we denote by l, m, n numbers proportional to the direction cosines of the direction (δ), we can easily see that *all straight lines having this direction* may be represented by the system of equations

$$\frac{x-X}{l}=\frac{y-Y}{m}=\frac{z-Z}{n}, \tag{1}$$

in which x, y, z are the running coördinates of a particular line of this two-parameter family (or *congruence*) of straight lines, and X, Y, Z are parameters fixing any such a particular line of the family (1).

If we put

$$U_1=\begin{vmatrix} m & n \\ y & z \end{vmatrix}, \quad U_2=\begin{vmatrix} n & l \\ z & x \end{vmatrix}, \quad U_3=\begin{vmatrix} l & m \\ x & y \end{vmatrix}, \tag{2}$$

which evidently satisfy the identical relation

$$lU_1+mU_2+nU_3\equiv 0, \tag{3}$$

and

$$c_1=\begin{vmatrix} m & n \\ Y & Z \end{vmatrix}, \quad c_2=\begin{vmatrix} n & l \\ Z & X \end{vmatrix}, \quad c_3=\begin{vmatrix} l & m \\ X & Y \end{vmatrix}, \tag{3'}$$

equations (1) become

$$U_1=c_1, \quad U_2=c_2, \quad U_3=c_3, \tag{4}$$

and *a pair of these equations* defines the line of congruence (1) which passes through the point (X, Y, Z).

Two problems of interest are the following:

PROBLEM I. Determine the cylindrical surface, of generators parallel to (δ), which shall pass through (*i.e.*, contain on its surface) a given curve Γ.

PROBLEM II. Determine the cylindrical surface, of generators parallel to (δ), which shall touch (*i.e.*, be tangent at a point of each of its generators to) a given surface S.

The solution of Problem II may be reduced to that of Problem I. For let the surface S be represented by the equation

$$F(x, y, z)=0. \tag{5}$$

At any point (x, y, z) lying on this surface a normal thereto has direction cosines which are proportional to the derivatives

$$\frac{\partial F}{\partial x}, \quad \frac{\partial F}{\partial y}, \quad \frac{\partial F}{\partial z}.$$

If now a line parallel to (δ) passes through this point it will be tangent to S, at this point, provided

$$l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z} = 0. \tag{6}$$

This equation represents another surface, and it is easy to see that the curve of intersection of the surfaces (5) and (6) is the locus of points at which rays of the congruence (1) are tangent to the surface (5) [the locus including multiple points of the surface (5)]. If this locus be taken for the curve Γ of Problem I, we will have, in the solution of that problem, the solution of Problem II.†

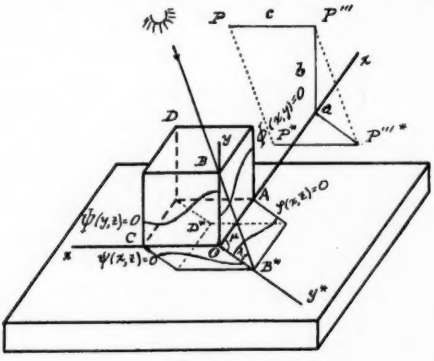

FIG. 1

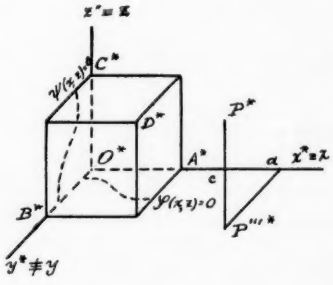

FIG. 2

Note. In order that we may visualize our procedure as much as possible, let us suppose that our three rectangular axes Ox, Oy, Oz lie along the edges of a cube and that the edges of this cube are of unit length. Furthermore let us think of the cube as resting on a horizontal table and of the sun's rays as casting a shadow of it upon this table (see pictorial view, Figure 1). Then the edges Ox and Oz, as well as the unit segments OA and OC on these edges are projected by the sun's rays into themselves, and the edge Oy is projected by these rays into the line Oy^* of the table, while the unit segment $\overline{OB}$ is projected into the segment $\overline{OB}^*$ on the table. It is easy to see that the shadow Oy^* of the edge Oy can have any direction in the plane of the table depending on the azimuth (μ) of the sun, while the segment $\overline{OB}^*$, into which the unit segment $\overline{OB}$ is projected, can be of any length depending on the altitude (λ) of the sun.

† The congruence of straight lines represented by equations (1) [or (i) of the second footnote] may be regarded as rays of light from an infinitely distant source, such as the sun. The curve of intersection of the surfaces (5) and (6) could then be regarded as the *line of shade* of the surface S which is produced by the source of light of rays (1). Furthermore, if the curve Γ of Problem I be regarded as a fine wire, then the cylindrical shadow of this wire represents the solution of this problem.

The projection (*i.e.*, the shadow) of the cube on the plane of the table will have the appearance, on that plane, which is shown in Figure 2. This form of *oblique parallel projection* is called *Cavalier perspective*, although usually this term is applied when the angles μ and λ have particular values.

2. Solution of Problem I. Let us suppose that the curve Γ of Problem I is given by the two equations

$$F_1(x, y, z) = 0, \qquad F_2(x, y, z) = 0. \tag{7}$$

The result of eliminating x, y, z from these two equations and equations (1) is an equation in the parameters X, Y, Z, and this equation represents the cylindrical surface sought in Problem I.

To be more specific, the two equations (7) may, in general, be (theoretically) solved for two of the variables in terms of the third, *i.e.*,

$$y = \mu(x), \qquad z = \nu(x).$$

If we put these expressions for y and z in equations (4) we obtain the three equations

$$\begin{vmatrix} m & n \\ \mu(x) & \nu(x) \end{vmatrix} = c_1, \qquad \begin{vmatrix} n & l \\ \nu(x) & x \end{vmatrix} = c_2, \qquad \begin{vmatrix} l & m \\ x & \mu(x) \end{vmatrix} = c_3,$$

of which, by (3), only two are independent, and the eliminant of these two is of the form

$$\Omega(c_i, c_j) = 0, \qquad i \neq j, \tag{8}$$

in which, by (3′), c_i, c_j are (linear) functions of X, Y, Z, and thus this equation represents the solution of Problem I. However, we may regard c_i, c_j as parameters fixing a line of congruence (1), and then, by (4), the cylindrical surface of Problem I is represented (in terms of the coördinates x, y, z) by the equation

$$\Omega(U_i, U_j) = 0, \qquad i \neq j. \tag{9}$$

On the other hand [still regarding c_i, c_j as the parameters fixing a line of the congruence (1)] the last equation, for an arbitrary function $\Omega(\xi, \eta)$ of its two arguments ξ, η, defines a one-parameter family of lines of the congruence (1) and thus this equation represents an arbitrary cylindrical surface of generators parallel to (δ). If now it be required to determine the form of the function $\Omega(\xi, \eta)$ so that each generator of the cylindrical surface (9) shall intersect a *given curve* Γ *of equations* (7), we have merely to proceed as in the preceding paragraph.*

* At this point it is interesting to note that the congruence of straight lines represented by the system of (finite) equations (1) may also be represented by the system of ordinary differential equations

$$\frac{dx}{l} = \frac{dy}{m} = \frac{dz}{n}, \tag{i}$$

2.1. If, in particular, the curve Γ lies in the xy-plane (see Figure 1), so that its equations may be written in the form

$$\Phi(x, y) = 0, \qquad z = 0, \tag{10}$$

we have from equations (1)

$$x = X - \frac{l}{n}Z, \qquad y = Y - \frac{m}{n}Z,$$

whence

$$\Phi\left(X - \frac{l}{n}Z,\ Y - \frac{m}{n}Z\right) = 0,$$

which, if we replace X, Y, Z by x, y, z, may be written in the form

$$\Phi\left(\frac{1}{n}U_2,\ -\frac{1}{n}U_1\right) = 0. \tag{10'}$$

This is the equation of the cylindrical surface, of elements parallel to (δ), which is determined by the curve (10).

To obtain the section of this cylindrical surface by the xz-plane, we put $y=0$ in the last equation and thus obtain for this section the equation

$$\Phi\left(x - \frac{l}{n}z,\ -\frac{m}{n}z\right) \equiv \phi(x, z) = 0. \tag{10''}$$

2.2. If the curve Γ lies in the yz-plane (see Figure 1), so that its equations may be written

$$\Psi(y, z) = 0, \qquad x = 0, \tag{11}$$

we obtain, in a manner similar to the above, the equation

$$\Psi\left(\frac{1}{l}U_3,\ -\frac{1}{l}U_2\right) = 0 \tag{11'}$$

for the cylindrical surface, of elements parallel to (δ), which is determined by the curve (11), and for the section of this surface by the plane $y=0$, the curve

$$\Psi\left(-\frac{m}{l}x,\ z - \frac{n}{l}x\right) \equiv \psi(x, z) = 0. \tag{11''}$$

---

of which, therefore, the solutions are given by a pair of the equations (4). Furthermore, the equation (9), for an arbitrary function Ω, represents the integral surfaces of the first order partial differential equation

$$l\frac{\partial f}{\partial x} + m\frac{\partial f}{\partial y} + n\frac{\partial f}{\partial z} = 0, \tag{ii}$$

of which the characteristics are represented by the above ordinary differential equations (i). Problems I and II are thus seen to be well known problems which arise in connection with the partial differential equation (ii).

2.3. If the curve Γ lies in the zx-plane, so that its equations may be written

$$\Theta(x, z) = 0, \qquad y = 0, \tag{12}$$

we obtain, as above, the equation

$$\Theta\left(-\frac{1}{m}U_3, \frac{1}{m}U_1\right) = 0 \tag{12'}$$

for the cylindrical surface, of elements parallel to (δ), which is determined by the curve (12), and the section of this cylinder by the plane $y=0$ is evidently the curve (12) itself.

2.4. If we replace the curve Γ by the point P of coördinates a, b, c (see Figure 1), we determine the projecting ray of this point by replacing X, Y, Z by a, b, c in equations (4), or in equations (1), and thus obtain for this ray the equations

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}. \tag{13}$$

To find the point in which this ray pierces the xz-plane, we put $y=0$ in these equations and thus find for the projection of $P(a, b, c)$ on the plane $y=0$ the point P^* of coördinates

$$\begin{cases} x = x^* = -\dfrac{\begin{vmatrix} l & m \\ a & b \end{vmatrix}}{m} = a - \dfrac{l}{m}b, \\ z = z^* = \dfrac{\begin{vmatrix} m & n \\ b & c \end{vmatrix}}{m} = c - \dfrac{n}{m}b. \end{cases} \tag{14}$$

If, in particular $c=0$, these coördinates become

$$x'''^* = x^*, \qquad z'''^* = -\frac{n}{m}b, \tag{14'}$$

and for the point $B(0, 1, 0)$ they become

$$x_B^* = -\frac{l}{m}, \qquad z_B^* = -\frac{n}{m}. \tag{14''}$$

From the sets of coördinates (14) and (14'), it follows that

$$a = x'''^* - \frac{l}{n}z'''^*, \qquad b = -\frac{m}{n}z'''^*, \qquad c = z^* - z'''^*. \tag{15}$$

Remark 1. Equations (14) enable us to obtain from the coördinates of a point $P(a, b, c)$ of space the coördinates (x^*, z^*) of the *projection* P^* of this point.

However, these equations are incapable of expressing uniquely the coördinates of P in terms of those of P^*. But if in addition to the coördinates (14) of the projection of P, we also have the coördinates (14′) of the projection of the foot P''' of the perpendicular from P to the xy-plane, then, as shown by equations (15), we have sufficient data for the determination of the coördinates (a, b, c) of P from the coördinates (x^*, z^*) and (x'''^*, z'''^*) of P^* and P'''^* (see Figures 1 and 2).

Instead of taking for the additional point the foot P''' of the perpendicular from P to the xy-plane, we might have taken the foot P'' of the perpendicular from P to the zx-plane or the foot P' of the perpendicular from P to the yz-plane.

From Remark 1 we infer that *a point in space cannot be adequately represented by a single point in the plane* (of projection), *but that it may be adequately represented by two* (properly related) *points of this plane*. Since a curve Γ of space is a one-parameter locus of points, a similar statement may be made for a curve.

For a point P to lie in the xy-plane it is necessary and sufficient that in the plane of projection $y=0$ (*i.e.*, in the plane of Figure 2) the points P^* and P'''^* should coincide, *i.e.*, $P^* \equiv P'''^*$. For P to lie in the xz-plane, the condition is $P^* \equiv P''^*$, and for P to lie in the yz-plane the condition is $P^* \equiv P'^*$. Similar conditions hold for curves which lie in the coördinate planes, and hence the curves (10″) and (11″) in the plane of projection $y=0$ (*i.e.*, in the plane of Figure 2) are, respectively, adequate representatives of the curves (10) and (11), (pictured in Figure 1), and (12) is its own representative.

Remark 2. We have just seen that a point, or a curve, which lies in one of the coördinate planes is adequately represented by its projection on the picture plane $y=0$ (*i.e.*, the plane of the paper in Figure 2). A general plane cuts these coördinate planes in straight lines (traces) which meet on the coördinate axes. Therefore straight lines, of two of the coördinate planes, which meet in the line of intersection of these planes determine a plane of space. Consequently two straight lines in the picture plane of Figure 2 which meet on one of the (*axonometric*) axes O^*x^*, O^*y^*, O^*z^* may together be regarded as an adequate representative of a plane of space.

A quadric surface is completely determined by its principal sections (*i.e.*, sections by its planes of symmetry) and if all of these sections lie in coördinate planes they are (as stated in the last paragraph of Remark 1) adequately represented by their projections on the picture plane, and thus together these projections adequately represent the surface. However, this "skeleton" of lines, while sufficient to represent the quadric surface, does not furnish a satisfactory picture of this surface. A more satisfactory picture may be obtained by adding to this "skeleton" the representatives in the picture plane of one or more properly chosen sections as well as the representative of the so-called *bounding curve*, concerning which something will be said in the next section.

A surface may ordinarily be satisfactorily represented in the picture plane by the representatives in this plane of sections made by equally spaced planes

parallel to the picture plane, together with the representative of the bounding curve.*

3. Solution of Problem II when S is a quadric surface. If, in particular, the surface S (of equation (5)) is a quadric surface, the equation (6) represents the polar plane of this surface with respect to an infinitely distant point in the direction (δ).

In order to treat our problem for such a surface let us introduce, for the moment, the homogeneous cartesian coördinates x_1, x_2, x_3, x_4, such that

$$x = \frac{x_1}{x_4}, \quad y = \frac{x_2}{x_4}, \quad z = \frac{x_3}{x_4},$$

and thus replace equation (5) by the homogeneous quadratic equation

$$Q \equiv \sum_1^4 a_{ij}x_ix_j = 0, \qquad a_{ij} = a_{ji}. \tag{16}$$

The polar plane of this quadric surface with respect to the point $Y(y_1, y_2, y_3, y_4)$ is

$$P \equiv \sum_1^4 a_{ij}x_iy_j = 0, \tag{17}$$

the left-hand member of which may be written in the form

$$P \equiv L_1x_1 + L_2x_2 + L_3x_3 + L_4x_4,$$

where

$$L_i = a_{i1}y_1 + a_{i2}y_2 + a_{i3}y_3 + a_{i4}y_4, \qquad (i = 1, 2, 3, 4). \tag{18}$$

The locus of points at which lines passing through the point Y are tangent to the surface (16) is the curve of intersection Γ of the surface (16) and the plane (17). The conical locus formed by such tangent lines may be found as follows:

Consider the line connecting the fixed point $Y(y_1, y_2, y_3, y_4)$ with a general point of space $X(x_1, x_2, x_3, x_4)$. The coördinates of a point of this line are

$$x_1 + \lambda y_1, \qquad x_2 + \lambda y_2, \qquad x_3 + \lambda y_3, \qquad x_4 + \lambda y_4,$$

where λ is a parameter. The condition that this point should lie on the quadric (16) is

$$\sum_1^4 a_{ij}(x_i + \lambda y_i)(x_j + \lambda y_j) \equiv \sum_1^4 a_{ij}x_ix_j + 2\lambda\sum_1^4 a_{ij}x_iy_j + \lambda^2\sum_1^4 a_{ij}y_iy_j = 0. \tag{19}$$

* See, for instance, the numerous beautiful pictures of this sort in Jahnke-Emde, Funktionentafeln (Tables of Functions) published in 1933 by B. G. Teubner, Leipzig and Berlin. See also §7, p. 156 of the author's paper in this MONTHLY, vol. 41, 1934, entitled, "Some frequently overlooked mathematical principles of descriptive geometry."

If the point Y does not lie on the quadric (16), this is a quadratic equation in λ and thus shows that the line YX pierces the surface (16) in two points. The condition that these two piercing points be coincident, and hence that the line YX be tangent to the quadric (16) is that the discriminant of equation (19) be zero, that is, that

$$\left(\sum_1^4 a_{ij}y_iy_j\right)\left(\sum_1^4 a_{ij}x_ix_j\right) - \left(\sum_1^4 a_{ij}x_iy_j\right)^2 = 0. \tag{20}$$

The point Y being fixed, this is an equation in the coördinates of the point X and thus represents the cone formed by those lines which pass through the point Y and are tangent to the quadric surface (16).*

If now we suppose the point Y to lie at infinity in the direction (δ), so that

$$y_1 : y_2 : y_3 : y_4 = l : m : n : 0, \tag{21}$$

the enveloping cone represented by equation (20) becomes a cylindrical surface of generators parallel to (δ), and thus we obtain a solution of Problem II for the quadric surface (16). If we further reintroduce our non-homogeneous coördinates by putting

$$x_1 : x_2 : x_3 : x_4 = x : y : z : 1, \tag{22}$$

we will have our solution, in non-homogeneous form, in the cartesian coördinates x, y, z.

In order to write equation (20) in another form let us denote the discriminant of the quadratic form Q by a, and the adjoint of this discriminant by A, so that

$$a = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}, \qquad A = \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} \\ A_{21} & A_{22} & A_{23} & A_{24} \\ A_{31} & A_{32} & A_{33} & A_{34} \\ A_{41} & A_{42} & A_{43} & A_{44} \end{vmatrix},$$

where A_{ij} is the cofactor of a_{ij} in a, and since $a_{ij} = a_{ji}$ it follows that $A_{ij} = A_{ji}$.

Assuming that $a \neq 0$, it can be shown that equation (20) may be written in the form:†

* This solution may be regarded as the dual of exercise 3, §53, p. 158 of Bôcher's Introduction to Higher Algebra.

† We may regard this solution as the dual of Theorem 2, §53, p. 158 of Bôcher's Higher Algebra. If we denote by β the bordered determinant on the left side of equation (23), it can be shown that

$$\beta \equiv a\left\{\sum_1^4 a_{ij}x_ix_j \cdot \sum_1^4 a_{ij}y_iy_j - \left(\sum_1^4 a_{ij}x_iy_j\right)^2\right\}$$
$$= \left(\sum_1^4 A_{ij}L_iL_j\right)\cdot \sum_1^4 a_{ij}x_ix_j - a(L_1x_1 + L_2x_2 + L_3x_3 + L_4x_4)^2$$

where L_i is given by equation (18). The first identity (but for the dual problem) suggests itself from exercise 4, §53, p. 158, Bôcher's Higher Algebra. The second identity comes from the first by means of Theorem 3, §54, p. 160, Bôcher's Higher Algebra.

$$(23)\qquad \begin{vmatrix} A_{11} & A_{12} & A_{13} & A_{14} & x_1 & y_1 \\ A_{21} & A_{22} & A_{23} & A_{24} & x_2 & y_2 \\ A_{31} & A_{32} & A_{33} & A_{34} & x_3 & y_3 \\ A_{41} & A_{42} & A_{43} & A_{44} & x_4 & y_4 \\ x_1 & x_2 & x_3 & x_4 & 0 & 0 \\ y_1 & y_2 & y_3 & y_4 & 0 & 0 \end{vmatrix} = 0,$$

in which we may again make the substitutions (21) and (22) in order to obtain another solution of Problem II, for the quadric surface (16), in terms of the cartesian coördinates x, y, z.

Let us now, in particular, consider the *central quadrics* and suppose these to be so situated, with respect to the rectangular cartesian axes Ox, Oy, Oz, that their *planes of symmetry shall coincide with the* coördinate planes.

In this case $a_{ij}=0$ if $i\neq j$. Let us further put $a_{44}=-1$, and make use of relations (21) and (22). Then

$$\sum_1^4 a_{ij}x_ix_j \equiv x_4^2(a_{11}x^2 + a_{22}y^2 + a_{33}z^2 - 1),$$

$$\sum_1^4 a_{ij}y_iy_j \equiv \rho^2(a_{11}l^2 + a_{22}m^2 + a_{33}n^2),$$

$$\sum_1^4 a_{ij}x_iy_j \equiv \rho x_4(a_{11}lx + a_{22}my + a_{33}nz),$$

where ρ and x_4 are the factors of proportionality in the proportions (21) and (22), and hence equation (20) becomes

$$(24)\qquad (a_{11}l^2+a_{22}m^2+a_{33}n^2)(a_{11}x^2+a_{22}y^2+a_{33}z^2-1)-(a_{11}lx+a_{22}my+a_{33}nz)^2=0,$$

which is the solution of Problem II for the central quadric surfaces

$$(25)\qquad a_{11}x^2 + a_{22}y^2 + a_{33}z^2 = 1,$$

[an ellipsoid, or an unparted or biparted hyperboloid].

Let us next consider the non-central quadrics and write their equations in the form

$$\sum_1^4 a_{ij}x_ix_j \equiv \left\{ \begin{array}{l} a_{11}x_1^2 \\ \qquad + a_{22}x_2^2 \\ \qquad\qquad\qquad + a_{34}x_3x_4 \\ \qquad\qquad + a_{43}x_4x_3 \end{array} \right\} = 0,$$

which by substitution (22) becomes

$$(26)\qquad a_{11}x^2 + a_{22}y^2 + 2a_{34}z = 0.$$

Then, by the further substitution (21)

$$\sum_1^4 a_{ij}y_iy_j \equiv \rho^2(a_{11}l^2 + a_{22}m^2),$$

$$\sum_1^4 a_{ij}x_iy_j \equiv \rho x_4(a_{11}lx + a_{22}my + a_{34}n),$$

and thus equation (20) becomes

$$(27)\quad (a_{11}l^2 + a_{22}m^2)(a_{11}x^2 + a_{22}y^2 + 2a_{34}z) - (a_{11}lx + a_{22}my + a_{34}n)^2 = 0,$$

which is the solution of Problem II for the non-central quadric surface (26) [a paraboloid, elliptic or hyperbolic, with vertex at the origin and planes xOz and yOz as planes of symmetry].

If in equations (24) and (27) we put $y=0$, we obtain, respectively, the equations

$$(\overline{24})\quad a_{11}(a_{22}m^2+a_{33}n^2)x^2-2a_{11}a_{33}lnxz+a_{33}(a_{11}l^2+a_{22}m^2)z^2=a_{11}l^2+a_{22}m^2+a_{33}n^2$$

and

$$(\overline{27})\quad a_{11}a_{22}m^2x^2 - 2a_{11}a_{34}lnx + 2a_{34}(a_{11}l^2 + a_{22}m^2)z - a_{34}^2n^2 = 0$$

as the equations of the *bounding curves** in the pictures of the quadric surfaces (25) and (26) respectively.

Equation (24) may be written in the form

$$(24')\quad (a_{11}l^2 + a_{22}m^2 + a_{33}n^2)(a_{11}x^2 + a_{22}y^2 + a_{33}z^2) - (a_{11}lx + a_{22}my + a_{33}nz)^2 = a_{11}l^2 + a_{22}m^2 + a_{33}n^2,$$

which by means of a well known identity of Lagrange† may be put into the form

$$(24'')\quad a_{22}a_{33}U_1^2 + a_{33}a_{11}U_2^2 + a_{11}a_{22}U_3^2 = a_{11}l^2 + a_{22}m^2 + a_{33}n^2,$$

where

$$U_1 = \begin{vmatrix} m & n \\ y & z \end{vmatrix}, \qquad U_2 = \begin{vmatrix} n & l \\ z & x \end{vmatrix}, \qquad U_3 = \begin{vmatrix} l & m \\ x & y \end{vmatrix}.$$

---

* Mr. Little calls these the *curves of visibility*.

† This identity is of the form

$$[L^2 + M^2 + N^2][X^2 + Y^2 + Z^2] - [LX + MY + NZ]^2 = \begin{vmatrix} M & N \\ Y & Z \end{vmatrix}^2 + \begin{vmatrix} N & L \\ Z & X \end{vmatrix}^2 + \begin{vmatrix} L & M \\ X & Y \end{vmatrix}^2.$$

If we put

$$L = \sqrt{a_{11}}\, l, \qquad M = \sqrt{a_{22}}\, m, \qquad N = \sqrt{a_{33}}\, n,$$
$$X = \sqrt{a_{11}}\, x, \qquad Y = \sqrt{a_{22}}\, y, \qquad Z = \sqrt{a_{33}}\, z,$$

we see that the left member of (24') is identical with that of (24'').

We also see that equation (27) may be written as follows:

$$(27') \qquad (a_{11}l^2 + a_{22}m^2)(a_{11}x^2 + a_{22}y^2) - (a_{11}lx + a_{22}my)^2 + 2a_{11}a_{34}l(lz - nx) + 2a_{22}a_{34}m(mz - ny) - a_{34}^2 n^2 = 0,$$

and this in turn becomes

$$(27'') \qquad a_{11}a_{22}U_3^2 + 2a_{22}a_{34}mU_1 - 2a_{11}a_{34}lU_2 - a_{34}^2 n^2 = 0,$$

where U_1, U_2, U_3 have the meanings given for equation (24'').

4. Corresponding treatment for orthographic projection. A *projection* being a perspection followed by a section, we have in the preceding pages [see equations (10''), (11''), (14), (14'), (14''), ($\overline{24}$), ($\overline{27}$)] found the oblique projection of certain curves, and certain points, by making a non-right section of the projecting rays. Let us now, however, make a *right section* and thus obtain an *orthographic projection* of these curves and points.

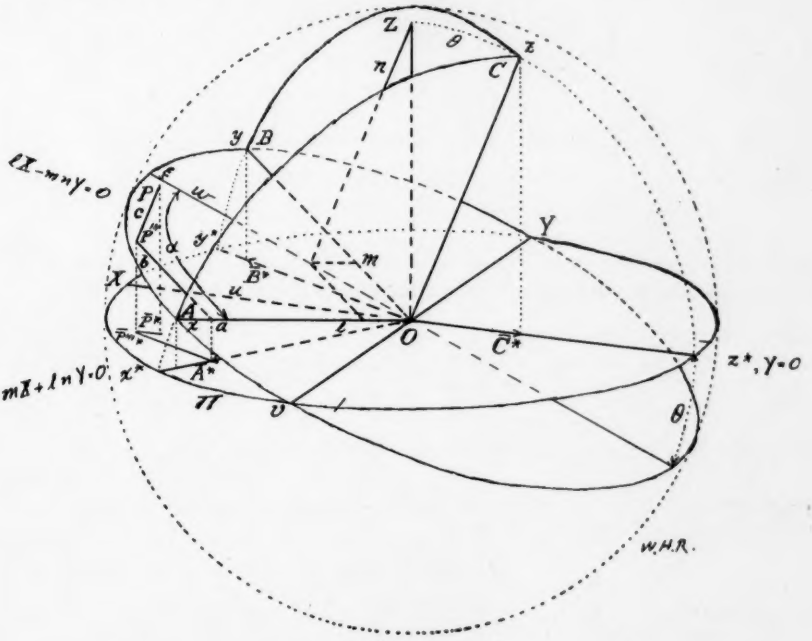

FIG. 3

To this end let us suppose that the plane π of this right section passes through the origin O of our set of axes Ox, Oy, Oz, and cuts the plane xOy in the line v (Figure 3). Let us next draw through O three lines u, w, s perpendicular to v, namely, u in the plane π, w in the plane xOy and s perpendicular to π. On these lines we will now take axes of common origin O and denote these by

OX, $O\xi$, and OZ, respectively. The senses on these axes are so chosen that when the set ξOY is rotated around Oz through an angle α this set will coincide with xOy, and when the set ξOz is rotated around v through an angle θ this set will coincide with XOZ. We thus have

$$(28)\qquad \begin{cases} X = \xi\cos\theta - z\sin\theta, \\ Z = \xi\sin\theta + z\cos\theta, \end{cases} \qquad \begin{cases} x = \xi\cos\alpha - Y\sin\alpha, \\ y = \xi\sin\alpha + Y\cos\alpha, \end{cases}$$

whence, by inversion,

$$(28')\qquad \begin{cases} \xi = X\cos\theta + Z\sin\theta, \\ z = -X\sin\theta + Z\cos\theta, \end{cases} \qquad \begin{cases} \xi = x\cos\alpha + y\sin\alpha, \\ Y = -x\sin\alpha + y\cos\alpha, \end{cases}$$

and consequently

$$(29)\qquad \begin{cases} X = (x\cos\alpha + y\sin\alpha)\cos\theta - z\sin\theta, \\ Y = -x\sin\alpha + y\cos\alpha, \\ Z = (x\cos\alpha + y\sin\alpha)\sin\theta + z\cos\theta, \end{cases}$$

or

$$(30)\qquad \begin{cases} x = (X\cos\theta + Z\sin\theta)\cos\alpha - Y\sin\alpha, \\ y = (X\cos\theta + Z\sin\theta)\sin\alpha + Y\cos\alpha, \\ z = -X\sin\theta + Z\cos\theta. \end{cases}$$

It is further evident from Figure 3, in which is shown the *segments* of lengths l, m, n defining the direction (δ) with respect to the set of axes Ox, Oy, Oz, that

$$(31)\qquad \sin\theta = \frac{\sqrt{l^2+m^2}}{\sqrt{l^2+m^2+n^2}}, \qquad \cos\theta = \frac{n}{\sqrt{l^2+m^2+n^2}},$$
$$\sin\alpha = \frac{m}{\sqrt{l^2+m^2}}, \qquad \cos\alpha = \frac{l}{\sqrt{l^2+m^2}}.$$

If, in particular, l, m, n are taken as the direction cosines of the direction (δ) with respect to Ox, Oy, Oz, so that

$$l^2 + m^2 + n^2 = 1,$$

we will have

$$(32)\qquad \sin\theta = \sqrt{l^2+m^2}, \qquad \cos\theta = n,$$
$$\sin\alpha = \frac{m}{\sqrt{l^2+m^2}}, \qquad \cos\alpha = \frac{l}{\sqrt{l^2+m^2}}.$$

For these expressions for θ and α the sets of equations of transformation (29) and (30) become

$$(33)\qquad \begin{cases} X = \dfrac{n(lx+my) - (l^2+m^2)z}{\sqrt{l^2+m^2}}, \\ Y = \dfrac{-mx+ly}{\sqrt{l^2+m^2}}, \qquad Z = lx+my+nz, \end{cases}$$

and

$$(34)\qquad \begin{cases} x = \dfrac{nlX - mY + l\sqrt{l^2+m^2}\,Z}{\sqrt{l^2+m^2}}, \\ y = \dfrac{mnX + lY + m\sqrt{l^2+m^2}\,Z}{\sqrt{l^2+m^2}}, \\ z = -\sqrt{l^2+m^2}X + nZ, \end{cases}$$

and hence we have

$$(35)\qquad \begin{cases} U_1 = \begin{vmatrix} m & n \\ y & z \end{vmatrix} = -\dfrac{mX + lnY}{\sqrt{l^2+m^2}}, \\ U_2 = \begin{vmatrix} n & l \\ z & x \end{vmatrix} = \dfrac{lX - mnY}{\sqrt{l^2+m^2}}, \\ U_3 = \begin{vmatrix} l & m \\ x & y \end{vmatrix} = \sqrt{l^2+m^2}\,Y. \end{cases}$$

Since the planes xOZ, yOZ, zOZ have the equations

$$U_1 = 0, \qquad U_2 = 0, \qquad U_3 = 0,$$

respectively, it follows that the orthographic projection, on the plane π, of the axes Ox, Oy, Oz are

$$(36)\qquad mX + lnY = 0, \qquad lX - mnY = 0, \qquad Y = 0,$$

respectively. See Figure 3, and also Figure 4, in which π coincides with the plane of the paper.

It further follows that in terms of the coördinates X, Y, Z* the equations of the projecting cylinders of the curves (10), (11), and (12) become, by (10′), (11′), and (12′),

$$(37)\qquad \Phi\left(\frac{lX - mnY}{n\sqrt{l^2+m^2}}, \qquad \frac{mX + lnY}{n\sqrt{l^2+m^2}}\right) \equiv \bar{\phi}(X, Y) = 0,$$

$$(38)\qquad \Psi\left(\frac{\sqrt{l^2+m^2}}{l}\,Y, \qquad -\frac{lX - mnY}{l\sqrt{l^2+m^2}}\right) \equiv \bar{\psi}(X, Y) = 0,$$

* The coördinates here denoted by X, Y, Z are *not* the parameters in equations (1).

$$(39)\qquad \Theta\left(-\frac{\sqrt{l^2+m^2}}{m}Y,\ -\frac{mX+lnY}{m\sqrt{l^2+m^2}}\right)\equiv\bar{\theta}(X,Y)=0,$$

respectively. These equations are free from Z because the projecting rays are perpendicular to the plane π. Thus these equations also represent the orthographic projections of the curves (10), (11), (12) respectively on the plane π.

Let us now obtain the orthographic projection on π $(Z=0)$ of the point P whose coördinates with respect to the axes Ox, Oy, Oz are a, b, c. To this end we put $x=a$, $y=b$, $z=c$ in equations (33), thus obtaining

$$(40)\qquad \begin{cases} X=\overline{X}^*=\dfrac{n(la+mb)-(l^2+m^2)c}{\sqrt{l^2+m^2}}, \\ Y=\overline{Y}^*=\dfrac{-ma+lb}{\sqrt{l^2+m^2}}. \end{cases}$$

If, in particular, $c=0$ these coördinates become

$$(41)\qquad \overline{X}'''^*=\frac{n(la+mb)}{\sqrt{l^2+m^2}},\qquad \overline{Y}_3'''^*=\overline{Y}^*,$$

and for the unit point A, for which $a=1$, $b=0$, $c=0$,

$$(42)\qquad X=\frac{nl}{\sqrt{l^2+m^2}},\qquad Y=\frac{-m}{\sqrt{l^2+m^2}},\qquad \text{coördinates of } \overline{A}^* \text{ (Figure 4)},$$

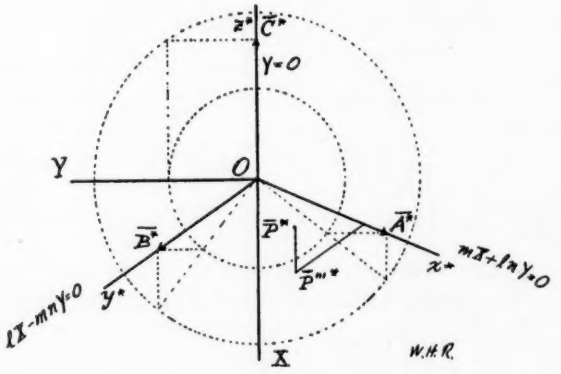

FIG. 4

for the unit point B, for which $a=0$, $b=1$, $c=0$,

$$(43)\qquad X=\frac{mn}{\sqrt{l^2+m^2}},\qquad Y=\frac{l}{\sqrt{l^2+m^2}},\qquad \text{coördinates of } \overline{B}^* \text{ (Figure 4)},$$

for the unit point C, for which $a=0$, $b=0$, $c=1$,

(44) $X = -\sqrt{l^2+m^2}, \quad Y = 0,$ coördinates of $\overline{C}^*$ (Figure 4).

From the sets of coördinates (40) and (41) it follows that

$$(45)\quad a = \frac{l\overline{X}'''^* - mn\overline{Y}'''^*}{n\sqrt{l^2+m^2}}, \qquad b = \frac{m\overline{X}'''^* + ln\overline{Y}'''^*}{n\sqrt{l^2+m^2}}, \qquad c = \frac{\overline{X}'''^* - X^*}{\sqrt{l^2+m^2}}.$$

Remark 3. Equations (40) enable us to obtain from the coördinates of a point $P(x=a, y=b, z=c)$ of space the coördinates $(\overline{X}^*, \overline{Y}^*)$ of the projection $\bar{P}^*$ of this point on π. However, from these equations we cannot obtain the coördinates of P from those of $\bar{P}^*$. But if in addition to the coördinates (40) of the point P^* we also have the coördinates (41) of the projection, $\bar{P}'''^*$, of the foot P''' of the perpendicular from P to the xy-plane, then, as shown by equations (45) we have sufficient data for the determination of the coördinates (a, b, c) of P from the coördinates $(\overline{X}^*, \overline{Y}^*)$ and $(\overline{X}'''^*, \overline{Y}'''^*)$ of $\bar{P}^*$ and $\bar{P}'''^*$ (see Figures 3 and 4). Since $\overline{Y}^* = \overline{Y}'''^*$, we see that the two points $\bar{P}^*$ and $\bar{P}'''^*$ of the picture plane π lie on a parallel to the (axonometric) axis Oz^*. Since, furthermore, we have by (42), (43), (44) the positions of the unit points $\overline{A}^*$, $\overline{B}^*$, $\overline{C}^*$, we can read from Figure 4 the coördinates of the point P in space. Instead of taking for the additional point the foot P''' of the perpendicular from the point P on the xy-plane, we might have taken the foot P'' of the point P on the xz-plane or the foot P' of the point P on the yz-plane. Thus here we draw conclusions similar to those stated in Remarks 1 and 2.

In order to obtain the equation, in terms of X, Y, Z, of the cylindrical surface, of generators parallel to (δ), which bounds the central quadric surface (25)

$$a_{11}x^2 + a_{22}y^2 + a_{33}z^2 = 1,$$

we have merely to replace U_1, U_2, U_3 in equation (24″) by the expressions (35). We thus obtain for this cylindrical surface the equation

$$(46)\quad \begin{aligned} &a_{22}a_{33}(mX + lnY)^2 + a_{33}a_{11}(lX - mnY)^2 + a_{11}a_{22}(l^2+m^2)^2Y^2 \\ &\qquad = (l^2+m^2)(a_{11}l^2 + a_{22}m^2 + a_{33}n^2), \end{aligned}$$

and since this equation contains no term in Z, it also represents the section of this cylinder by the plane π (*i.e.*, $Z=0$).

Similarly for the paraboloid (26)

$$a_{11}x^2 + a_{22}y^2 + 2a_{34}z = 0$$

we obtain from equation (27″) the equation

$$(47)\quad \begin{aligned} &a_{11}a_{22}(l^2+m^2)^{3/2}Y^2 - 2a_{22}a_{34}m(mX + lnY) - 2a_{11}a_{34}l(lX - mnY) \\ &\qquad - a_{34}^2n^2\sqrt{l^2+m^2} = 0, \end{aligned}$$

and this also is the equation of the section of this cylinder by the plane π $(Z=0)$. Equations (46) and (47) are thus the equations of the *bounding curves* in the orthographic pictures of the quadric surfaces (25) and (26) respectively.

COLLINEATIONS AND CENTRAL PROJECTIONS

P. H. DAUS, University of California at Los Angeles

1. Introduction. The following well known theorem is basic in any study of projective geometry. We indicate two usual proofs. (Figures 1 and 2.)

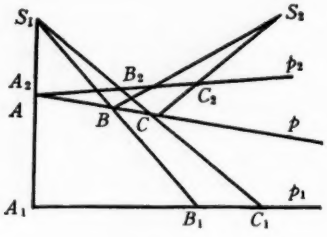

FIG. 1

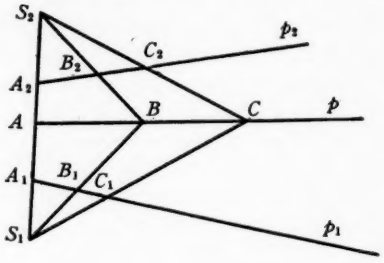

FIG. 2

THEOREM 1. *Any three points A_1, B_1, C_1 of a line p_1 can be transformed into three points A_2, B_2, C_2 of another line p_2 by means of two centers of perspectivity.*

In Fig. 1 we select S_1 arbitrarily on A_1A_2, and p arbitrarily through A_2. Then S_2 is determined as the intersection of BB_2 and CC_2. The proof indicated in Figure 2 is more symmetrical; S_1 and S_2 are selected as arbitrary points on A_1A_2, and p is determined as the line BC.

The corresponding two-dimensional problem, fundamental in the study of collineations, may be considered* by reduction to the one-dimensional case, or the following space analog of the above theorem.

THEOREM 2. *Any complete quadrangle $A_1B_1C_1D_1$ of a plane π_1 can be transformed into any complete quadrangle $A_2B_2C_2D_2$ of π_2 by means of three perspectivities.*†

If the given planes are coincident, it is obvious we could reduce the construction to one involving distinct planes by an additional perspectivity. But it is desirable to consider this problem as one to be solved entirely in the plane. Accordingly it is the purpose of this article to present proofs of the following theorem, these proofs being generalizations of those indicated in Figures 1 and 2.

THEOREM 3. *Any complete quadrangle $A_1B_1C_1D_1$ can be transformed into any complete quadrangle $A_2B_2C_2D_2$ in the same plane by means of three*‡ *central projections.*

* See, for example, F. Enriques (English translation by H. R. Phalen), Lessons in Projective Geometry, p. 115; Reye, Geometrie der Lage, Part 2, p. 5; Patterson, Projective Geometry, p. 253.

† Cremona, Elements of Projective Geometry, translated by C. Leudesdorf, 3rd ed., pp. 78–79.

‡ Veblen and Young, Projective Geometry, vol. 1, p. 74, state that a finite number are required and actually use five central projections in the proof. Reye, *l.c.*, p. 20, makes the transformation of one quadrangle into the other depend upon metrical considerations.

2. Definitions. First let us define a few terms used in the foregoing theorems and elsewhere in this paper, since the same terminology is not used by all authors. A *collineation* between two planes is a 1-1 projective correspondence which transforms a point into a point, a line into a line, and maintains incidence properties between points and lines. The simplest type of a collineation between distinct planes is a *perspectivity*, whereby the planes π_1 and π_2 are sections of the same bundle of lines and planes whose center is S. The simplest type of collineation between two coincident planes is called a *central projection*,* and may be defined as follows. If the plane π is projected from two distinct centers S_1 and S_2 upon the plane $\pi_1 \equiv \pi_2$, the correspondence between the points and lines of π_1 and those of π_2 is called a central projection. It is characterized by the properties that the line s, the intersection of π and π_1, and the point S, the intersection of S_1S_2 and π_1, remain invariant; corresponding points are collinear with S, and corresponding lines are concurrent with s. Because of these properties, the correspondence is completely determined by planar constructions if the center S, axis s, and any pair of corresponding points (or lines) are given. If S is not on s, the central projection is known as an *homology*, while if S is on s, it is called an *elation*.

It is apparent that a chain of perspectivities or of central projections establishes a collineation between the first and last members of the chain. In proving, conversely, that any collineation can be established by such a chain, we require such results as given in Theorem 2 or Theorem 3.

3. Construction of the collineation. We make the proof of Theorem 3 depend upon the following lemma, which is an immediate consequence of Desargues's theorem on perspective triangles.

LEMMA. *If $A_2B_2C_2$ and $A_3B_3C_3$ are triangles in the same plane, there exists a triangle ABC which is a central projection of both.*

For (Figure 3) suppose D is an arbitrary point in the given plane, while s_2 and s_3 are arbitrary lines through D. Let the sides B_2C_2, C_2A_2, A_2B_2 cut s_2 in L_2, M_2, N_2, respectively. Similarly determine L_3, M_3, N_3. The three lines L_2L_3, M_2M_3, N_2N_3 form a triangle ABC perspective to the given triangles with respect to the axes s_2 and s_3; and hence ABC is perspective to $A_2B_2C_2$ and $A_3B_3C_3$ with respect to the centers S_2 and S_3.

Therefore $A_2B_2C_2$ may be transformed into $A_3B_3C_3$ by two central projections which keep D invariant.

This lemma and the corresponding configuration are completely self-dual. Hence we could have selected S_2 and S_3 arbitrary and by means of a triangle abc determine the axes s_2 and s_3, but we shall not carry this idea further here.

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* For a discussion and classification of central projections, see Doehlmann, Geometrische Transformationen, Göschens Lehrbücherei, vol. 15, 2nd ed., pp. 104–118. Also see, in particular, Veblen and Young, *l. c.*, p. 72, where the term *perspective collineation* is used. Patterson, *l. c.*, uses the term *planar perspective*.

We are now in a position to establish Theorem 3. For if $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ (Figure 3) are the given quadrangles, we may take S_1 as any point on D_1D_2, and s_1 as any line in the plane and so centrally project $A_1B_1C_1D_1$ into $A_3B_3C_3D_2$. Now using D_2 as the point D above, we apply the construction of the Lemma, and thus establish the following chain of central projections:

$$(1) \qquad A_1B_1C_1D_1 \overset{S_1}{\wedge} A_3B_3C_3D_2 \overset{S_3}{\wedge} ABCD_2 \overset{S_2}{\wedge} A_2B_2C_2D_2.$$

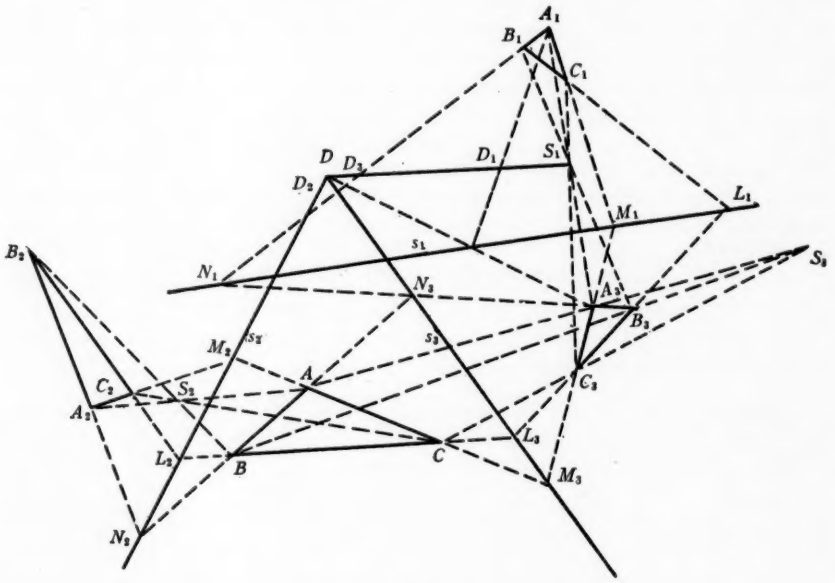

FIG. 3

4. Special cases and classification. It is well known that the collineation is determined by the quadrangles, and we have shown how to determine the correspondence by three central projections when the quadrangles are given in general position. Every collineation has at least one real self-corresponding point,* and if this point is given, the lemma indicates directly that two central projections (homologies H or elations E) suffice. The type† of collineation, as determined by the fixed elements, may be specified by the relative positions of S_i and s_i. As in the lemma, we select s_i as distinct lines through the real self-

* For a geometric construction of the self-corresponding points, see Veblen and Young, *l. c.*, p. 295. Briefly the construction is as follows: Let a point $A_2 \equiv B_1$ have correspondents A_1 and B_2. Projective pencils at A_1 and A_2, and at B_1 and B_2 determine conics which intersect at $A_2 \equiv B_1$ and again at the self-corresponding points, and there is always one such real intersection.

† Veblen and Young, *l. c.*, p. 106.

corresponding point, and consider possible positions of S_1 and S_2. If S_1 and S_2 are not both on one s_i, the collineation has three distinct fixed points and is of type I; it may be a product H_1H_2, HE, or E_1E_2, depending upon whether S_i is, or is not, on the corresponding s_i. If S_1 and S_2 are both on s_2, say, while S_1 is not on s_1, then we have type II, with only two fixed points, and the collineation is a product HE. If, however, S_1 and S_2 are both on s_2 with S_1 also on s_1, we have type III, with only one fixed point, and the collineation is a product E_1E_2. These three types together with H, E, and the identity complete the classification.

5. Second proof of Theorem 3. There might be given two or three self-corresponding points in the collineation, but unless additional conditions are stated, two central projections will be required. One such special case which we use in the sequel is when we are given two self-corresponding points and it is required to transform $A'B'CD$ into $A''B''CD$. If $A'B'$ and $A''B''$ meet on CD, then one central projection suffices, an homology or elation depending upon whether $S \equiv A'A'' - B'B''$ is not or is on CD. This special case enables us to give a proof of Theorem 3 which is more symmetrical than the one given, and is essentially a generalization of Figure 2.

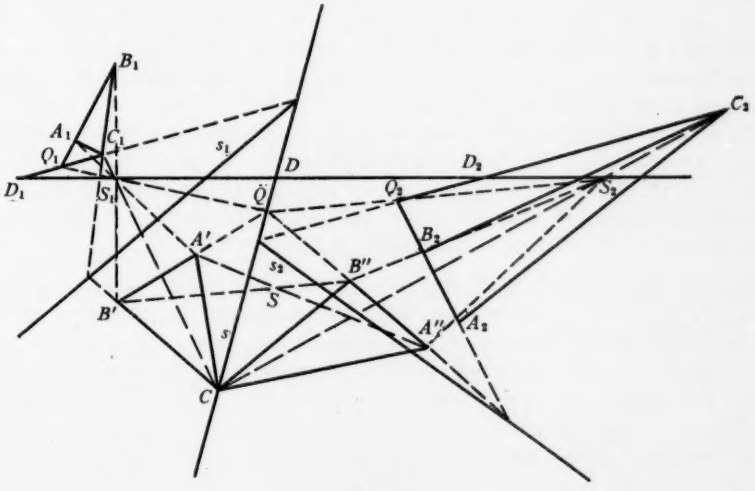

FIG. 4

Let S_1 and S_2 be arbitrary points on D_1D_2 (Figure 4). Let $S_1C_1 - S_2C_2 = C$, and $S_1Q_1 - S_2Q_2 = Q$, where $A_iB_i - C_iD_i = Q_i$, $(i = 1, 2)$. Let CQ intersect D_1D_2 in D. Take s_i through the intersection of CD and C_iD_i. If then we project $A_iB_iC_iD_i$ using S_i and s_i, $A'B'$ and $A''B''$ meet at Q on CD, so that $A'B'CD$ is a central projection of $A''B''CD$ with axis $s = CD$ and center $S = A'A'' - B'B''$. Hence we have the following chain of central projections to prove Theorem 3:

$$A_1B_1C_1D_1 \overset{S_1}{=\!\wedge} A'B'CD \overset{S}{=\!\wedge} A''B''CD \overset{S_2}{=\!\wedge} A_2B_2C_2D_2. \tag{2}$$

With a little ingenuity and a little more patience, the interested reader can improvise other constructions by taking S_1 and S_2, or s_1 and s_2, in special positions, or by other special devices. We may also leave as exercises for the reader the dual constructions.

PYTHAGOREAN SETS OF NUMBERS

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O. E. BROWN, Case School of Applied Science

1. Introduction. Students of high school geometry know that $3^2+4^2=5^2$, and every teacher of mathematics knows several other sets of numbers which satisfy the equation $x^2+y^2=z^2$. If $a_1, a_2, \cdots, a_m; b_1, b_2, \cdots, b_n$ are any $m+n$ integers satisfying the equation

$$a_1^2 + a_2^2 + \cdots + a_m = b_1^2 + b_2^2 + \cdots + b_n^2 \tag{1}$$

the set $(a_1, \cdots, a_m; b_1, \cdots, b_n)$, written briefly as (a, b), is called a *pythagorean set of numbers of dimensions* (m, n). The numerical value of each member of (1) is called the *norm* of the set. The numbers on either side of the semicolon constitute *partial sets*. For example, (3, 4, 5; 7, 1) is a pythagorean set of dimensions (3, 2) and norm 50. The partial sets are (3, 4, 5) and (7, 1).

Parametric formulas have been devised* which give all the pythagorean sets of dimensions (2, 1), (3, 1), (2, 2), and $(n, 1)$. We give, in this paper, an algorithm which gives all pythagorean sets of any dimensions (m, n) whatever. It consists of a set of transformations each of which carries a given pythagorean set (a, b) over into another pythagorean set (A, B).

2. The trivial transformations. Certain of these transformations may be spoken of as *trivial*. They consist of operations obviously allowable such as adjoining zero to or deleting it from either partial set, adjoining the same integer to or deleting it from both partial sets, permuting the elements in either partial set, interchanging the two partial sets, and changing the sign of any element.

For example, under trivial transformations the set (8, 1; 7, 4) becomes any one of the sets (8, 0, 1; −4, 7), (8, 1, 2; 7, 4, 2), and (4, 7; 1, 8).

To be sure, the trivial transformations involving the adjunction or deletion of elements change the dimensions of the pythagorean set to which they are applied.

3. Transformations T_h. Assume now a set of integers (a, b) of dimensions (m, n) satisfying (1). Let us seek new integers (A, B), also satisfying (1), of the form

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* L. E. Dickson, History of the Theory of Numbers, vol. 2, p. 165.

$$
(2)\qquad
\begin{aligned}
A_i &= s - a_i, && (i = 1, \cdots, k),\\
A_i &= a_i, && (i = k+1, \cdots, m),\\
B_i &= s - b_i, && (i = 1, \cdots, h),\\
B_i &= b_i, && (i = h+1, \cdots, n),
\end{aligned}
$$

where h, k, and s are to be determined later. Since (A, B) is to be a pythagorean set we have

$$\sum_{i=1}^{k}(s - a_i)^2 + \sum_{i=k+1}^{m} a_i^2 = \sum_{i=1}^{h}(s - b_i)^2 + \sum_{i=h+1}^{n} b_i^2,$$

which, by use of (1), reduces to

$$(3)\qquad (k - h)s^2 - 2\left(\sum_{i=1}^{k} a_i - \sum_{i=1}^{h} b_i\right)s = 0.$$

The roots of (3) are $s=0$, which leads to a trivial transformation, and

$$(4)\qquad s = \frac{2}{k-h}\left(\sum_{i=1}^{k} a_i - \sum_{i=1}^{h} b_i\right).$$

The latter is an integer if $k-h=\pm 1$ or ± 2. The case $k-h=2$ has been found sufficient for our purposes; so we define T_h as the transformation furnished by (2) with s defined by (4), with $1 \leqq h \leqq n$, and with $k=h+2$.

In order to illustrate the use of the transformations T_h let us apply T_1 to the (trivial) set (2, 1, 0; -2, 1). We have $k=3$ and

$$
\begin{aligned}
s &= a_1 + a_2 + a_3 - b_1 = 2 + 1 + 0 + 2 = 5,\\
A_1 &= s - a_1 = 5 - 2 = 3,\\
A_2 &= s - a_2 = 5 - 1 = 4,\\
A_3 &= s - a_3 = 5 - 0 = 5,\\
B_1 &= s - b_1 = 5 + 2 = 7,\\
B_2 &= b_2 = 1.
\end{aligned}
$$

We thus obtain the non-trivial set (3, 4, 5; 7, 1). By use of the trivial transformations this last pythagorean set may be changed to the set (3, 4, 5; -1, 7) which is transformed by T_1 into the set (10, 9, 8; 14, 7).

It is the object of this paper to show that any existing non-trivial pythagorean set may be obtained from a trivial set by the proper sequence of these transformations T_h and trivial transformations.

4. The completeness of the method. To show that this method will give all existing pythagorean sets we shall show, first, that each transformation of the set has an inverse also in the set, and second, that by these transformations any non-trivial pythagorean set (a, b) may be transformed into a pythagorean

set (A, B) whose norm is smaller than the norm of (a, b). Having proved these statements we need only argue that, by continually decreasing the norm, we may reduce any set to one whose norm is zero and, by applying the inverses of these transformations in the reverse order, we can obtain the original set by transforming the trivial set $(0; 0)$.

That each of the five trivial transformations has an inverse of its own type is obvious. We shall show now that any T_h is its own inverse. For, let $T_h(a, b) = (A, B)$ and $T_h(A, B) = (\alpha, \beta)$. In applying T_h to (a, b) we have

$$s = \sum_{i=1}^{h+2} a_i - \sum_{i=1}^{h} b_i.$$

In applying T_h to (A, B) we have

$$S = \sum_{i=1}^{h+2} A_i - \sum_{i=1}^{h} B_i = \sum_{i=1}^{h+2} (s - a_i) - \sum_{i=1}^{h} (s - b_i) = 2s - \sum_{i=1}^{h+2} a_i + \sum_{i=1}^{h} b_i = s.$$

Hence

$$\alpha_i = s - A_i = s - (s - a_i) = a_i, \quad (i = 1, \cdots, h + 2),$$
$$\beta_i = s - B_i = s - (s - b_i) = b_i, \quad (i = 1, \cdots, h),$$

and $(\alpha, \beta) = (a, b)$.

To transform a non-trivial set (a, b) so that the new set (A, B) shall have a smaller norm, suppose trivial transformations have been applied such that

$$b_1 > a_1 \geqq a_2 \geqq \cdots \geqq a_m \geqq 0, \tag{5}$$

$$b_1^2 = b_1^2 + \cdots + b_h^2, \tag{6}$$

whence $b_i = 0$, $(i = 2, \cdots, h)$ if $h > 1$. Choose h so that

$$b_1 > a_1 + \cdots + a_h, \tag{7}$$

$$b_1 < a_1 + \cdots + a_h + a_{h+1} + a_{h+2}. \tag{8}$$

This is plainly possible unless

$$b_1 \geqq a_1 + \cdots + a_m,$$

in which case we should have

$$b_1^2 + \cdots + b_n^2 \geqq b_1^2 \geqq (a_1 + \cdots + a_m)^2 \geqq a_1^2 + \cdots + a_m^2.$$

Since (1) holds, this could only be true with all signs read as equals and, hence, we should have $n = m = 1$, contrary to the assumption that (a, b) is non-trivial.

Let us now compare the norm of the new set (A, B) with that of (a, b). We have

$$\sum_{i=1}^{n} B_i^2 = \sum_{i=1}^{h} (s - b_i)^2 + \sum_{i=h+1}^{n} b_i^2$$

$$= hs^2 - 2s\sum_{i=1}^{h} b_i + \sum_{i=1}^{n} b_i^2$$

$$= hs^2 - 2sb_1 + \sum_{i=1}^{n} b_i^2 .$$

Whence

$$\sum_{i=1}^{n} B_i^2 - \sum_{i=1}^{n} b_i^2 = s(hs - 2b_1). \tag{9}$$

Now from (7) and (8) we have

$$b_1 < a_1 + \cdots + a_{h+2} < b_1 + a_{h+1} + a_{h+2}.$$

Subtracting b_1 we have

$$0 < s < a_{h+1} + a_{h+2}. \tag{10}$$

Now from (5) and (7)

$$b_1 > a_1 + \cdots + a_h \geqq ha_{h+1} \geqq ha_{h+2}.$$

Hence, by these relations and (10)

$$2b_1 > h(a_{h+1} + a_{h+2}) > hs,$$

or

$$hs - 2b_1 < 0. \tag{11}$$

Since one factor of the right-hand member of (9) is positive by (10), and the other factor is negative by (11) the product is negative and

$$\sum_{i=1}^{n} B_i^2 < \sum_{i=1}^{n} b_i^2 .$$

This concludes the proof.

DEFINITIONS AND PROPERTIES OF THE MEDIAN, QUARTILES, AND OTHER POSITIONAL MEANS

E. L. DODD, University of Texas

1. Introduction. The literature on the median, quartiles, deciles, and other "iles" is extensive. It is easy to give precise meanings to these positional means or division marks when we are dealing with continuous frequency. Thus, with unit area between a frequency curve and the X-axis, the third decile is that abscissa for which the area to its left is 0.3, while the area to its right is 0.7.

The approach to the discrete case does not seem so obvious. In practice, data are often grouped; and certain interpolative methods are used for locating the quartiles and other iles—with results generally satisfactory. But any regrouping would change the position of the iles. Theoretically, this is somewhat objectionable. It suggests the desirability of defining positional means in some way independent of a choice of the range of the groups.

The purpose of this paper is, first, by a minimizing process to define the positional means in the discrete case where the data or variates are n real numbers

$$x_1 \leqq x_2 \leqq \cdots \leqq x_{n-1} \leqq x_n. \tag{1}$$

This fixes such a mean as a single number or as a set of numbers which, geometrically speaking, forms an interval.

But sometimes it may seem desirable to focus attention upon some point in such an interval. For the median of an even number of variates, Dunham Jackson* used a limit process to accomplish this result. It will be shown that this limit process may be used for the other positional means in such cases of indeterminacy.

Finally, a few properties of positional means will be set forth—some fairly well known or at least subconsciously perceived.

2. Positional means as points or intervals to effect minimization. One of the outstanding characteristics of the median is that it minimizes the sum of the absolute deviations from it for a given set of measurements or variates. Each such deviation $|x-x_i|$ is here counted but once. By a natural extension of this principle, in order to locate the first quartile which is designed to place three times as much frequency to the right as to the left, we may count each of the left deviations three times, while counting each of the right derivations only once. Passing at once to the general case, we note that the k-iles are designed to divide frequency into k parts approximately equal. Thus, the r-th k-ile should split frequency into two parts bearing to each other the ratio of r to $k-r$. Hence, preparatory to minimization, these numbers in reverse order should be used as weights—the left deviations to be weighted by $k-r$, and the right deviations by r.

Given, then, n real numbers, x_i ordered as in (1). These may be thought of as abscissas of n points on a line. Then, with x as a variable abscissa, consider the

* Note on the median of a set of numbers, Bulletin of the American Mathematical Society, vol. 27, 1921, pp. 160–164.

distances from x to each of the n "points" x_i, these distances regarded as positive. And let I_t be the interval from x_t to x_{t+1}. Thus

$$I_t: \qquad x_t \leqq x \leqq x_{t+1}. \tag{2}$$

With x in I_t, the distances from x to $x_1, x_2, \cdots, x_t$ will be called anterior distances, and will be designated by δ_j; the other distances will be called posterior distances, and will be designated by Δ_j. If $x \leqq x_1$, then all these distances are posterior; if $x \geqq x_n$, then all these distances are anterior. Now let

$$F(x) = (k-r)\sum \delta_j + r\sum \Delta_j, \qquad 0 \leqq r \leqq k. \tag{3}$$

For example, if x is in I_t given by (2), then

$$\begin{aligned} F(x) = {} & (k-r)[(x-x_1) + \cdots + (x-x_t)] \\ & + r[(x_{t+1}-x) + \cdots + (x_n - x)]. \end{aligned} \tag{4}$$

THEOREM I. *Given $F(x)$ in (3) with $0<r<k$, it follows that (1°) either $F(x)$ takes its least value when x has some value x_t with t an integer nearest to $\frac{1}{2}+nr/k$; or (2°) there is some interval I_t given by (2), with $t=nr/k$, such that $F(x)$ is constant in I_t, and has a larger value outside I_t than in I_t,—a case of weak minimum.*

DEFINITION 1. With $0<r<k$, the r-th k-ile of $x_1, x_2, \cdots, x_n$ is the number or "point" x_t which minimizes $F(x)$ or it is the interval I_t which gives to $F(x)$ a weak minimum.

DEFINITION 2. The maximum or greatest number in a set will be called the k-th k-ile for every positive integer k; and the minimum or least number will be called the 0-th k-ile.

Proof of Theorem. Case 1. Suppose that rn is not a multiple of k. In this case, an integer t is determined by the inequalities,

$$t - 1 < rn/k < t. \tag{5}$$

That is, t is the next integer above the fraction rn/k, and t is also either equal to $rn/k+1/2$ or it is the integer nearest thereto. From (5), it follows that

$$kt - k < rn < kt. \tag{6}$$

From (4) with $x=x_t$, we obtain, since $x_t - x_t = 0$,

$$\begin{aligned} F(x_t) = {} & (k-r)([x_t - x_1) + \cdots + (x_t - x_{t-1})] \\ & + r[(x_t - x_t) + (x_{t+1} - x_t) + \cdots + (x_n - x_t)]. \end{aligned} \tag{7}$$

With $c>0$, consider now $F(x_t-c)$ as related to $F(x_t)$. As the x in $F(x)$ is moved from the point x_t a distance of c to the left, at least $(n-t+1)$ posterior distances are each increased by c, and not more than $(t-1)$ anterior distances are decreased by c. Then

$$\begin{aligned} F(x_t - c) &\geqq F(x_t) + r(n-t+1)c - (k-r)(t-1)c \\ &= F(x_t) + (rn - kt + k)c > F(x_t), \end{aligned} \tag{8}$$

since $c>0$, and by (6) $rn-kt+k>0$. Likewise,

$$F(x_t + c) \geqq F(x_t) + (k - r)tc - r(n - t)c > F(x_t). \tag{9}$$

Here, then, $x=x_t$ is the value which minimizes $F(x)$.

Case 2. Suppose nr is a multiple of k; and $x_{t+1}>x_t$, where $t=nr/k$. Let

$$x_t \leqq x < x' \leqq x_{t+1}; \qquad d = x' - x. \tag{10}$$

Then

$$F(x') = F(x) + (k - r)td - r(n - t)d = F(x), \tag{11}$$

since, by hypothesis, $kt=rn$. But, with $c>0$,

$$\begin{aligned} F(x_t - c) &\geqq F(x_t) + r(n - t + 1)c - (k - r)(t - 1)c \\ &= F(x_t) + kc > F(x_t), \end{aligned} \tag{12}$$

and

$$\begin{aligned} F(x_{t+1} + c) &\geqq F(x_{t+1}) + (k - r)(t + 1)c - r(n - t - 1)c \\ &= F(x_{t+1}) + kc > F(x_{t+1}). \end{aligned} \tag{13}$$

Hence, if $t=nr/k$, the interval I_t gives to $F(x)$ a weak minimum.

Case 3. Suppose n is a multiple of k; and $x_{t+1}=x_t$, where $t=nr/k$. This may be regarded as a degenerate case arising when the interval I_t in Case 2 shrinks to a point. In this case $F(x)$ is minimized by the single value $x=x_t=x_{t+1}$. Here nearest to $1/2+nr/k$ are the two integers t and $t+1$.

This completes the proof of the theorem. But, in connection with Definition 2, it may be noted that if in (3) we take $r=k$, then $F(x)$ takes a weak minimum for $x\geqq x_n$. Likewise, if in (3) we take $r=0$, $F(x)$ takes a weak minimum for $x\leqq x_1$.

3. Unique values for positional means as obtained by a limit process. Under Case 2—see (11)—the interval I_t is the r-th k-ile. Nevertheless, when desirable, it is possible to pick out of this interval a single value for the r-th k-ile, following the method that Dunham Jackson used, to assign a single value to the median of an even number of variates. This method involves the use of a new function

$$G(x, p) = (k - r)\sum \delta_i^p + r\sum \Delta_i^p, \qquad p > 1, \tag{14}$$

in place of (3). And we now vizualize a set of variates where $x_1, x_2, \cdots, x_t$, are each actually set down $k-r$ times, while $x_{t+1}, \cdots, x_n$ are each set down r times. If now $x_t<x<x_{t+1}$, there are to the left of x exactly $r(n-t)$ points; and also exactly the same number to the right. This parallels the situation discussed by Jackson. We may then conclude that

$$\lim_{p\to 1} G(x, p) = \mu, \tag{15}$$

a unique value, which may also be found from

$$[(\mu - x_1)(\mu - x_2) \cdots (\mu - x_t)]^{k-r} = [(x_{t+1} - \mu) \cdots (x_n - \mu)]^r. \tag{16}$$

4. Some properties of positional means. Let $y=f(x_1, x_2, \cdots, x_n)$ be a mean of the real numbers $x_1, x_2, \cdots, x_n$. The median, quartiles, and other iles are all means, not only general or *substitutive** means satisfying the requirement that

$$f(c, c, \cdots, c) = c; \tag{17}$$

but they are *internal* means satisfying

$$\text{Minimum } (x_i) \leqq f(x_1, x_2, \cdots, x_n) \leqq \text{Maximum } (x_i). \tag{18}$$

Including the minimum and the maximum, there are $(k+1)$ k-iles, symmetrically situated with respect to the "center" of the data in the following sense: If x_t is the r-th k-ile, then the $(k-r)$-th k-ile is the t-th variate from the right end; viz., x_{n-t+1}.

It is natural to ask why in finding the r-th k-ile, when nr is not a multiple of k, it should turn out that for t in x_t, the integer next *above* rn/k must be taken rather than the integer next *below*. This peculiarity seems to be associated with the fact that the difference between the subscripts of x_1 and x_n is $n-1$, and not n. As a remedy, let us conceive that hypothetical variates $x_{1/2}$ and $x_{n+1/2}$ are added to the set, with the difference between the subscripts equal to n. Then the rule for t can be roughly expressed thus: To get the r-th k-ile, go *up* t steps from $x_{1/2}$, or go *down* $n-t$ steps from $x_{n+1/2}$. Or, precisely: Take for t the integer *nearest* to $1/2+nr/k$.

5. Simulvariance. Illustration: If each x_i is increased by a constant c, each positional mean is *simultaneously* increased by c.

DEFINITION 3. A function $f(x_1, x_2, \cdots, x_n)$ will be said to be simulvariant to a transformation $y=\psi(x)$ if

$$f\{\psi(x_1), \psi(x_2), \cdots, \psi(x_n)\} = \psi\{f(x_1, x_2, \cdots, x_n)\}. \tag{19}$$

Simulvariance may be thought of as a special case of *cogredience*. Or, in case ψ has an inverse ψ^{-1}, we may write briefly $\psi^{-1}f\psi=f$, exhibiting a sort of *invariance* of f to the joint action of ψ and ψ^{-1}. Here f is a function of n variables; but ψ of only one.

THEOREM II. *Positional means, as defined in Definitions* 1 *and* 2 *are simulvariant to all continuous monotone increasing transformations.*

Proof. It is given that $\psi(x') \geqq \psi(x)$ if $x' > x$. If, then, $y_i=\psi(x_i)$, the ordered points $x_1 \leqq x_2 \leqq \cdots \leqq x_{n-1} \leqq x_n$, are carried over into $y_1 \leqq y_2 \leqq \cdots \leqq y_{n-1} \leqq y_n$, maintaining the same order. It may happen, of course, that an interval $x_t \leqq x \leqq x_{t+1}$, where $x_t < x_{t+1}$ is carried over into a point with $y_t=y_{t+1}$, so that a mean originally not single-valued becomes single-valued.

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* O. Chisini, Sul concetto di media, Periodico di Matematiche, ser. 4, vol. 9, 1929, pp. 106–116.

E. L. Dodd, Internal and external means arising from the scaling of frequency functions, Annals of Mathematical Statistics, vol. 8, 1937, pp. 12–20.

E. L. Dodd, Regression coefficients as means of certain ratios, This MONTHLY, vol. 44, 1937 pp. 306–308.

The simulvariance just noted ceases to be valid if in place of an interval I_t mentioned in Definition 1, there is substituted the unique value obtained by the Jackson method in (15) or the crude but more convenient $(x_t+x_{t+1})/2$. However, when either of these two unique values replaces the interval I_t, the positional means are still simulvariant to the translation $y=x+c$, and to the stretching transformation $y=cx$ with c a constant. These means are then *translative* and *homogeneous*. Indeed, it will be seen at once that (16) is unaltered by translation applied *simultaneously* to the x's and to μ; for if $y_i=x_i+c$ and $M=\mu+c$, then $M-y_i=\mu-x_i$. Moreover, if stretching is used, $y_i=cx_i$ and $M=c\mu$; and thus $M-y_i=c(\mu-x_i)$. Set up now in M and y the equation analogous to (16). This then reduces to (16) itself, since we are now considering the case $t=nr/k$, and hence $t(k-r)=(n-t)r$.

As regards simulvariance, the positional means rate much higher than the arithmetic, geometric, harmonic, exponential, and power means for all of which the simulvariance is *extremely* limited.

The positional means are also somewhat more flexible than the mode—or modes in multimodal distributions—for the latter while simulvariant to continuous *increasing* transformations are not simulvariant to continuous *monotone increasing* transformations.

The median itself is simulvariant to all continuous monotone increasing transformations and continuous monotone *decreasing* transformations. It would seem that this property just about characterizes the median. At least the question arises: If a single-valued mean f, satisfying (17), of an odd number of variables is simulvariant to all the transformations just mentioned, as simulvariance is defined in (19), is f the median?

6. Transitivity and associativeness. In certain respects, the positional means are inferior to the power means and other related means. For example, the median is *not transitive*—as is well known. The median of the median of the sets (1, 1, 1), (1, 2, 2), and (1, 2, 2) is 2, whereas the median of the nine given numbers is 1. Moreover, the median is *not associative*. Thus, the median of 1, 1, 2, 2, 2, is 2. But, if each of the first three numbers is replaced by the median of that set of numbers, the median is changed to 1.

7. Summary. Definitions of quartiles and other positional means for grouped data are arbitrary to the extent that a change in the scheme of grouping changes the values assigned to these means. For ungrouped data by a minimizing process, precise definitions are here given to all such means. A mean, however, may be an interval. In such an interval a unique value may be determined by a limit method due to Jackson. As regards simulvariance, the positional means are more flexible than the modes, and are immensely superior to most means in common use. These positional means, however, lack transitivity and associativeness.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A FUNCTION-THEORETICAL PARADOX

E. S. PONDICZERY, New York City

The object of this note is to indicate a pseudo-proof of the following rather surprising

THEOREM. *Let $f(z)$ be analytic in the region D. Then $f(z) \equiv 0$ in D.*

Let C be any circle in D, composed of the two semicircles C_1 and C_2. We write

$$\frac{1}{2\pi i}\int_C \frac{f(t)dt}{t-z} = \frac{1}{2\pi i}\left(\int_{C_1} + \int_{C_2}\right)\frac{f(t)dt}{t-z} = f_1(z) + f_2(z).$$

By Cauchy's integral formula,

$$f_1(z) + f_2(z) \equiv f(z), \tag{1}$$

for z inside C. On the other hand, for any z whatever outside C, by Cauchy's integral theorem,

$$f_1(z) + f_2(z) \equiv 0. \tag{2}$$

But $f_1(z)$ is analytic except for z on C_1, and $f_2(z)$ is analytic except for z on C_2. Extending $f_1(z)$ and $f_2(z)$ analytically from some point outside C to the interior of C, we preserve the identity (2). Applying this to (1), we have $f(z) \equiv 0$ inside C, and hence throughout D.

Since any constant is analytic everywhere, we obtain as a corollary the proposition "$1 = 0$."

ON THE ISOSCELES TRIANGLE

V. THÉBAULT, Le Mans, France

The following proof of this well known theorem is believed to be new:*

When a triangle has two equal internal (or external) bisectors, the triangle is isosceles.

1. Let ABC be *any* triangle, BE, CF any two straight lines, BE', CF' the perpendiculars to BE and CF at B and C meeting AC and AB at E' and F'. Further let M and Q, N and P be the intersections of BE' and CF' with the parallels to BE, CF drawn through A and C, A and B respectively. Finally let A_1, B_1, C_1 be the midpoints of BC, CA, AB.

* This MONTHLY has already given several proofs of this theorem: 1895, pp. 157, 189; 1917, p. 344.

Then, we have, the areas being taken with appropriate sign,

$$BME = BAE, \qquad BEQ = BEC,$$
$$CFN = FAC, \qquad CPF = CBF,$$

hence

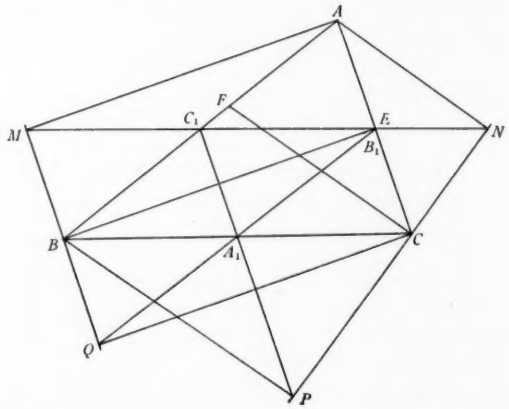
$$MEQ = BME + BEQ = BAE + BEC = ABC,$$
$$PFN = CFN + CPF = FAC + CPF = ABC,$$

and therefore

$$\tfrac{1}{2}MQ \cdot BE = MEQ = ABC = PFN = \tfrac{1}{2}PN \cdot CF.$$

It follows that *BE and CF are inversely proportional to MQ and NP.*

This property subsists for BE', CF' and the segments $M'Q'$, $N'P'$ determined on BE, CF by the parallels to BE', CF' drawn through A and C, A and B.



2. When BE, CF are the internal bisectors of the angles B, C of any triangle and BE', CF' the external bisectors of the same angles, we obtain the following theorem:

In any triangle, the lengths of the internal (or external) bisectors are inversely proportional to the projections of the opposite sides on the corresponding external (or internal) bisectors.

Further, in the right angled triangle ABM,

$$\text{angle } MC_1B = 2MAC_1 = 180° - 2C_1BM = 180° - 2(90° - \tfrac{1}{2}B) = B,$$

and M lies on B_1C_1; therefore, by analogy, N, P, Q are respectively on C_1B_1, C_1A_1, B_1A_1.*

* In any triangle, the projections of the vertices on the six bisectors are four by four on the straight lines joining the midpoints of the sides.

The quadrilateral $BCPQ$ being inscribed in the circle having BC as diameter, *$MNPQ$ is also inscribed in a circle.*

3. If now the internal bisectors BE and CF are equal, $MQ=NP$, according to the foregoing theorem. Hence $MNPQ$ is an isosceles trapezium and so also is $BCPQ$; therefore

$$90^\circ - \tfrac{1}{2}B = QBC = BCP = 90^\circ - \tfrac{1}{2}C$$

and $B=C$, $AB=AC$.

NOTE ON DEGENERATE CONICS

V. F. IVANOFF, San Francisco

In a well known textbook on analytical projective geometry, the author writes about the degeneration of the ellipse and hyperbola into a straight line. Denoting the ends of the major axis of either conic by P and P', and letting b approach zero, we have, in the words of the author:

"Now, the conic, quà locus, is the segment PP' (that is one segment or the other) taken twice; but an ordinary point equation cannot express a terminated portion of a line; it can express only the whole line."

As a matter of fact, an ellipse or hyperbola degenerated in the way prescribed by the author is not a segment, but the whole line taken twice. If the conic be $b^2x^2 \pm a^2y^2 = a^2b^2$, then for the real values of x, with $|x| > a$ for the ellipse, and $|x| < a$ for the hyperbola, the values of y are imaginary. As b approaches zero, the norm of y, which is $b/a\{\mp(a^2-x^2)\}^{1/2}$ also approaches zero, and in the limit we have $y=0$, that is these imaginary points become real.

A similar argument shows that the parabola becomes a straight line taken twice when the parameter P becomes zero.

Note by the Editor. In elementary plane analytic geometry a one-to-one relationship is set up between points in a plane and pairs of *real* numbers. At a later stage of mathematical development we consider pairs of numbers at least one of which is imaginary, and use geometric language in discussing the new system. It is wise to keep clearly in mind which of these two mathematical systems is in use in any discussion.*

If the book referred to by Mr. Ivanoff was using the first system, then the limits of the points on the conic are the points of a *part* of the x-axis. If it was using the second system the limits of the points on the conic are the points on the *whole of the* x-axis. In the former case there are no imaginary points to approach limiting positions; in the latter case there are imaginary points whose limit points are real.—E. J. M.

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* See George Salmon, A Treatise on Conic Sections, 1911, par. 82; C. A. Scott, Modern Analytical Geometry, 1894, pp. 45–46; W. C. Graustein, Introduction to Higher Geometry, 1930, pp. 135, 137.

RECENT PUBLICATIONS

EDITED BY TOMLINSON FORT, Lehigh University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Advanced Calculus. By W. B. Fite. New York, The Macmillan Company, 1938. 12+399 pages. $5.00.

Analisi matematica algebrica ed infinitesimale. By B. Levi. Bologna, Zanichelli, 1937. 7+541 pages. L. 80.

Analytic Geometry. By W. W. Burton. (General editor, B. H. Brown.) New York, Harcourt Brace and Company, 1938. 11+262 pages. $2 50.

Analyse des courbes, surfaces et fonctions usuelles. By P. Appell. Vol. I. Fifth Edition. Entirely revised. Paris, Gauthier-Villars, 1937. 8+296 pages.

Beiträge und Deckungsrücklagen in der Lebensversicherung. By C. Boehm and E. Rose. (Versicherungsmathematische Aufgabensammlung, published by the Deutscher Aktuarverein, No. 1.) Leipzig and Berlin, Teubner, 1937. 11+75 pages. RM 1.65.

Comptes-Rendus du Deuxième Congrès International de Récréation Mathématique. Brussels, Librairie du "Sphinx," 1937. 105 pages, as follows:

Les relations de l'enseignement des mathématiques avec les autres enseignements. By A. Decerf. (Conference taking place at the Deuxième Congrès de Récréation Mathématique.) Paris, Vuibert; Brussels, Librairie du "Sphinx," 1938. 24 pages. 5 f.

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Computation and Trigonometry. By H. J. Gay. New York, The Macmillan Company, 1938. 7+231 pages. $1.90.

Elements of Statistical Method. By A. E. Waugh. New York and London, McGraw-Hill Book Company, 1938. 15+381 pages. $3.50.

Essentials of the Mathematics of Investment. By P. R. Rider. New York, Farrar and Rinehart, Inc. 1938. 10+162 pages. $2.00.

Geometrical Optics, By J. L. Synge. An introduction to Hamilton's method. (Cambridge Tracts in Mathematics and Mathematical Physics, edited by G. H.

Hardy and E. Cunningham, No. 37.) Cambridge, University Press; New York, Macmillan Company, 1937. 9+110 pages. $2.00.

Historical Development of the Graphical Representation of Statistical Data. By H. G. Funkhouser. Osiris, Studies on the history and philosophy of science and on the history of learning and culture, edited by George Sarton. Brussels. Vol. III, Part I, 1937. 136 pages.

Introduction to Mathematical Probability. First Edition. By J. V. Uspensky. McGraw-Hill Book Company, New York and London, 1937. 9+411 pages.

La Spéculation et le Calcul des Probabilités. By L. Bachelier. Paris, Gauthier-Villars, 1938. 7+49 pages. 20 f.

Mathematical Tables. British Association for the Advancement of Science. Vol. 6: Bessel Functions. Part I: Functions of Orders Zero and Unity. Cambridge, University Press, 1937. 20+288 pages. $14.00.

New Analytic Geometry. By P. F. Smith, A. S. Gale, and J. H. Neeley. Alternate edition. Boston, Ginn and Company, 1938. 10+36 pages. $2.00.

New Applied Mathematics. By S. J. Lasley and M. F. Mudd. Revised Edition. New York, Prentice-Hall, Inc., 1937. 23+433 pages. $1.60.

Plane Geometry and Its Reasoning. By H. C. Barber and G. Hendrix. (General editor, B. H. Brown.) New York, Harcourt Brace and Company, 1937. 10+310 pages. $1.40.

Segmental Functions. Text and Tables. By C. K. Smoley. Scranton, Smoley, 1937. 53+439 pages.

Technique de la méthode des moindres carrés. By H. Mineur. (Monographies des Probabilités, publiés sous la direction de E. Borel, No. 2.) Paris, Gauthier-Villars, 1938. 8+93 pages. 50 f.

The Number System of Algebra. By H. B. Fine. Treated theoretically and historically. Reprint from 1890 edition. New York, G. E. Stechert and Company, 1937. 9+131 pages. $2.00.

Theory and Applications of Finite Groups. By G. A. Miller, H. F. Blichfeldt, and L. E. Dickson. New York, G. E. Stechert and Company, 1938. 17+390 pages. $4.00.

Umwandlung von Lebensversicherung. By C. Boehm, P. Lorenz and J. Staniszweski. (Versicherungs mathematische Aufgabensammlung, published by the Deutscher Aktuarverein, No. 2.) Leipzig and Berlin, Teubner, 1937. 14+52 pages. RM 1.65.

REVIEWS

Analytic Geometry. By W. W. Burton. New York, Harcourt Brace and Company, 1938. 11+262 pages. $2.50.

The author has set forth as an objective the writing of a text which shall be not only an introduction to the calculus but also a training in the essential methods of analytic geometry. He has written with the student of average ability in mind, but has included much material which will challenge the ability of the better student.

The text includes in order the regulation chapters on coördinates (rectangular); the straight line; equations and loci; the circle; the parabola; the ellipse; the hyperbola; transformation of coördinates; polar coördinates; tangents and normals; poles, polars and diameters; higher plane curves; cartesian coördinates in space; the plane; the straight line; and surfaces.

The text contains abundant problem material, eighty per cent of which has been carefully selected and graded to meet the needs of the average student. The answers have been given for one half of the problems.

The printing is good and the figures are clear. It appears to be a teachable text.

However, there are numerous errors which should be corrected in later editions. On page 27 the author uses "rhombus" where "parallelogram" probably was intended. In both the ellipse and the hyperbola the latus rectum should be defined as "the chord drawn through the focus perpendicular to the major axis." In defining polar on page 168 the author defines polar properly as a locus but immediately follows it, without proof, with this statement: "Hence the polar is a straight line and the pole is a point." He is also careless in his statements about horizontal and vertical asymptotes. For instance, on page 60, we find the sentence: "Here the only value of y that will cause x to increase without limit is zero."

On page 186 the figure for the spiral has been rotated through 180°, probably by the printer. This will be confusing to the student.

In several places some of the sentences are rather involved and should be recast. Such a sentence is found on page 183: "Hence the discussion of $x=\sin y$ is the same as that for the sine curve interchanging the variables and the graph is the same with the axes interchanged."

R. P. STEPHENS

Géométrie Vectorielle. By Ad. Mineur. Volume 1, *Algèbre vectorielle.* Fourth edition. Brussels, Albert Vanderlinden, 1937, 14+110 pages. 25 Belgian fr.

Since the previous editions were in autographic form with a distribution mostly among the author's students, the present edition is the first to be reviewed either in the MONTHLY or the *Bulletin*. There are to be two other volumes in the series, the second to be on the calculus of vectors and the third to contain further theory and exercises.

The author is Professor of Geometry and Mechanics in the Faculty of Sciences of the University of Brussels. As one of the older men in his field in Belgium, his influence on the younger generation of applied mathematicians in his country is considerable. In addition, he was a student of Junius Massau, who bears to Belgian vector analysis the same relation that Gibbs does to the American variety. At a time approximately contemporaneous with the work of Gibbs, Massau pioneered in the development of a simple and practical vector analysis and applied it to mechanics. We thus have in the present series the culmination of the work of the important Belgian school founded by Massau.

The present volume gives a careful and detailed treatment of the theory of projections, scalar and vector products, theory of moments, and linear vector functions. Perhaps the greatest difference in the treatment from that of the Gibbs school is to be found in the point of view concerning the nature of vector itself. Whereas in texts in English it is customary to restrict the term vector to mean *free* vector, Mineur uses vector only in the sense of *fixed* vector, requiring for its specification a line of action and a point of application as well as a magnitude and direction. There are at least two advantages of the latter procedure. In the first place, there is the logical contradiction of having to fix a vector, as in vector fields. If a vector is by definition free, how can it be fixed? This is a point which is apt to confuse the beginner. Secondly, statements on page 1 of many books to the contrary, a force is *not* a vector if by vector is meant free vector. From the point of view of fixed vectors, however, one is led naturally to such important topics in statics as the Theorem of Varignon, couples, and the equivalence of non-concurrent force systems; topics conspicuous for their absence in most texts in English. The theory of fixed vectors is made to include that of free vectors by use of the simple notion of equipollence (represented analytically by the sign of equality).

The book of Mineur is a thorough and practical account of vector algebra which should be of particular interest to teachers of introductory courses in vector analysis and mechanics.

P. W. KETCHUM

Projective Geometry. By B. C. Patterson. New York, John Wiley and Sons, 1937. 14+276 pages. $3.50.

This book seems to be the most satisfactory text in synthetic projective geometry that we have. It is admirably designed for a first course in the subject; the material is well chosen, and in compass is somewhat more than would ordinarily be covered in a semester course. The author has recognized the fact that different teachers of the subject have personal preferences as to selection and order of topics, and has provided for much flexibility.

The author has been true to his principles, and has developed the main body of his material on a sound projective basis. This is in contrast not only to the older English approach, where everything is metric, and a conic is defined as the projection of a circle; but to some recent American texts in alleged "pure projective geometry" in which the basic theorems are proved by metric relations. Professor Patterson starts with a set of nine axioms of incidence; for sound pedagogical reasons the list is not reduced to the minimum possible. There are no axioms of order, but the matter is covered by a brief discussion appealing mainly to intuition.

By developing the subject in this way, using Desargues's theorem and harmonic relations as the point of departure, the author has to sacrifice the very useful theorem that cross-ratio is projectively invariant. It is unfortunate that there is no simple way of establishing this theorem projectively; but one cannot

have everything. He develops metric interpretations of the projective geometry in a separate chapter, and throughout the work he makes careful distinctions between the projective and the metric.

The main line of development follows traditional lines: Desargues's theorem and harmonic relations, projective ranges and pencils (Poncelet's definition), projective generation of conics, the theorems of Pascal and Brianchon, poles and polars, applications, and projectivity in two-dimensional forms. An innovation in an elementary text, and one of doubtful wisdom, is a chapter on von Staudt's theory of imaginaries. Throughout the work, theorems are accurately stated; and proofs are for the most part satisfactory, though in some cases they could be made more concise.

The subject of involution in a one-dimensional form seems not to have received its due. Instead of being developed early, as a special case of a superposed projectivity, it is first mentioned in connection with poles and polars, and its full treatment comes in a late chapter. The projective theory and the metric interpretations should be utilized from the beginning.

The troublesome topic of foci of a conic is well handled. The foci are first defined as intersections of tangents from the circular points; but the alternative definition in terms of real elements, as points where the involution of conjugate lines consists of perpendiculars, is also given and is the basis of most of the discussion.

The converse of the theorem of Desargues is flatly stated to follow by the principle of duality; the reader is "challenged" to accept the statement. In offering as an alternative the modification of the proof given for the first part of the theorem, the author overlooks the very simple proof based on finding two triangles which have, under the hypothesis, a perspective center.

The proof of the theorem of Pascal fails to emphasize sufficiently that the theorem is implicit in the projective generation of the conic, and is indeed a mere restatement of it. The fault (if it be such) is not so serious as that of a text of some years ago, in which, after Pascal's theorem was established, there were given two proofs each for several following theorems: (a) by the projective generation; (b) by Pascal's theorem. Obviously the two were essentially identical.

The special or "limiting" cases of the theorems of Pascal and Brianchon are most regrettably proved by continuity and limits. Sound proofs are given in an appendix. Of course in practice one uses continuity to *suggest* the special theorems, but the latter should be proved independently.

The exercises are numerous and well chosen. Perhaps more could have been done with the metric relations and properties of conics derived by metric interpretation of projective ranges and pencils.

The only misprint noted is an obvious one in the statement of the Fundamental Theorem (page 71), which reads "$D' = D'$" instead of "$D' = D_1'$."

R. A. JOHNSON

Lessons in Elementary Analysis. 2nd edition. By G. S. Mahajani. Poona, Aryabhushan Press, 1934. 12+264 pages.

The Application of Moving Axes Methods to the Geometry of Curves and Surfaces. By G. S. Mahajani. Poona, Aryabhushan Press, 1937. 7+60 pages.

According to the preface of the first of these books, "The modern calculus course for the B.A. degree of most of the Universities in India may be divided into three parts, *viz.*, (i) analysis, (ii) geometrical and graphical applications of calculus, and (iii) differential equations." This book was written to supply the material for the first of these parts. Some idea of the standards of the above course can be gained from the following list of chapter headings: The arithmetic continuum, Theory of limits, Infinitesimals, Continuity, Mean value theorems, Taylor's theorem, Integrals, Mean value theorems (integrals), Infinite integrals, Uniform convergence, Inversion of operations, Fourier series. There is also an addendum containing discussions of beta and gamma functions, power series, Taylor's theorem for functions of several variables, Euler's theorems on homogeneous functions, and implicit functions. In addition to the excellent explanations there are numerous exercises in each chapter, many of which might serve to stimulate further study.

The topics are treated quite rigorously, and while it is obvious that they cannot be discussed fully in so short a book, sub-topics are so well chosen and the presentation is so coherent that a surprisingly large amount of useful material is made accessible to the reader. The author clearly has had the learner in mind at all times. Ideas that are often difficult for the student on first acquaintance, perhaps because their places in the general scheme of things are not made plain, are here presented in such a way that they may be grasped immediately. The author frequently points out in advance what he is going to do and why it is necessary. He is never obscure and never makes the futile mistake of using a fallacious method to save time or avoid difficulties. The book is made even more useful by the fact that, while there is coherence in the whole work, each separate chapter can be studied as an independent unit. It is unfortunate that there is no bibliography.

The second book is, in substance, the essay which the author, while an undergraduate at St. John's College, submitted in 1923 for the Adams Memorial Prize. While it did not win the award it was among those "highly commended." The title suggests the contents quite accurately. The author applies the method advantageously to the discussions of the rectifying developable, the polar developable, the osculating sphere, evolutes, Bertrand's curves, curves with common binormals, Meusnier's theorem, geodesic curvature, geodesic torsion, lines of curvature, geodesic lines, asymptotic lines, Joachimsthal's theorem, and Euler's theorem. The treatment is concise and clear. This is the sort of essay that can with benefit be put in the hands of students that they may round out their knowledge of a topic which is perforce treated somewhat sketchily in a course. A bibliography would be helpful.

H. R. Cooley

Graphical Calculus. By Theodore R. Running. Ann Arbor, Michigan. George Wahr, 1937. 81 pages. $1.90.

It is just possible that, with the advent of calculus, the mathematicians of the Leibnitz-Newton era were sure that the death knell of approximate methods had been sounded and henceforth the science of mathematics was to follow a path of lily-white purity straight through that region known as rigor. Modern requirements in the field of technical and industrial research, however, have shown that a compromise with rigidity must be made if we expect to get definite results from experimental data.

Professor Running's *Graphical Calculus* is a fit companion to his earlier treatise on *Empirical Formulas*. One finds here no attempt to get the actual equation of a curve in which the parameters have been adjusted to best represent the data, but a means of determining graphically and with minimum effort new individual values for the tabulated measurements which conform to the set as a whole from the measurements themselves without the use of formulas, moments, or least squares.

By using the principle that the area in square units under the derived curve is equal to the corresponding increase in the ordinate of the integral curve, a graphical method is devised to produce either curve from the other. This is what the author calls graphical differentiation and integration. All the applications are based on this simple idea. The whole plan is particularly suited to the treatment of tabulated empirical values obtained by experiment.

To familiarize the reader with the principle he cites two interesting examples. First, on an automobile trip the time and mileage were recorded simultaneously every few minutes, giving data from which the space-time curve was plotted. The derived curve from this is the velocity-time curve—an example of graphical differentiation. Second, on another trip the velocity and time were recorded and the data were plotted. This velocity-time curve was considered as the derived curve and the graph from this produced the space-time curve—an example of graphical integration.

Perhaps the most important topics discussed and of most general application in modern research are graduation of data and graphical interpolation. The major part of the book is devoted to these two subjects.

For graduation of data a derived curve in the form of a broken line is drawn and from this the integral curve is formed which, of course, will consist of a series of connected parabolic arcs. As most empirical curves can be pretty well represented by a succession of parabolic arcs this is a short and effective method of smoothing out the data. In cases where the derived curve cannot be well represented by a broken line it is shown that the data can be graduated effectively by using the second derived curve from which the derived curve emerges as a series of parabolic arcs, which in turn produces the integral curve as a series of connected third degree curves representing a much more involved set of observed values.

Graphical interpolation is accomplished by plotting the desired intermediate

ordinate from the corresponding ordinate of the derived curve after the data have been graduated.

Among other topics in the book there is shown a method of solving equations which employs a parabola through three points on the curve, whose equation is given, where the abscissas are very near each other and the root lies between two of them. This gives a more rapid approximation than that found in Newton's Method but the amount of necessary mathematical calculation does not seem to be materially lessened.

The book deserves a place on the shelf of everyone who is interested in tabulated experimental data.

J. H. OGBURN

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to F. W. Owens, 462 East Foster Ave., State College, Pa.

BOOKS

27. *Projective Ornament.* By Claude Bragdon. New York, Knopf, 1915. $2.00. On design and fourth dimensional figures.
28. *A Romance of Many Dimensions.* By A. Abbott. Boston, Little, Brown, 1927. $1.50. An amusing and instructive presentation of many-dimensional spaces not too difficult for high school students of geometry.
29. *Poetry and Mathematics.* By S. M. Buchanan. New York, The John Day Company, 1929. $2.50. The chapters are: Poetry and mathematics, Figures, Numbers, Proportions, Equations, Functions, Symbols, Tragedy, Comedy. The influence of mathematical concepts on art, and of art on mathematics, by a logician.
30. *Number, The Language of Science.* By Tobias Dantzig. New York, Macmillan, 1930. $2.50. A somewhat historical treatment of the development of the number system by the human race. One of the most readable of the popular books on mathematics.
31. *The Elements of Euclid.* No. 891, Everyman's Library. New York, Dutton. $.80. Books I–VI and XI and XII. A reproduction of Todhunter's reproduction of Robert Simson's translation (1756) of the Greek texts of Theon. Designed to be read as literature, the presentation is revealing to Americans brought up on a different type of textbook.
32. a) *Non-euclidean Geometry,*
    b) *Galois and the Theory of Groups,*
    c) *The Einstein Theory of Relativity,* Part I. By H. G. and L. R. Leiber, 228 Clinton Avenue, Brooklyn, New York, $1.00 each. Very brief expositions, amusingly illustrated.

33. *New Mathematical Pastimes.* By P. A. MacMahon, New York, Macmillan, 1936. $2.50. "Generalized dominoes," of all patterns in a multitude of shapes and colors, together with a general theory of constructing others. Contains a bibliography of mathematical amusements.

CLUB REPORTS
1936–37

Delta Pi Sigma, University of Arizona

President, R. Greenwood; Vice-President, Dr. R. F. Graesser; Corresponding Secretary, J. B. Henry; Recording Secretary-Treasurer, Evellyn Caullvine. Organized in 1932, this fraternity has for its purpose the promotion of scholarship and the encouragement of mathematical studies, pure and applied. Dinner meetings with a speaker on some mathematical topic are held bi-monthly. One public lecture is sponsored each year. Two or three social meetings are held. A cup is awarded each year to the student who has completed, with the highest grade average, all mathematics courses through the calculus.

Pi Mu Epsilon, Marquette University

Director, C. Doda; Secretary, Louise Denker; Treasurer, W. Nelson. This chapter continued the publication of its monthly bulletin which is a lively sheet calculated to arouse the interest of all students in the work of the chapter. Besides taking an active part in the Intercollegiate Mathematics Association of Milwaukee, and its regular meetings, initiation, and banquet, the chapter has held monthly open meetings with topics which included: Chess and mathematics; Infra red photography in medicine; Mathematicians of antiquity; Earthquakes; Quantitative determination of protombin; Music and mathematics. The chapter conducted its annual Frumveller Contest in May. The prize for this contest is a year's tuition at Marquette University. This year the chapter sponsored the organization of the Mathematics Club of Marquette University, membership in which is open to all of the student body interested in mathematics. The activities of this new group will hereafter be reported separately.

Mathematical Society, Trinity College

After a lapse of several years the Pascal Circle has been reorganized under a new name. President, Edith Sullivan; Vice-President, Mary Reges; Secretary, Margaret Collins. Monthly meetings were held with topics including: Pythagoras and the Pythagoreans; Problems of a young teacher of mathematics; Synthetic division with higher degree divisors; Mathematical correspondence Story of the Bernoullis; Some general theorems of modern geometry.

Delta X, University of Kansas City

President, Catherine Torbert; Vice-President, C. Pavolich; Secretary, D. Kavorinos; Treasurer, H. Bootman; Faculty Advisers, W. A. Luby, D. T. Sigley. The club issued a printed program for the year. Its monthly meetings are held in the evenings. Topics discussed included: Mendelian law of heredity; Use and abuse of statistics; Mathematics in chemistry; First table of logarithms; On the theory of numbers; Use of theory of errors in physical data; Psychology as a statistical science; Rotations of regular polygons.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 333. *Proposed by O. E. Henry, Pittsburgh Public Schools.*

The equations, $SLED+SNOW=RIDE$, and $SLED-SNOW=BOB$, are the results of coding the digits of numbers by assigning to each digit a different letter which replaces it throughout the equations. Identify the numerals and determine whether or not the solution is unique.

E 334. *Proposed by J. R. Musselman, Western Reserve University.*

The corners of two rectangles, named counterclockwise, are A, B, C, and D, and D, E, F, and G. (They share the corner D.) Prove or disprove that the midpoints of the segments AG and CE, together with the centers of the given rectangles, constitute the corners of another rectangle.

E 335. *Proposed by J. H. Edmonston, Washington, D. C.*

In the proof of the binomial theorem for positive, integral exponents, use is made of the identity, ${}_{n+1}C_r = {}_nC_r + {}_nC_{r-1}$. Prove the generalized proposition,

$$ {}_{n+t}C_r = \sum_{k=0}^{t} {}_tC_k \cdot {}_nC_{r-k}, \quad \text{for} \quad t \leqq r. $$

E 336. *Proposed by W. B. Campbell, Drexel Institute.*

The equation, $x^2-bx+c=0$, has two positive integer roots, p and q. Let r equal p with its (two) digits interchanged, and s equal q with its (two) digits interchanged. Show that the following four statements are consistent, and lead to a unique solution, and that any three of them lead to the same unique solution. (a) In the equation whose roots are r and s, the two digits of b are interchanged. (b) In the revised equation, $b=78$. (c) In the revised equation, c is unchanged. (d) r and s differ by 6.

E 337. *Proposed by V. Thébault, Le Mans, France.*

(S) and (T) are two fixed intersecting spheres, and (X) and (Y) are two variable spheres, tangent to each other at P, and each tangent to both (S) and (T). Show that the locus of P is the surfaces of two more spheres which cut each other orthogonally. Examine the cases in which (S) and (T) are internally or externally tangent, or have no points in common.

E 338. *Proposed by C. W. Trigg, Eagle Rock High School, Los Angeles.*

There are only two numbers which are permutations of the ten digits, which are also the products of three consecutive integers. Find them and show that there are no others.

SOLUTIONS

E 299 [1937, 598]. *Proposed by V. Thébault, Le Mans, France.*

If a certain three-digit number is divided by the product of its digits, the quotient will be its hundreds digit. Find the number and show that the solution is unique.

Solution by W. C. Rufus, University of Michigan.

The statement is equivalent to the equation, $100x+10y+z=x^2yz$.

Division by x shows that the product, xyz, exceeds 100, so x, y, and z must each be greater than 1.

For an integer solution, z must divide $100x+10y=10(10x+y)$. If z divides $10x+y$, assume $10x+y=nz$. Then, dividing the original equation by z gives $10n+1=x^2y$, so that x^2y ends in 1. Now x^2 must end in 1, 4, 5, 6 or 9, so y must be 1 or 9. But we have already excluded the possibility $y=1$. Hence if $10x+y=nz$, $y=9$, and x^2 ends in 9. So x would be 3 or 7. If x were 3, the original equation would become $390+z=81z$, which is impossible because 390 is not a multiple of 80. If x were 7, the original equation would become $790+z=441z$, which is also impossible because 790 is not a multiple of 440.

Consequently, z is not a factor of $10x+y$; so it must be a factor of 10, and hence be 2 or 5. If we replace z by 2 in the original equation, and solve for x, the roots are irrational for every possible value of y. Therefore z must be 5. Then x and y must be odd, as an even factor would give 0 in the units place. Putting $z=5$ in the original equation and solving for x gives $x=\left[10+\sqrt{(100+2y^2+y)}\right]/y$, in which y is limited to 3, 5, 7 or 9. Only $y=3$ rationalizes this expression. This gives $x=7$, the only integral value. The number sought is 735, and the solution is unique.

Also solved by W. E. Buker, Fred Discepoli, William Douglas, Daniel Finkel, E. P. Starke, C. W. Trigg, B. C. Zimmerman, Samuel Zuckerman, and the proposer.

E 300 [1937, 598]. *Proposed by Daniel Finkel, New York City.*

When n is a positive integer, the coefficients in the binomial theorem may be divided into three groups as follows:

1) ${}_nC_0, {}_nC_3, {}_nC_6, {}_nC_9, \ldots$
2) ${}_nC_1, {}_nC_4, {}_nC_7, {}_nC_{10}, \ldots$
3) ${}_nC_2, {}_nC_5, {}_nC_8, {}_nC_{11}, \ldots$

Two of these three sets are composed of identical terms. Prove that the sum of the terms in the remaining set differs from the sum of the terms in either of the equal sets by unity.

Solution by Emma Lehmer, Bethlehem, Pennsylvania.

Denote the sum of the terms in the ith set by $S_{i-1}(i=1, 2, 3)$. Then, if we substitute for x in the binomial expansion

$$(1 + x)^n = \sum_{j=0}^{n} {}_nC_j \cdot x^j$$

the three cube roots of unity, $1, \omega$, and ω^2, we get three equations for S_0, S_1, and S_2, namely

$$S_0 + S_1 + S_2 = 2^n,$$
$$S_0 + S_1\omega + S_2\omega^2 = (-1)^n\omega^{2n},$$
$$S_0 + S_1\omega^2 + S_2\omega = (-1)^n\omega^n.$$

Solving these for S_i, we get

$$S_i = (1/3)[2^n + (-1)^n(\omega^{n+i} + \omega^{2n+2i})].$$

Since $\omega^{n+i}+\omega^{2n+2i}$ is equal to 2 or -1 according as $n+i$ is or is not a multiple of 3, we obtain

$$S_i = (2/3)[2^{n-1} + (-1)^n] \text{ if } n + i \text{ is a multiple of 3,}$$

and

$$S_i = (1/3)[2^n - (-1)^n] \text{ otherwise.}$$

From these results, of course, the stated theorem follows immediately.

Also solved by E. P. Starke, C. W. Trigg, and the proposer.

E 301 [1937, 598]. *Proposed by D. L. MacKay, Evander Childs H.S., New York.*

If in triangle ABC, $B-C=90°$, and S and T are the intersections of the internal and external bisectors of angle A with the side BC, prove:

a) $\sin A = (b^2-c^2)/(b^2+c^2)$,

b) ST is twice the altitude from A,

c) a^2 is the harmonic mean of $(b-c)^2$ and $(b+c)^2$.

Solution by B. C. Zimmerman, Corozal, British Honduras.

Since $B-C=90°$ and $A+2C=90°$, we have $\sin^2 B = \cos^2 C$, and from the law of sines it follows that

$$\frac{b^2 - c^2}{b^2 + c^2} = \frac{\sin^2 B - \sin^2 C}{\sin^2 B + \sin^2 C} = \frac{\cos^2 C - \sin^2 C}{\cos^2 C + \sin^2 C} = \cos 2C = \sin A,$$

which proves part (a) of the problem.

Since $A+2C=90°$, angle $AST = C+\frac{1}{2}A = 45° =$ angle STA. Therefore the altitude from $A = SH = HT$, so that $ST = 2HA$, proving part (b). (H is the foot of the altitude from A.)

Since $\sin^2 A = (a^2/b^2)\sin^2 B$, it follows from our proof of part (a) that

$$\frac{(b^2-c^2)^2}{(b^2+c^2)^2} = \sin^2 A = \frac{a^2}{b^2}\sin^2 B = \frac{a^2}{b^2}\cdot\frac{b^2(\sin^2 B+\sin^2 C)}{b^2+c^2} = \frac{a^2(\cos^2 C+\sin^2 C)}{b^2+c^2}.$$

Consequently, $a^2=(b^2-c^2)^2/(b^2+c^2)$, which is equivalent to the statement,

$$\frac{1}{(b-c)^2}-\frac{1}{a^2}=\frac{1}{a^2}-\frac{1}{(b+c)^2},$$

which proves statement (c) that a^2 is the harmonic mean between $(b-c)^2$ and $(b+c)^2$.

Also proved by C. B. Barker, Lois E. Bell, W. E. Buker, W. B. Clarke, R. W. Cowan, Fred Discepoli, William Douglas, L. M. Kelly, E. P. Starke, C. W. Trigg, Maud Willey, Samuel Zuckerman, and the proposer.

E 302 [1937, 598]. *Proposed by F. A. Alfieri, New York City.*

If A, B, and C are the angles of a plane triangle, prove that

$$\cot A+\cot B+\cot C=\frac{1}{2}\left[\frac{\sin A}{\sin B\cdot\sin C}+\frac{\sin B}{\sin C\cdot\sin A}+\frac{\sin C}{\sin A\cdot\sin B}\right].$$

Solution by R. W. Cowan, College of St. Scholastica, Duluth, Minn.

$$\begin{aligned}
&\frac{1}{2}\left[\frac{\sin A}{\sin B\sin C}+\frac{\sin B}{\sin C\sin A}+\frac{\sin C}{\sin A\sin B}\right]\\
&=\frac{1}{2}\left[\frac{\sin(B+C)}{\sin B\sin C}+\frac{\sin(C+A)}{\sin C\sin A}+\frac{\sin(A+B)}{\sin A\sin B}\right]\\
&=\frac{1}{2}\left[\frac{\sin B\cos C}{\sin B\sin C}+\frac{\cos B\sin C}{\sin B\sin C}+\frac{\sin C\cos A}{\sin C\sin A}\right.\\
&\qquad\left.+\frac{\cos C\sin A}{\sin C\sin A}+\frac{\sin A\cos B}{\sin A\sin B}+\frac{\cos A\sin B}{\sin A\sin B}\right]\\
&=\frac{1}{2}\left[\cot C+\cot B+\cot A+\cot C+\cot B+\cot A\right]\\
&=\cot A+\cot B+\cot C.
\end{aligned}$$

Solution by the proposer.

By the sine law, $\sin A=ak$, $\sin B=bk$, $\sin C=ck$, where k is a proportionality factor. By the cosine law, $\cos A=(b^2+c^2-a^2)/2bc$, . . ., where A, B, C, a, b, c are the angles and opposite sides of the triangle. Therefore,

$$\begin{aligned}
\cot A+\cot B+\cot C&=\frac{b^2+c^2-a^2}{2abck}+\frac{c^2+a^2-b^2}{2abck}+\frac{a^2+b^2-c^2}{2abck}\\
&=\frac{a^2+b^2+c^2}{2abck}=\frac{ak}{2bck^2}+\frac{bk}{2ack^2}+\frac{ck}{2abk^2}
\end{aligned}$$

$$= \frac{1}{2}\left[\frac{\sin A}{\sin B \sin C} + \frac{\sin B}{\sin C \sin A} + \frac{\sin C}{\sin A \sin B}\right].$$

Also solved by E. F. Allen, C. B. Barker, Lois E. Bell, E. O. Box, W. E. Buker, Arthur Danzl, Fred Discepoli, Abraham Franck, L. M. Kelly, R. F. Schnepp, C. E. Snoke, Jr., E. P. Starke, C. W. Trigg, Maud Willey, and Samuel Zuckerman.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3872. *Proposed by Ellis R. Ott, University of Buffalo.*

One man has m coins and another has n. They match coins until one player has won all the coins. Find the average number of tosses required to end the game.

3873. *Proposed by G. T. Coate, Student, Tulane University.*

Evaluate the nth order determinant whose elements are given by

$$e_{11} = e_{1j} = e_{i1} = 1; \quad \text{and} \quad e_{ij} = e_{i-1,j} + e_{i,j-1}, \qquad 1 < i, j \leqq n.$$

3874. *Proposed by V. Thébault, Le Mans, France.*

If a conic is inscribed in a triangle so that one of its principal axes passes through the circumcenter of the triangle, the auxiliary circle of the conic corresponding to the axis considered is tangent to the nine point circle of the triangle, and conversely.

Note. This theorem completes a proposition of M. Weill, *Nouvelles Annales de Mathématique*, 1880, p. 253.

3875. *Proposed by John Tom Hurt, Texas A. and M. College.*

Let $h_1, h_2, \cdots, h_k, \cdots$ be a sequence of positive numbers which tend to infinity and which are such that the sum of their reciprocals is a divergent series. Let $H_n^{-1} = (1+h_1)(1+h_2)\cdots(1+h_{n+1})$, and let the elementary symmetric functions of $h_1, h_2, \cdots, h_n$ be $B_{n,k}$, that is, $B_{n,k}$ is the sum of all products taken k at a time formed without repetitions from the first n of the h's. Prove that $\sum_{n=p}^{\infty} H_n B_{n,n-p} = 1$, for $p = 0, 1, 2, 3, \cdots$.

3876. *Proposed by Otto Dunkel, Washington University.*

Prove the following assertions: Let

$$S_1(12)S_2(23)\cdots(i-1\ i)S_i, \tag{1}$$

be a product of transpositions $T=(ab)$, where S_h denotes a partial product of transpositions no one of which contains the element h, or S_h may be the identity I and may be omitted. Then (1) is equal to

$$S_1S_2'S_3''\cdots S_i^{(i-1)}(23)(34)\cdots(i-1\ i)(i1), \tag{2}$$

where $S_h^{(h-1)}$ is a product of transpositions no one of which contains the element h. The total number of transpositions in (1) is not altered by this transformation (A) to the form (2).

If a given product of transpositions contains a consecutive sequence (1) with S_i followed by $(i\ 1)$, then this part of the product reduces to

$$S_1S_2'S_3''\cdots S_i^{(i-1)}(23)(34)\cdots(i-1\ i), \tag{3}$$

and the total number of transpositions in the original product has been reduced by two by this transformation (B).

Repeated applications of (A) and (B) to a given product of r transpositions reduce it either to the identity or to a product of cycles $C_aC_bC_c\cdots$, where $C_a=(a_1a_2)\ (a_2a_3)\cdots(a_{i-1}a_i)$ and where no two cycles have an element in common. If s is the number of transpositions in this reduced form, then $r-s$ is precisely twice the number of times (B) has been applied in the reduction. It then follows that the reduced form must contain the minimum number of transpositions with which it is possible to represent the given substitution.

SOLUTIONS

3784 [1936, 309]. *Proposed by J. M. Feld, New York City.*

Prove that for any positive integer k the following determinant is zero:

$$\begin{vmatrix} \frac{1}{2!} & \frac{1}{3!} & \frac{1}{4!} & \cdots & \frac{1}{(2k+1)!} & \frac{1}{(2k+2)!} \\ 1 & \frac{1}{2!} & \frac{1}{3!} & \cdots & \frac{1}{(2k)!} & \frac{1}{(2k+1)!} \\ 0 & 1 & \frac{1}{2!} & \cdots & \frac{1}{(2k-1)!} & \frac{1}{(2k)!} \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & \frac{1}{2!} & \frac{1}{3!} \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{2!} \end{vmatrix}.$$

Solution by J. S. Frame, Brown University, Providence, R. I.

Using a slightly different notation from the proposer, let $f(n)$ denote the determinant obtained by replacing in the one given $2k+1$ by n, $f(0)=1$. Then the minor of the kth element of the first row is $f(n-k)$, so that for $n>0$, these determinants must satisfy the recursion formula:

$$f(n) - \frac{1}{2!}f(n-1) + \frac{1}{3!}f(n-2) - \cdots + \frac{(-1)^n}{(n+1)!}f(0) = 0.$$

The coefficient of $f(n-k)$ in this recursion formula is seen to be the coefficient of x^k in the expansion of $(1-e^{-x})/x$. Since this function is developable in a power series with constant term unity, its reciprocal also has a power series expansion (convergent for $x<2\pi$) which is of the form

$$x/(1-e^{-x}) = F(x) = \sum_{n=0}^{\infty} a_n x^n = 1 + \frac{1}{2}x + \frac{B_1}{2!}x^2 - \frac{B_2}{4!}x^4 + \frac{B_3}{6!}x^6 - \cdots$$

where B_n is the nth Bernoulli number (by one definition of these numbers), and where the coefficients a_n are determined by $F(x)\cdot(1-e^{-x})/x=1$. We easily find that $a_0=1$, $a_1=\frac{1}{2}$, and that a_n is determined by the same recursion formula as $f(n)$. Since this determination is unique, $f(n)=a_n$. Furthermore, since

$$F(x) - \frac{x}{2} = \frac{x}{2}\,\frac{(1+e^{-x})}{(1-e^{-x})} = \frac{x}{2}\operatorname{ctnh}\frac{x}{2}$$

is an even function, the coefficients of the remaining odd powers of x vanish.

Thus $f(2k+1)=0$, k a positive integer; and this proves the proposition. Also $f(2k)=(-1)^{k-1}B_k/(2k)!$

Solved also by Frank Ayres, Jr., H. T. Davis, and the proposer.

3785 [1936, 309]. *Proposed by V. Thébault, Le Mans, France.*

The six spheres of similitude of four spheres with centers at the vertices of a tetrahedron and with radii proportional to given numbers pass through two fixed points on a diameter of the circumsphere of the tetrahedron.

Solution by the Proposer.

We have shown (*Mathesis*, 1932, supplement) that the locus of the centers S of spheres (S) for which the powers of the vertices A, B, C, D of a tetrahedron are of the form kl^2, km^2, kn^2, kp^2, where l^2, m^2, n^2, p^2 are given numbers and k is a variable parameter, is the straight line Δ through the center O of the circumsphere (O) and perpendicular to the plane π with the barycentric coördinates l^2, m^2, n^2, p^2, with respect to the given tetrahedron. The plane π is the radical plane of (O) and (S).

A sphere (S) is orthogonal to the corresponding spheres with centers A, B, C, D, and also to the six spheres of similitude of these four spheres. Hence these six spheres intersect in two points V, W on the line Δ so that $OV\cdot OW=R^2$,

where R is the radius of (O). Since the plane π is fixed as k varies, the proposition of the problem follows.

Editorial Note. The theorem on which the above solution is based was used also by the proposer in his solution of 3752 [1937, 403] and in a note following that solution the theorem was extended to n dimensions. By the use of the notation and numbered results in that note, it is easy to extend the theorem of this problem in the same manner. We exclude the trivial case in which all the R_i's are equal. The equation of a sphere of similitude is

$$R_i^2(\mathbf{x} - \mathbf{b}_j)^2 - R_j^2(\mathbf{x} - \mathbf{b}_i)^2 = 0. \tag{12}$$

If $R_i = R_j$, (12) is the equation of the radical plane of the two equal spheres at the vertices B_i and B_j, and it is easily verified that this radical plane contains the straight line (8) $\mathbf{x} = \mathbf{c} + k\mathbf{d}/2$. The straight line (8) cuts the sphere (12) in two points P_1, P_2 given by the solution of a quadratic in k, which reduces after taking out the factor $R_i^2 - R_j^2$, here assumed to be different from zero, to

$$\left(\mathbf{c} + k\,\frac{\mathbf{d}}{2}\right)^2 - kR_0^2 = 0. \tag{13}$$

Hence the $(n+1)n/2$ spheres of similitude pass through these two points, which may be real, imaginary, or coincident. We easily verify that $CP_1 \cdot CP_2 = \mathbf{c}^2 = R^2$, where C is the circumcenter and R is the circumradius. The centers of the spheres (12) are easily seen to lie in the plane (9) $\mathbf{x} \cdot \mathbf{d} = R_0^2$, the radical plane of the system of spheres

$$\mathbf{x}^2 - 2\mathbf{c} \cdot \mathbf{x} - k(\mathbf{x} \cdot \mathbf{d} - R_0^2) = 0,$$

which are orthogonal to the $n+1$ spheres at the vertices corresponding to the parameter k.

3786 [1936, 310]. *Proposed by V. Thébault, Le Mans, France.*

With the vertices of a tetrahedron $ABCD$ as centers spheres are described such that the square of the radius of the sphere for a given vertex is equal to half the sum of the squares of the lengths of the edges through that vertex. (a) The sphere (w), orthogonal to the four spheres, has for its center the Monge point Ω of the tetrahedron. (b) Express the radius of the sphere (w) as a function of the radius of the circumsphere (o) and the lengths of the edges. (c) Show that the polar plane of the centroid G of the tetrahedron, with respect to the sphere (w), is the transform of the radical plane of the spheres (o) and (w) in the transformation of similitude $(G, 3)$.

Solution by L. M. Kelly, Lawrence, Mass.

The point Ω is the radical center of the four spheres since its power with respect to each is $R^2 - S/4$, where S is the sum of the squares of the lengths of the edges of the tetrahedron. Hence the Monge point is the center of the orthogonal sphere and the above expression is the square of the radius of the sphere. See

Court's *Solid Geometry*, p. 202. The formula on page 183 of that text gives the distance of G from the radical plane as

$$\frac{R^2 - [R^2 - S/4]}{2\Omega 0} = \frac{S}{8\Omega 0}.$$

If GN is the distance from G to the polar plane, we have $\Omega G \cdot \Omega N = R^2 - S/4$; and

$$GN = \frac{R^2 - S/4}{\Omega 0/2} - \frac{\Omega 0}{2} = \frac{3S}{8\Omega 0}.$$

Hence the ratio of the two distances is 1/3.

Solved also by the proposer.

Editorial Note. The proof of the following generalization is easy:

Given in space of n dimensions, $n \geqq 2$, $n+1$ points A_i as vertices of a non-degenerate simplex, and at each vertex A_i as center a sphere of radius R_i, a necessary and sufficient condition that the Monge point M of the simplex shall be the center of a sphere, real or imaginary, orthogonal to the $n+1$ spheres is that

$$(1) \qquad (n-1)R_i^2 - (n+1)(GA_i)^2 = K, \text{ a constant},$$

where G is the centroid of the simplex. The radius r of the orthogonal sphere is given by

$$(2) \qquad r^2 = \frac{4R^2}{(n-1)^2} - \frac{2S}{(n+1)(n-1)^2} - \frac{K}{n-1},$$

where R is the radius of the circumsphere of the simplex and S is the sum of the squares of its edges.

Taking G as the origin of vectors let $\mathbf{a}_i$ be the vector of the vertex A_i, let $\mathbf{m}$ and $\mathbf{c}$ be the vectors of the Monge point M and circumcenter C, and let s denote the sum of the squares of the distances of the vertices from G. We have the following relations:

$$(3) \qquad \begin{gathered} S = (n+1)s, \qquad (n-1)\mathbf{m} = -2\mathbf{c}, \\ [(n-1)\mathbf{m} + \mathbf{a}_i + \mathbf{a}_j] \cdot (\mathbf{a}_i - \mathbf{a}_j) = 0, \qquad (n+1)[R^2 - \mathbf{c}^2] = s. \end{gathered}$$

The second and third equations were derived in the solution of 3757 [1937, 604]. If e_{ij} is the length of the edge A_iA_j then $(\mathbf{a}_i - \mathbf{a}_j)^2 = e_{ij}^2$, and

$$(4) \qquad \begin{aligned} \sum_{j=1}^{n+1} e_{ij}^2 &= (n+1)\mathbf{a}_i^2 - 2\mathbf{a}_i \cdot \sum_{j=1}^{n+1} \mathbf{a}_j + \sum_{j=1}^{n+1} \mathbf{a}_j^2, \qquad e_{ii} = 0, \\ &= (n+1)\mathbf{a}_i^2 + s, \quad \sum_{j=1}^{n+1} \mathbf{a}_j = 0. \end{aligned}$$

$$S = \sum_{i<j} \sum e_{ij}^2 = \frac{(n+1)s + (n+1)s}{2} = (n+1)s.$$

The last equation of (3) is obtained by the summation of each member of $(\mathbf{a}_i-\mathbf{c})^2=R^2$ from $i=1$ to $i=n+1$. The power r_i^2 of M with respect to the sphere at A_i is

$$r_i^2 = \mathbf{a}_i^2 - 2\mathbf{a}_i\cdot\mathbf{m} + \mathbf{m}^2 - R_i^2; \tag{5}$$

and in order to have $r_i^2=r_j^2$, we must have

$$\mathbf{a}_i^2 - \mathbf{a}_j^2 - 2\mathbf{m}\cdot(\mathbf{a}_i - \mathbf{a}_j) - R_i^2 + R_j^2 = 0.$$

With the third equation of (3) this reduces to

$$(n-1)R_i^2 - (n+1)\mathfrak{a}_i^2 = (n-1)R_j^2 - (n+1)\mathbf{a}_j^2 = K. \tag{6}$$

This proves (1). Setting now $r_i^2=r^2$ in (5) and using (6), we have

$$r^2 = -\frac{2}{n-1}\mathbf{a}_i^2 - 2\mathbf{a}_i\cdot\mathbf{m} + \mathbf{m}^2 - \frac{K}{n-1}.$$

The summation of each member from $i=1$ to $i=n+1$ gives

$$(n+1)r^2 = -\frac{2s}{n-1} + (n+1)\mathbf{m}^2 - \frac{n+1}{n-1}K. \tag{7}$$

Then by the use of (3) this reduces to the desired result in (2).

The equations of the orthogonal sphere (M) and the circumsphere (C) are respectively

$$(\mathbf{x}-\mathbf{m})^2 = r^2, \qquad (\mathbf{x}-\mathbf{c})^2 = R^2.$$

The polar of G with respect to (M) is $\mathbf{m}\cdot\mathbf{x}-(\mathbf{m}^2-r^2)=0$, and this reduces by (7) to

$$\mathbf{m}\cdot\mathbf{x} = \frac{2s+(n+1)K}{(n+1)(n-1)}. \tag{8}$$

The radical plane of (M) and (C) reduces by (3) and (7) to

$$\mathbf{m}\cdot\mathbf{x} = \frac{s+K}{(n+1)(n-1)}. \tag{9}$$

These two planes are cut by their normal, the straight line through G, C, M in N_1 and N_2, respectively; and, if we set $K=ps$, we have at once

$$\frac{GN_1}{GN_2} = \frac{2+p(n+1)}{1+p}. \tag{10}$$

In order to obtain the particular case of an extension of the problem to n dimensions we set $(n-1)R_i^2$ equal to the sum of the squares of the edges issuing from A_i, that is

$$(n-1)R_i^2 = s + (n+1)\mathbf{a}_i^2,$$

and this requires $K=s$, and $p=1$. The above results are then

$$r^2=\frac{4R^2-S}{(n-1)^2}, \qquad \frac{GN_1}{GS_2}=\frac{n+3}{2}.$$

For $n=3$ we have the results required in the problem.

The solution of 3752 [1937, 402] considered a variable system of $n+1$ spheres with centers at the vertices of the simplex and with radii $\sqrt{k}R_i$. It was shown that the locus of the radical centers of the system is a straight line through C as k varies. If we select for the values of R_i those given by (1) in this note for any chosen value of K, the straight line locus passes through C, G, and M.

3789 [1936, 378]. *Proposed by J. M. Feld, New York City.*

Show that the equation of the ellipse can be written in the form

$$[\bar{a}(z-c)-b(\bar{z}-\bar{c})][\bar{b}(z-c)-a(\bar{z}-\bar{c})]+(a\bar{a}-b\bar{b})^2=0,$$

where a, b, c are fixed complex numbers, z is a variable complex number, and $\bar{z}$ is the conjugate of z. Find the center, the lengths of the semi-axes, and the position of the major axis.

Solution by E. P. Starke, Rutgers University.

Referring to Boyce's solution of problem 3718 [1936, 441], the elimination of t from equation (2) and its conjugate relation gives the equation now proposed. Hence quoting his results, we have here an ellipse with center at c, with $|a|+|b|$ and $||a|-|b||$ as lengths of semi-axes, and with the bisector of the angle between the vectors a and b at the origin as the direction of the principal axis.

Solved also by W. B. Campbell, J. W. Cell, J. F. Locke, C. E. Springer, F. Welker, and the proposer.

Editorial Note. Starke gave an independent solution; but, after the appearance of the solutions of 3718 in print, he supplied the above as containing the essentials of his solution. The proposer referred to his note in this MONTHLY *On the geometry of clocks*, 1930, p. 368 as the origin of this problem.

3790 [1936, 378]. *Proposed by V. W. Adkisson, University of Arkansas.*

If $F(a, b, c, x)$ represents the hypergeometric series, show that

$$F(s,-p,p+s,1)=\frac{(s+p-1)(s+p-2)\cdots(p+1)}{2(s+2p-1)(s+2p-2)\cdots(2p+1)},$$

where p is in general complex, $R(p)>0$, and s is a positive integer. In particular show that

$$F(1,-p,p+1,1)=1/2, \qquad R(p)>0.$$

Solution by J. W. Cell, North Carolina State College.

From Whittaker and Watson *Modern Analysis* (4th Ed.), page 282, we take the formula

$$F(a, b, c, 1) = \Gamma(c)\Gamma(c - a - b)/\Gamma(c - a)\Gamma(c - b),$$

provided that $R(c-a-b)>0$. We substitute the corresponding quantities for a, b, and c and find that

$$F(s, -p, p + s, 1) = \Gamma(p + s)\Gamma(2p)/\Gamma(p)\Gamma(2p + s)$$

which easily simplifies to the required results by the aid of the standard formula $\Gamma(x+1)=x\Gamma(x)$.

Solved also by J. F. Locke and the proposer.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

The Institute for Advanced Study and Princeton University are continuing the policy of reproducing notes for some of the series of lectures in mathematics given during the year. The tentative list of new editions to be photolithographed is: Bohnenblust, Integral equations; Hecke, Dirichlet series, modular functions and quadratic forms; Hurewicz, Dimension theory; Lefschetz, Algebraic geometry; Mayer, Topological group-systems; Thomas, Differential geometry; Wintner, Infinite convolutions. Some of the notes from former years are still available. Second editions or reruns have been made for the following: Bohnenblust, Real variables, 1936–7, (slightly revised); Bochner, Complex variables II, 1935–6; Goedel, On undecidable propositions of formal mathematical systems, 1933–4; and Lefschetz, Topology, 1934–5. Order blanks with titles and prices will be ready in September. Inquiries and orders should be addressed to *Princeton Mathematical Notes, Fine Hall, Princeton, N. J.*

The Cowles Commission for Research in Economics has issued a preliminary announcement of the fourth annual research conference on Economics and Statistics, to be held at Colorado Springs, Colorado, from July 5 to July 29, 1938. Two lectures will be given each morning except on Saturdays and Sundays. Suitable living accommodations are available in the dormitories at Colorado College at reasonable rates. All persons who plan to attend the Conference should notify the Cowles Commission as soon as possible. A final program giving the dates of lectures with a list of subjects and speakers will be available on request about June 1. The tentative program includes addresses by H. T. Davis, Mordecai Ezekiel, H. E. Jones, A. P. Lerner, V. D. Reed, Gerhard Tintner, Arne Fisher, H. P. Hartkemeier, D. H. Leavens, Francis McIntyre, Joseph Mayer, Horst Mendershausen, C. F. Roos, R. G. D. Allen, Alfred Cowles 3rd, Margaret Joseph, Horace Secrist, A. Wald, R. J. Watkins, T. O.

Yntema, S. v. Ciriacy-Wantrup, E. L. Dodd, M. M. Flood, F. L. Griffin, J. Neyman, Henry Schultz, and E. J. Working.

The Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics, which is preparing a report on "The Place of Mathematics in Secondary Education," is issuing a Preliminary Report in two parts. The first part has already been issued, and it is expected that the second part will appear about July 1. The report is being issued in a preliminary form in order to secure the comments and suggestions of teachers, administrators, and educators. Persons desiring copies can obtain them by writing to the Chairman, Professor K. P. Williams, Indiana University, Bloomington, Indiana, at ten cents a copy for each part (coins preferred). The membership of the Joint Commission is as follows: For the Association: A. A. Bennett, Brown University; H. E. Buchanan, Tulane University; F. L. Griffin, Reed College; C. A. Hutchinson, University of Colorado; H. F. MacNeish, Brooklyn College; U. G. Mitchell, University of Kansas. For the Council: William Betz, Rochester Public Schools; M. L. Hartung, Progressive Education Association; G. H. Jamison, State Teachers College, Kirksville, Mo.; Ruth Lane, University High School, Iowa City; J. A. Nyberg, Hyde Park High School, Chicago; Mary A. Potter, Supervisor of Mathematics, Racine, Wisconsin; W. D. Reeve, Columbia University.

Models of second order surfaces and a few models of more advanced types made by the late Professor R. P. Baker will be available for purchase. Descriptions of the models may be had upon application to Dr. Frances E. Baker, Mount Holyoke College, South Hadley, Mass.

Assistant Professor R. H. Knox of the Virginia Military Institute is on leave of absence and is at the University of Michigan.

NOTICE ABOUT FORWARDING THE MONTHLY

A member or subscriber wishing to have his copy of the June-July MONTHLY forwarded to a summer address should leave four cents postage with his local post office or carrier, with instructions for remailing.

THE SEMICENTENNIAL OF THE AMERICAN MATHEMATICAL SOCIETY

The Semicentennial Celebration of the founding of the American Mathematical Society will be held from Tuesday, September 6, to Friday, September 9, at Columbia University, New York, N. Y. This jubilee meeting is of unusual interest both in the scientific program arranged and in the excursions, exhibits and social features planned for that week. By special action of the Council of the Society, members of the Mathematical Association of America are invited to participate in all the activities on the same basis as members of the Society. The copy of the final program and a reservation blank will be sent in due course to members of the Association. The Mathematical Association is omitting a program of its own in order that the entire week may be devoted to the Society's celebration.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The following results of the William Lowell Putnam Mathematical Competition held April 16, 1938, have been determined in accordance with the rules of the Competition agreed to by the representatives of the Mathematical Association and the trustees of the William Lowell Putnam Intercollegiate Memorial Fund. The contestants were known to Association officials and to the reader only by number up to the time of this announcement.

The first prize, five hundred dollars, is awarded to the department of mathematics of the UNIVERSITY OF TORONTO, Toronto, Ontario. The members of the team were A. J. COLEMAN, I. KAPLANSKY, N. S. MENDELSOHN; to each of these is awarded a prize of fifty dollars.

The second prize, three hundred dollars, is awarded to the department of mathematics at the UNIVERSITY OF CALIFORNIA, Berkeley. The members of the team were S. P. FRANKEL, W. M. KINCAID, C. W. LIPPMAN; to each of these is awarded a prize of thirty dollars.

The third prize, two hundred dollars, is awarded to the department of mathematics at Columbia University, New York. The members of the team were S. W. BENSON, B. A. JACOBSOHN, JOSEPH STATSINGER; to each of these is awarded a prize of twenty dollars.

The five persons ranking highest in the examination, arranged alphabetically, were ROBERT GIBSON, Fort Hays Kansas State College; I. KAPLANSKY, University of Toronto; G. W. MACKEY, Rice Institute; M. J. NORRIS, College of St. Thomas, BERNARD SHERMAN, Brooklyn College. Each of these will receive a prize of fifty dollars. The order of the names in this list has no relation to their rank in the examination.

The following is a list of all colleges and universities which entered teams in the Competition. (This list is arranged alphabetically, and the order of the names here has no relation to the rank of the teams in the examination.) Adelphi College, Amherst College, Brooklyn College, Centenary College, Colby College, College of New Rochelle, College of St. Thomas, College of the City of New York, Columbia University, Cornell University, Dalhousie University, Fontbonne College, Heidelberg College, Lehigh University, Luther Collge, Maryville College, Massachusetts Institute of Technology, Middlebury College, Northeastern University, Northwestern University, Oberlin College, Pomona College, St. Olaf College, Shurtleff College, Swarthmore College, the universities of Buffalo, California, California at Los Angeles, Denver, Illinois, Pittsburgh, Saskatchewan, Tennessee, Toronto, Vermont, Washington, Western Ontario; Vassar College, Webb Institute of Naval Architecture, Western Reserve University, Whitman College, Yale University.

In addition to these teams,* there were thirty-seven individual contestants from these and twenty-five other institutions, making a total of one hundred sixty-three individuals representing sixty-seven institutions.

W. D. CAIRNS, *Secretary-Treasurer*

* The Division of Mathematics at Harvard University made out the examination paper this year and, in view of that fact, decided that Harvard should not enter a team this year.

THE MATHEMATICAL ASSOCIATION OF AMERICA

The following forty-six persons have been elected to membership in the Association on applications duly certified:

H. C. AYRES, Ph.D.(California) Instr., Univ. of Alabama, University, Ala.

AARON BAKST, Ph.D.(California) Research asso., Teachers Coll., Columbia Univ., New York, N. Y.

W. G. BANKS, JR., A.M.(Virginia) Instr., Centenary Coll., Shreveport, La.

MURIEL BOWDEN, A.M.(Columbia) Headmistress, St. Agatha School., New York, N. Y.

R. D. BROWN, Jr. Engr. in charge, Patent Dept., Philadelphia Storage Battery Co., Philadelphia, Pa.

G. F. CRAMER, Ph.D.(Missouri) Asst. Prof., Tulane Univ., New Orleans, La.

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W. D. CAIRNS, *Secretary-Treasurer*

THE APRIL MEETING OF THE SOUTHEASTERN SECTINO

The sixteenth annual meeting of the Southeastern Section of the Mathematical Association of America was held at the Georgia School of Technology, Atlanta, Georgia, on Friday and Saturday, April 1–2, 1938. There were in attendance more than two hundred persons from fifty-five institutions, including the following seventy-three members of the Association: H. C. Ayres, D. H. Ballou, J. G. Barnes, D. F. Barrow, Helen Barton, W. S. Beckwith, C. W. Bruce, Iris Callaway, T. C. Carson, Edna J. Cofield, W. B. Coleman, H. M. Cox, Forrest Cumming, U. P. Davis, B. F. Dostal, F. G. Dressel, L. A. Dye, E. D. Eaves, Floyd Field, R. L. Fritz, H. K. Fulmer, Leslie J. Gaylord, A. M. Gignilliat, M. E. Gillis, J. T. Hains, C. L. Hair, W. T. Hanson, D. C. Harkin, R. A. Hefner, G. W. Hess, Ruby U. Hightower, P. R. Hill, H. K. Holt, J. A. Hyden, J. B. Jackson, Rosa L. Jackson, H. T. Karnes, F. W. Kokomoor, G. B. Lang, Curtis Ledford, F. A. Lewis, S. W. McInnis, A. N. McPherson, J. F. Messick, Verdie F. Miller, W. L. Miser, J. S. Morrel, F. D. Murnaghan, W. P. Ott, D. D. Peele, Z. M. Pirenian, R. A. Purviance, W. W. Rankin, B. P. Reinsch, G. E. Reves, H. A. Robinson, Douglas Rumble, J. A. L. Saunders, W. H. Sears, Jr., W. E. Sewell, T. M. Simpson, D. M. Smith, H. E. Spencer, A. L. Starrett, F. H. Steen, R. P. Stephens, Ruth W. Stokes, S. Helen Taylor, Louise Thompson, M. J. Turner, D. L. Webb, M. C. Wicht, F. L. Wren.

Sessions were held Friday afternoon and evening and Saturday morning. Professor J. B. Jackson, Chairman of the Section, presided, except Friday evening and part of Saturday morning when the Section was divided into subgroups according to the nature of the papers presented. Subgroups were presided over by Vice-Chairman J. A. Hyden, Iris Callway, T. M. Simpson and J. F. Messick. On Friday evening a joint dinner with the Georgia Academy of Science was held in honor of the visiting speaker, Professor F. D. Murnaghan of Johns Hopkins University. At this time Dean Floyd Field presided.

At the business session on Saturday the following officers were chosen for 1938–39: Chairman, J. A. Hyden, Vanderbilt University; Vice-Chairman, D. M. Smith, Georgia School of Technology; Secretary-Treasurer, H. A. Robinson, Agnes Scott College. The next meeting was scheduled for March, 1939, at The

Citadel in Charleston, S. C. A resolution was passed relative to the loss sustained by the Section in the passing of Professor J. D. Bond.

The following twenty-nine papers were presented:

1. "Some aspects of the teaching of mathematics in Japan and Korea" by Dr. W. P. Parker, Pyengyang, Korea, introduced by the Secretary.

2. "On the factorization of matric polynomials" by Professor H. S. Thurston, University of Alabama, introduced by the Secretary.

3. "Some inequalities connected with exponential functions" by Professor W. E. Sewell, Georgia School of Technology.

4. "Eighteenth century mathematics" by Professor W. W. Rankin, Jr., Duke University.

5. "The new Georgia high school requirement in mathematics" by Dean R. P. Stephens, University of Georgia.

6. "A class of real functions" by Professor J. M. Thomas, Duke University, by title.

7. "Moments of the point binomial" by Professor F. L. Wren, Peabody College.

8. "The basic ideas of arithmetic and algebra" by Professor F. D. Murnaghan, Johns Hopkins University.

9. "Certain formulas in compound interest and annuities" by Professor Z. M. Pirenian, University of Florida.

10. "A Cremona transformation of order 13" by Professor L. A. Dye, The Citadel.

11. "Some properties of the solution of linear difference equations" by Dr. H. C. Ayres, University of Alabama.

12. "Finite deformations of an elastic solid" by Professor F. D. Murnaghan, Johns Hopkins University.

13. "Conics in complex coördinates" by Professor Verdie F. Miller, Young Harris College.

14. "A study of the triangle by means of trilinear coördinates" by Professor G. W. Hess, Howard College.

15. "On volumes bounded by cylindrical surfaces" by Professor J. D. Mancill, University of Alabama, by title.

16. "Epitome equation of geometry" by Professor D. C. Harkin, Alabama Polytechnic Institute.

17. "Non-vanishing Laplace integrals" by Professor D. H. Ballou, Georgia School of Technology.

18. "Stieltjes integral equations" by Dr. F. G. Dressel, Duke University.

19. "The solution of a certain linear partial differential equation of first order" by Dr. L. D. Rodabaugh, University of Alabama, introduced by the Secretary.

20. "First lessons in definite integrals" by Professor W. L. Miser, Vanderbilt University.

21. "Functions of a generalized angle" by Dr. I. M. Hostetter, Howard College, introduced by the Secretary.

22. "Reduction of polynomial fractions to sums of partial fractions" by Professor F. H. Steen, Georgia School of Technology.

23. "A set of defining relations for the simple group of order 3420" by Professor F. A. Lewis, University of Alabama.

24. "A theorem on resultants of polynomials" by Professor J. S. Morrel, Vanderbilt University.

25. "Rings in many-valued logics" by Dr. D. L. Webb, Georgia School of Technology.

26. "Mathematical accuracy in a physical experiment" by Professor P. R. Hill, University of Georgia.

27. "Pretended accuracies in mathematical problems and computations" by Professor B. P. Reinsch, Florida Southern College.

28. "Measurement of student achievement in mathematics" by H. M. Cox, secretary to the Examiners, University System of Georgia.

29. "Discussion: high school requirements in mathematics" led by Professor D. M. Smith, Georgia School of Technology.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Dr. Parker showed how the Japanese and Koreans are overcoming certain difficulties in learning mathematics due to their past training such as would result from rote learning, a clumsy number system, and an over emphasis on classics.

2. Professor Thurston determined the number of ways in which a polynomial $F(X)$ may be factored as a product of linear factors, $X-X_i$, where the X_i are integral rational functions of a given matrix. The question of similar factorization was considered.

3. The exponential function may be defined as the limit of a binomial expansion, or as a Taylor expansion. The rates at which these expansions approach the function and each other were discussed by Professor Sewell.

4. Professor Rankin described the development of mathematics during the eighteenth century, showing that the builders of that period were mostly interested in the application of the new methods of analysis, analytical geometry, and calculus to the study of physical phenomena. The ideal rigor of the eighteenth century mathematicians was that of the ancient Greeks.

5. Dean Stephens discussed the new Georgia regulation for graduation from a senior high school which requires one unit in "general mathematics including the elements of algebra," but no geometry. From those preparing for college will come the future leaders. This special group should be given special consideration and should be advised to take more mathematics as a part of their preparation. Seemingly a great deal of experimentation is going on in many of the schools. This should be discouraged except in laboratory institutions. The required core program should consist of those subjects whose educational value has been thoroughly proved by past experience.

6. Let $f(x)$ be bounded and differentiable on the interval $a \leqq x \leqq b$. Let its

derivative satisfy $|f'(x)| \leqq M|f(x)|$, where M is a positive constant. Professor Thomas proved that if f has a zero on the interval, it is identically zero there. He applied the result to the uniqueness proof for the solution of systems of ordinary differential equations of the first order.

7. Professor Wren developed recurrence formulas which expressed for the point binomial, moments of any given order about q^n, p^n, and the true mean in terms of moments of the same type but lower order.

8. Professor Murnaghan discussed the properties of the positive integers which serve to define them and then passed to the fractions and negative numbers, pointing out the seeming paradoxes connected with them. The central position assumed by the commutative law of multiplication was emphasized and attention was called to the existence of non-commutative algebras and to their important applications in the domain of physics.

9. Professor Pirenian presented several formulas by which long divisions in the mathematics of finance may be replaced by multiplications, and long multiplications by additions or subtractions. These formulas were found useful in constructing certain tables and solving certain problems.

10. Professor Dye used two (1, 1) correspondences to generate this transformation. The first was between the quadrics of a pencil and the lines of a linear system of lines; the second related the conics cut from a quadric by planes through the associated line, to the points of a fixed line. The transformation was capable of generalization in two ways.

11. Dr. Ayres discussed the solution of a boundary value problem for an nth order nonhomogeneous linear difference equation. The boundary conditions were n linear relations involving differences of order $n-1$. A Green's function for the reduced equation was derived and led by means of a summation to a unique solution of the problem under consideration.

12. Professor Murnaghan gave an account of a recent theory of elasticity proposed by him in which the basic simplicity assumption of the classical theory of elasticity, namely, that the deformation is infinitesimal, is not made. The exact equations connecting the stress and strain tensors were given. The theory found confirmation by experiment when applied to the case of uniform hydrostatic pressure. Recent experiments by Bridgman on the compressibility of metals and of various rocks show that the theory predicts, to within the limits of experimental error, the experimental results over a range extending to a pressure of 45,000 atmospheres. A remarkable feature was that only one constant appeared in the formula.

13. Professor Miller gave the solution of problem 3789 of the MONTHLY (June–July, 1936) together with a generalization to all conics.

14. Professor Hess made a thorough study of the various properties of a triangle by means of trilinear coördinates, using the given triangle for reference.

15. The determination of certain volumes common to two given cylindrical surfaces was discussed by Professor Mancill. It was shown how a slight alteration of the usual procedure of the integral calculus greatly simplified the problem.

16. Professor Harkin gave various geometric and trigonometric interpretations to an identity attributed to Euler, but known in the tenth century by the Arab El-Sābī.

17. Professor Ballou gave a set of conditions on the determining function of a Laplace integral sufficient to ensure that the integral shall not vanish in the half-plane of convergence.

18. In this paper Dr. Dressel presented a short history of a Stieltjes integral equation, indicated when such an equation can be changed to an ordinary Fredholm equation, and finally pointed out some unsolved problems concerning the equation.

19. Dr. Rodabaugh proved an existence theorem for a solution of a linear partial differential equation in a simply connected region of the xy-plane.

20. Professor Miser exhibited several examples of definite integrals which can be evaluated as limits of sums. By such simple examples the students may learn to calculate areas, etc., as limits of sums and obtain a better comprehension of a definite integral.

21. The single parameter family of extremal arcs in space determined by a fixed vector at a point defines a Finsler space which may be considered as a generalization of a plane. It was the purpose of this note to find a measure of the generalized angle formed by two of the members of the family determining the plane, and to discuss functions of such an angle analogous to the trigonometric functions.

22. Professor Steen gave a simple method for obtaining the coefficients in any of the various types of partial fraction expansions.

23. Professor Lewis discussed the defining relations of the simple group of order 3420.

24. By means of linear dependence, necessary and sufficient conditions for the existence of common divisors of two polynomials were derived. Professor Morrel obtained formulas for such divisors in terms of certain minors of the conventional Sylvester determinant.

25. The rings which M. H. Stone has shown to exist in the case of Boolean algebra were extended by Dr. Webb to the algebra of many-valued logics.

26. Professor Hill discussed a physical experiment in which any desired degree of accuracy, however great, may be secured.

27. Professor Reinsch called attention to the frequent occurrence of errors in data and computation in science textbooks, and proposed remedies for the situation.

28. Mr. Cox presented a statistical study illustrating the achievement in mathematics of the freshmen and sophomore students of thirty Georgia colleges. The same battery of tests was used by a large number of Georgia high schools so that the sampling of students entering college can be analyzed.

29. Until the regulations for graduation from high school are changed, colleges will have to give special consideration to students entering with only one mathematics unit. It is imperative that those qualified to study mathematics,

not only be given an ample opportunity to do so, but be prevented from placing upon themselves a regrettable handicap by neglecting it.

H. A. ROBINSON, *Secretary*

THE MARCH MEETING OF THE MICHIGAN SECTION

The fifteenth annual meeting of the Michigan Section of the Mathematical Association of America was held at the University of Michigan, Ann Arbor, Michigan, on Saturday, March 19, 1938. The chairman of the Section, Professor V. G. Grove, presided. The Section was most fortunate in having among its attendants Professor E. J. Moulton, the invited speaker, and Professor W. D. Cairns, the national Secretary.

The attendance was about one hundred, including the following forty-seven members of the Association: N. H. Anning, W. L. Ayres, W. D. Baten, F. A. Beeler, W. M. Borgman, Jr., J. W. Bradshaw, J. B. Brandeberry, W. D. Cairns, R. V. Churchill, C. J. Coe, A. H. Copeland, Max Coral, S. E. Crowe, Wayne Dancer, P. S. Dwyer, C. H. Fischer, J. D. Fitzpatrick, W. B. Ford, R. E. Gaskell, T. N. E. Greville, V. G. Grove, T. H. Hildebrandt, L. A. Hopkins, E. E. Ingalls, L. S. Johnston, L. C. Karpinski, A. E. Lampen, Theodore Lindquist, C. E. Love, E. D. McCarthy, D. D. Miller, D. C. Morrow, E. J. Moulton, A. L. Nelson, L. F. Ollmann, L. C. Plant, J. E. Powell, G. Y. Rainich, E. D. Rainville, C. C. Richtmeyer, T. R. Running, M. E. Shanks, G. G. Speeker, T. O. Walton, Fern Welker, R. L. Wilder, Margarete C. Wolf.

Sixty-nine persons attended the noon luncheon at the Michigan Union. At the business meeting which followed, W. L. Ayres was elected chairman and P. S. Dwyer was elected secretary-treasurer. After some discussion it was decided to have but one meeting during the coming year.

The luncheon meeting was also addressed by Professor W. D. Cairns, who assisted the Section in its deliberations and indicated the rôle the Association should fill with reference to (a) a greater correlation with the interests of secondary school teachers, and (b) the preparation of good college teachers.

The following seven papers were presented at the morning and afternoon sessions:

1. "Angles between two flat subspaces in n-dimensions" by Professor G. Y. Rainich and R. S. Phillips, University of Michigan. Presented by Mr. Phillips.

2. "A certain class of variation problems" by Morris Friedman, Wayne University, introduced by Professor A. L. Nelson.

3. "The Fibonacci series and allied trigonometric identities" by Professor L. S. Johnston, University of Detroit.

4. "The solution of a problem in card matching" by Dr. T. N. E. Greville, University of Michigan.

5. "An algebra of Dedekindian numbers" by J. C. Rumball, Wayne University, introduced by Professor A. L. Nelson.

6. "Some remarks on maxima and minima" by Professor C. J. Coe, University of Michigan.

7. "Mathematical accuracy" by Professor E. J. Moulton, Northwestern University, by invitation.

Abstracts of these papers follow, the numbers corresponding to the numbers in the the list of titles:

1. Mr. Phillips gave a vectorial treatment of the problem of characterizing the relative position of two flat subspaces of a hyperspace. This had been treated analytically by Jordan and synthetically by Schoute. The method consists in finding stationary values of scalar products of pairs of unit vectors—one in each subspace. These quantities are obtained as the roots of a polynomial which appears as a determinant, some of whose elements are scalar products of unit vectors of the two subspaces.

2. In this paper Mr. Friedman discussed the calculus of variations problem involving the integral $I=\int_{x_0}^{x_1} y^{1/n}(1+y'^2)^m dx$ where m and n are real numbers such that $m>0$, $n<0$. He showed that there exist certain regions of the plane, depending on m and an arbitrarily chosen point P_0, the points of which can be joined to P_0 by one and only one extremal, and discussed properties of these extremals.

3. Professor Johnston exhibited several classical identities of trigonometry which have either exact or very closely related parallels in which the arguments are numbers of the Fibonacci sequence instead of trigonometric functions. He also exhibited several functional equations solvable either in terms of trigonometric functions of the arguments or in terms of the Fibonacci sequence which are identifiable by the same arguments. Also shown were some well known expressions for trigonometric functions of specific angles which are derivable from the same Fibonacci sequence.

4. Dr. Greville obtained the exact frequency distribution of matchings which occurs when two shuffled decks of cards, each consisting of five duplicates of each of five designs, are matched against each other, card by card. This paper has been published in the *Journal of Parapsychology*, Vol. II, No. 1, under the title, *Exact Probabilities under the Matching Hypothesis.*

5. Sections of an everywhere dense set of transfinite ordinal numbers were made defining a set of Dedekindian numbers. Mr. Rumball utilized the properties of the transfinite ordinals in establishing the corresponding properties of Dedekindian numbers. The method is analogous to the method used by Richard Dedekind in defining the real number system by cuts of the rational numbers.

6. Professor Coe discussed the problem of determining the extrema of a function $\phi(x, y)$ under the condition $f(x, y)=0$. The problem was reduced to finding those linear elements which are simultaneously tangent to some curve of the family $\phi(x, y)=$constant and to the curve $f(x, y)=0$. This reduction was shown to be a fertile source of problems on maxima and minima and to lead to an immediate solution of a large percentage of such problems ordinarily treated in the calculus.

7. To say that something is stated or proved with "mathematical accuracy" is usually considered the highest of praise. Professor Moulton undertook to illustrate how the efforts of mathematicians to attain precision have not been entirely successful. His examples were taken from elementary courses, referring in particular to definitions of a straight line and of a circle, to a method of finding points of intersections of curves, to the definition of sin x as used in trigonometry and in calculus, to a proof of a law of the mean, and to an application of Duhamel's Theorem. He advocated the use of more precise statements, but recognized the impossibility of attaining absolute precision, particularly in elementary courses.

P. S. DWYER, *Secretary*

THE FIFTH ANNUAL MEETING OF THE OKLAHOMA SECTION

The fifth regular meeting of the Oklahoma Section of the Mathematical Association of America was held at the Central High School in Oklahoma City, Oklahoma, on Friday morning, February 11, 1938, Professor H. L. Hall presiding.

The number in attendance was sixty-one, including the following twenty members of the Association: E. F. Allen, J. C. Brixey, J. H. Butchart, N. A. Court, R. C. Dragoo, W. V. N. Garretson, Esther Gassett, F. C. Gentry, Dot Jeannette Gifford, H. L. Hall, J. O. Hassler, E. E. Heimann, J. E. LaFon, Clarence McCormick, Dora McFarland, W. T. Short, C. E. Springer, E. B. Wedel, B. S. Whitney, J. H. Zant.

The following officers were elected for the next year: Chairman, W. T. Short, Oklahoma Baptist University; Vice-Chairman, J. H. Butchart, Phillips University; Secretary, C. E. Springer, University of Oklahoma. The next meeting will be held at Oklahoma City in February 1939 in connection with the Oklahoma Education Association.

The following papers were presented:

1. "Some recent advances in astrophysics" by Dr. H. S. Mendenhall, Oklahoma Agricultural and Mechanical College, introduced by Professor Allen.

2. "Quaternary Cremona groups of ternary type" by Professor F. C. Gentry, University of Oklahoma.

3. "M-points generated by similar triangles" by Professor J. H. Butchart, Phillips University.

4. "On numbers equal to the sum of their aliquot parts increased by the number of their aliquot parts" by Professor J. C. Brixey, University of Oklahoma.

5. "The right triangle as an aid in the teaching of integration" by Professor W. T. Short, Oklahoma Baptist University.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Dr. Mendenhall reviewed the developments in astrophysics of general

interest which have taken place during the last decade. He discussed quantitative, as well as qualitative, chemical analyses of the solar and stellar systems.

2. Professor Gentry considered the possibility of determining involutions in space by means of webs of quartic surfaces of degree two. Thirteen cases were found in which the base of the web contained simple base points and a double curve. In each case, by allowing the simple base points to vary while the remainder of the base was held fixed, these involutions were used to generate groups of Cremona transformations simply isomorphic to certain well known linear groups.

3. Professor Butchart proved for the sets of points (a_i), (b_i), (x_i), $(i=1, \cdots, n)$ that if the triangle $a_ib_ix_i$ is given in shape, then the centroids of the three sets determine a triangle of the same shape. Complex coördinates were used. Certain propositions from Johnson's *Modern Geometry* were proved as corollaries to the theorem.

4. Professor Brixey discussed the equation $S(n)=n$, where n is a positive integer and where $S(n)$ denotes the sum of the divisors of n, which are less than n, increased by the number of such divisors. Writing n in the form $n=2^{\alpha}p_1^{e_1}p_2^{e_2}, \cdots$, where $p_1, p_2, \cdots$ are distinct odd primes, he showed that α is odd and $e_1, e_2, \cdots$ are even. The only power of 2 which satisfies $S(n)=n$ is $n=2$, and n contains no $p_i \leqq 2^{\alpha+1}-1$. $S(n) \neq n$ if $n=2^{\alpha}p_1^{e_1}p_2^{e_2}$. The only solution less than 2,004,002 is $n=2$.

5. Professor Short exhibited several examples of integrals each containing one radical expression and showed the advantage of placing the radical expression on a suitably chosen side of a right triangle at the outset. The desired trigonometric substitution to effect the integration is then evident from the triangle.

C. E. SPRINGER, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The eighteenth regular meeting of the Southern California Section of the Mathematical Association of America was held at Pomona College, Claremont, California, Saturday, March 26, 1938. Professor A. D. Michal presided.

The attendance was about sixty, including the following twenty-six members of the Association: L. J. Adams, O. W. Albert, L. D. Ames, Harry Bateman, Clifford Bell, E. T. Bell, A. L. Buchman, Jessie R. Campbell, P. H. Daus, D. C. Duncan, Iva B. Ernsberger, Harriet E. Glazier, G. H. Hunt, C. G. Jaeger, Glenn James, G. R. Kaelin, I. Maizlish, A. D. Michal, W. B. Orange, Lena E. Reynolds, G. E. F. Sherwood, Marcus Skarstedt, C. W. Trigg, S. E. Urner, W. M. Whyburn, Euphemia R. Worthington.

The following officers were elected for the next year: Chairman, W. M. Whyburn, University of California at Los Angeles; Vice-Chairman, L. J. Adams, Santa Monica Junior College; Program Committee: Marcus Skarstedt, Whittier College, and Morgan Ward, California Institute of Technology. The meeting was tentatively scheduled for March 4, 1939, at Whittier College.

The following five papers were read:

1. "Mortality tables—their construction, graduation, and use" by A. L. Buckman, Occidental Life Insurance Co.

2. "Analysis of cosmic rays with the help of terrestrial and solar magnetic fields" by Professor P. S. Epstein, Professor of Theoretical Physics, California Institute of Technology, introduced by Professor Michal.

3. "Using the unusual—instructional values in the neglected exceptional cases, with examples" by Professor S. E. Urner, Los Angeles Junior College.

4. "Stability from the algebraic, analytical, and technical standpoint" by Professor Harry Bateman, California Institute of Technology.

5. "Some uses of matrix algebra in differential equations" by Professor W. M. Whyburn, University of California at Los Angeles.

Abstracts of the papers follow, the numbers corresponding to those in the list of titles.

1. Some applications of mathematics to the field of actuarial science were illustrated by Mr. Buckman, in his paper on the construction, graduation, and use of mortality tables. Formulas were given to obtain crude rates of mortality for population mortality tables and for mortality tables on insured lives. Comparison was made of various methods of graduation, including Orloff's weighting of crude data by probable errors, summation formulas, osculatory interpolation, Makeham's formulas and the Henderson-Whittaker Formula A. Some demographic interpretations of population mortality tables were given as well as the solution of a complicated tax problem requiring the determination of the value to each of five heirs of an estate left in trust.

2. Professor Epstein discussed the mathematical problems arising in connection with the motion of charged cosmic ray particles in the magnetic fields of the earth and sun. Particular attention was devoted to the question of the influence of the solar magnetic field upon the terrestrially observed phenomena, a subject recently advanced by Vallarta. The speaker pointed out that this cause might produce measurable diurnal variations of cosmic ray intensities in high magnetic latitudes and presented theoretically calculated curves for the magnitude of this effect. The comparison with the observations of Millikan and Neher (made in Saskatoon, Canada) was as yet inconclusive as the predicted differences lie on the verge of the experimental error.

3. Professor Urner believes: (1) The care used in setting up the conditions under which a theorem is true, is wasted unless the student sees clearly what could happen if the conditions were *not fulfilled*. (2) The proper frame of mind for apprehension of mathematical truth is one of *questioning*. (3) The exceptional cases constitute one of the most powerful tools for the creation of *interest*, in that they are likely to partake of the *spectacular*. Simple examples were listed, under such headings as equivalence of equations and systems of equations, linear dependence, intersections of curves whose equations are in polar coördinates, continuity, multiple-valued functions, existence of a derivative, implicit functions, end-points, maxima and minima, and graphical treatment of complex solutions of equations in two variables.

4. Professor Bateman first reviewed the theory of the stability of equilibrium largely from the historical standpoint, starting with the work of Huygens. Following Duhem the conditions for the stability of a floating body were obtained by considering all possible variations. Mention was made of recent results relating to the stability of ships. The algebraic problem connected with the stability of motion of a vehicle of transportation was discussed from a general standpoint and an account was given of some related developments. Some remarks were made also on the stability problems depending on the asymptotic forms of solutions of differential and integral equations. An attempt was made to indicate some of the branches of astronomy, biology, physics, and engineering in which problems of stability are of major importance.

5. The use of matrix algebra in studies of ordinary differential equations was discussed by Professor Whyburn. The major parts of the paper have appeared in an article by Birkhoff and Langer in the *Proceedings of American Academy of Arts and Science*, vol. 57, pp. 51–128, and in an article by the present author in the *American Journal of Mathematics*, vol. 56 (1934), pp. 587–592. Some applications of the notation in connection with Green's Function were discussed.

P. H. DAUS, *Secretary*

DETERMINATION OF A MAYAN UNIT OF LINEAR MEASUREMENT

G. F. CRAMER, Tulane University

The directors who were preparing for the Chicago Fair decided about 1930 to reproduce for the Fair the partly ruined buildings of the Nunnery Quadrangle at Uxmal in Northern Yucatan. The reproductions were to be constructed as nearly as possible exactly as the original buildings were when new. The careful measurements and scale drawings necessary for this work were made by members of an expedition working under the direction of the Department of Middle American Research of Tulane University.

This paper is concerned with the problem of using the measurements which appear on these plans to determine the size of the previously unknown unit of length which was used in the original structures. The measurements were arranged in order of size and all duplicates were discarded before the following work was started. The set of over a thousand dimensions appearing on the plans was thus cut down to 286.

The whole investigation was based on the assumption that any set of measurements of this sort would be almost certain to show a more frequent occurrence of multiples of the unit than of multiples of any other quantity not commensurable* with the unit. The set would, however, probably contain

* In this paper, two numbers are called "to all practical purposes not commensurable" if few or none of their common multiples lie within the range of the recorded measurements. The words commensurable and not commensurable were used in that sense throughout the paper.

many measurements which were not intended to be multiples of the unit and others which were inaccurately measured.

One plausible method of attacking the problem is that of dividing each measurement of the set by the integers 1, 2, 3, $\cdots$ and arranging the results in the form of a table with the original numbers in the first row, their halves in the second row, their thirds in the third row, and so on. Any entries appearing with unusual frequency would be suspected of being commensurable with the unit. However, this method is probably more suitable as an aid to guessing the unit than as a means of verifying the guess. In the present work, the above method was not used to justify any final conclusions, but the tabulation did show a remarkable number of entries which were almost exactly whole numbers of feet or whole numbers of inches, and this in spite of the fact that fractions of an inch were not dropped in making the measurements of the ruins.

The other method used requires a more extended discussion. Let us suppose that we are given a set of M numbers in which there is no particular predominance of multiples of any one quantity. If we plot these numbers on an X-axis, we shall find a more or less random distribution of points with no regular, periodic repetition of clusters of points. Let us now choose a number N of any convenient size and plot on the same axis the numbers kN for successive integral values of k. Next, we choose a number $2n$ which is less than N, and lay off a short interval of length n on each side of each of the division-points kN. Now, it seems reasonable that the number of points of the set M lying in one or another of these short intervals should be approximately $(2n/N)M$, since a point is, presumably, just as likely to fall in one position as in any other nearby position on the axis. We shall define the *predicted fraction* of the M numbers as $p=(2n/N)$.

If, on the contrary, our set of M numbers contains a great many which were intended to be multiples of some unit, we shall expect the plotted points to show a tendency to occur in regularly spaced clusters with centers at those points which represent the multiples of the unit which was used. In our practical application of this, we would probably find this clustering more noticeable if we did not discard all of the duplicate measurements, but it can be detected even in the smaller set of only 286 measurements. If N and n are chosen as before, we can determine how many of the numbers of the set M differ from multiples of N by an amount n or less, that is, how many of the points of the set lie in one or another of the short intervals previously referred to. Suppose an actual count shows m such numbers in the set M. We define the *actual fraction* of the M numbers as $P=m/M$. We also define the *ratio* as $R=P/p$, where p is the predicted fraction. It is clear that, if the chosen value of N is nearly the same as the unknown unit, then the corresponding value of R will be relatively high. If, on the contrary, N is not nearly commensurable with the unit, then the points which represent multiples of N will miss most of the clusters and the corresponding value of R will probably be less than one. In case the N is very nearly equal to the unit, we can show that the values of R which correspond to $N'=2N$ and

$N''=3N$ should be, roughly, the same as that of the R which was found for N. According to the statements above, we should expect a set of M random numbers to give values of R not greatly different from 1 for all N's of reasonable size, but should expect a set of M measured quantities to give unusually high values of R for some particular N_1 and its multiples. In the latter case, any N commensurable with N_1, but not a multiple of N_1 would be likely to give an R greater than 1, but not as great as the R's corresponding to N_1 and its multiples. Any N not commensurable with N_1 should, of course, give a value of R less than 1.

In the actual numerical work on the set of 286 numbers, the value $n=\frac{1}{4}$ inch was used because the men who made the measurements stated that their errors, particularly on the smaller measurements, were probably less than that. The value of R was calculated for each of a considerable number of different values of N chosen at rather short intervals. Since practical considerations seemed to indicate the necessity of some commonly used unit less than a foot, no values of N greater than 12 inches were tried. The results of this work as shown in Table I seem to show that the most likely unit is very close to 3 inches. We notice that the R's which correspond to the values 3, 6, 9, and 12 of N are considerably larger than any of the other R's. The values of R which correspond to values of N which are commensurable with 3 but not divisible by 3 are also somewhat high, but, as was mentioned above, this was to be expected. The 3 was chosen rather than the 6, 9, or 12 on account of some facts which are to be mentioned in the conclusion.

TABLE I

N	$\frac{3}{4}$	1	$1\frac{1}{2}$	$1\frac{3}{4}$	2	$2\frac{1}{4}$	$2\frac{1}{2}$	$2\frac{3}{4}$	3	$3\frac{1}{4}$
R	0.97	1.29	1.14	0.85	1.39	0.94	0.93	1.00	1.48	0.97

N	$3\frac{1}{2}$	$3\frac{3}{4}$	4	$4\frac{1}{2}$	5	6	7	8	9	10	12
R	0.78	1.13	1.33	1.19	1.19	1.58	.98	1.28	1.57	1.26	1.68

The small number of measurements in the set M made it advisable to try the above process on a set of 286 telephone numbers. The last two digits of each number were pointed off for decimal places, and the other work was done in the same way as before. The results appear in Table II. We notice that the values of R do not differ from 1 by nearly so much as was the case in the previous work. The values of R are, in this case, much more nearly what we should expect from a set of random numbers.

TABLE II

N	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$	5
R	0.95	0.89	1.06	0.92	0.98	1.07	1.28	0.82	0.80

N	6	7	8	9	10	12
R	0.88	1.12	1.06	0.94	0.84	1.01

As a check of the validity of the process used, the same method was used on a

set of 286 measurements of door and window openings, cut stones, steps, and many other objects about the campus at Tulane. All of these last measurements were made in centimeters without observing the equivalent measurement in feet and inches at all. The objects used were, it is true, objects which probably had to be measured in some way during their construction, but there was never any intentional selection of objects whose dimensions would be likely to give favorable results. Furthermore, the plans used in computing Table I contained many measurements of similar objects. In Table III, the value of n was $\frac{1}{4}$ cm. This table shows a very marked tendency to emphasize the multiples of 1 inch and of 2 inches in our modern construction. The values 2.54 and 5.08 of N gave values of R which are of the same order of size as those found in Table I for $N=3$ and its multiples. In Table III, however, the other values of N which were used were, to all practical purposes, "not commensurable" with 2.54 and hence there were no other very high values of R occurring in the table.

TABLE III

N	1	$1\frac{1}{2}$	2	2.54	3	$3\frac{1}{2}$	4	5	5.08
R	0.96	1.04	0.98	1.57	0.98	1.03	0.86	1.15	1.81
N	6	7	8	9	10				
R	0.75	1.03	0.54	0.88	1.19				

In conclusion, it should be mentioned that these Mayan buildings from which Table I was computed showed frequent repetition of 5′0″. Since this is 20 times 3″, and since the Mayan number system used the base 20, it seems quite likely that there was a large unit of 60″ as well as the smaller unit. If this is true, then it is clear that our 3″ estimate could hardly be in error by as much as 1/20 in. Our throwing away all duplicate measurements undoubtedly weakened the results, since many multiples of 3″ were repeated again and again in the plans. Moreover, we could have gotten larger values of R than we did by using the value 1/8 instead of the value $\frac{1}{4}$ for n, as there were relatively few of the measurements which involved eighths. The method worked fairly well in spite of our having chosen the least favorable conditions. The 3″ unit seems more likely than the 6″, 9″, or 12″ unit, first because of its more convenient size, and second because it would be rather hard to explain the very high value of R for $N=3$ on any other grounds. It is not likely that halves and thirds of a unit would be as popular among workmen as whole numbers would be. There was quite frequent repetition, however, of the measurement $\frac{3}{4}$″. This did not result in a correspondingly high value of R for that value of N because hardly any other odd multiples of $\frac{3}{4}$″ occurred at all and duplicates were not counted.

THE ESTIMATION OF THE TOTAL FISH POPULATION OF A LAKE

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The purpose of this note is to discuss and compare as to fundamental assumptions four different methods which have been developed for the estimation of the fish population of a given lake from a sample census. Although the solutions outlined are adaptations of well known statistical procedures, it has seemed worthwhile to point out their application to this particular problem of estimation in the hope that the general methods may prove useful in other types of population studies.

In order to obtain the sample fish census, which may or may not be confined to a particular species, seining stations are established at various points on the given lake, samples taken at periodic intervals (usually about 24 hours), the fish tagged and replaced. Records are kept for each sample of the total number, the number of new fish, and the number of recaptures. If it be assumed that each sample be drawn at random and that the population of the lake remain constant throughout the period under consideration, it is possible to obtain an estimate of the total number of fish in the lake from the data of the census.

If N denote the total population of the lake, M_i the number of tagged fish in the lake when the ith sample is drawn, t_i the total number, r_i the number of recaptures, and d_i the number of untagged fish in the ith sample, then M_i/N is the probability of drawing a marked fish in a single trial in the ith sample, and, by the binomial distribution,

$$p_i = C_{r_i}^{t_i}\left(\frac{M_i}{N}\right)^{r_i}\left(1-\frac{M_i}{N}\right)^{d_i}$$

is the probability of drawing r_i recaptures and d_i new fish in the ith sample. It follows that $P=\prod_{i=1}^{n}p_i$ is the probability of obtaining the exact figures that were obtained in a census of n samples. The method of maximum likelihood,* which has been applied to this expression by Evelyn Hull and M. H. Ingraham, gives as an estimate of N the positive real root of the nth order equation

$$\sum_{i=1}^{n}\frac{d_iM_i}{N-M_i}=\sum_{i=1}^{n}r_i,$$

which can be expanded in the form

$$\sum_{i=1}^{n}\frac{d_iM_i}{N}\left(1+\frac{M_i}{N}+\cdots\right)=\sum_{i=1}^{n}r_i. \tag{1}$$

Approximations to the root N may be obtained by taking a finite number of terms in (1). If N_k be the positive root of the equation using k terms of the left

* Fisher, R. A., On the mathematical foundations of theoretical statistics, Philosophical Transactions of the Royal Society of London, series A, vol. 222, pp. 309–368.

member, then the sequence $\{N_k\}$ is a monotonic increasing sequence bounded above by N. The accuracy with which N_k approximates the root N can be estimated by calculating an upper bound of N which is dependent upon N_k.

The remainder after k terms of the series $\sum_{r=0}^{\infty}(M_i^r/N^r)$ is

$$R_k = \frac{M_i^{k-1}}{N^{k-1}} \frac{M_i}{N - M_i}.$$

Since $N_k < N$ and $M_i \leqq M_n$,

$$R_k < \frac{M_i^{k-1}}{N^{k-1}} \frac{M_i}{N_k - M_i} \leqq \frac{M_i^{k-1}}{N^{k-1}} \frac{M_n}{N_k - M_n}$$

provided k be chosen large enough that $N_k > M_i$. Now (1) can be written in the form

$$(\sum r_i)N^k - (\sum d_iM_i)N^{k-1} - \cdots - \sum d_iM_i^k - N^{k-1}\sum d_iM_iR_k = 0,$$

and it follows that an upper bound for N is given by the positive root of

$$\text{(1a)} \quad (\sum r_i)N^k - (\sum d_iM_i)N^{k-1} - \cdots - \sum d_iM_i^k - \sum\left(\frac{d_iM_i^{k+1}}{N_k - M_i}\right) = 0,$$

and a greater upper bound by the positive root of

$$\text{(1b)} \quad (\sum r_i)N^k - (\sum d_iM_i)N^{k-1} - \cdots - \sum d_iM_i^k - \frac{M_n}{N_k - M_n}\sum d_iM_i^k = 0.$$

Since all terms except the last in each of these equations are involved in the equation from which N_k is obtained, very little additional computation is necessary in finding these bounds of N after N_k has been determined. The roots of (1a) and (1b) decrease as k increases so that the required root N of (1) can be fixed between satisfactorily narrow limits by taking k large enough. In actual practice, it is usually sufficient to find N_3 and the corresponding upper bound from (1b). The computational labor in obtaining approximations higher than N_3 is often prohibitive.

Certain special cases of (1) are of interest. If the equation be written in the form

$$\sum_{i=1}^{n} \frac{t_iM_i - r_iN}{N - M_i} = 0,$$

it is apparent that if M_i is negligible compared with N the root of (1) is approximately

$$\text{(2)} \qquad \frac{\sum_{i=1}^{n} t_iM_i}{R},$$

where $R=\sum_{i=1}^{n} r_i$. Furthermore, if $M_i = M$ for all i,

$$N = M\frac{\sum t_i}{\sum r_i}. \tag{3}$$

This formula is applicable to the data of experiments in which the number tagged is held constant after a certain point. The method has the disadvantage that the data taken before M becomes constant are not utilized.

A second method suggested to me by M. H. Ingraham can be applied when M_i/N is small and r_i may be assumed to follow the Poisson law of distribution. In such cases,

$$P = \prod_{i=1}^{n} \frac{\left(\frac{M_i t_i}{N}\right)^{r_i} e^{-M_i t_i/N}}{r_i!}$$

is the probability of obtaining r_i recaptures in a total of t_i for all i. We find that P is a maximum when

$$N = \frac{\sum_{i=1}^{n} t_i M_i}{R},$$

a result which agrees with (2). It is interesting that this formula can also be derived from a simple consideration of the expected number of recaptures in the ith sample,

$$\epsilon(r_i) = \frac{t_i M_i}{N}, \qquad (i = 1, 2, \cdots, n).$$

The expected number in n samples is

$$\sum_{i=1}^{n} \epsilon(r_i) = \sum_{i=1}^{n} \frac{t_i M_i}{N},$$

and, if $\epsilon(r_i)$ is replaced by r_i, formula (2) results.

A fourth estimate of N can be obtained by minimizing χ^2, the Pearson measure of goodness of fit.* By definition,

$$\chi^2 = \sum_{i=1}^{n} \frac{(m_i' - m_i)^2}{m_i}$$

where m_i' is the observed and m_i the expected frequency in the ith sample. It is evident that χ^2 is a measure of the agreement between observation and theory. The expected number of recaptures in the ith sample is $t_i M_i/N$ so that, in this case,

* Pearson, Karl, Philosophical Magazine, 5th series, vol. 50, 1900, pp. 157–175.

$$\chi^2 = \sum_{i=1}^{n} \frac{\left(\frac{t_i M_i}{N} - r_i\right)^2}{\frac{M_i t_i}{N}}.$$

This is a minimum when

$$N = \left[\frac{\sum_{i=1}^{n} M_i t_i}{\sum_{i=1}^{n} \left(\frac{r_i^2}{M_i t_i}\right)} \right]^{1/2}. \tag{4}$$

Since the χ^2 test is based on the assumption of fairly large cell frequencies, this method will not give reliable results if r_i is small.

The formulas developed here have been applied to various types of data with quite consistent results. We shall examine in particular three sets of data which exhibit markedly different characteristics. Census I concerns the number of black crappies in a large lake (5,600 acres) and is taken from unpublished data by D. H. Thompson of the Illinois State Natural History Survey. In Census II, which was made by C. Juday and C. L. Schloemer on a lake of about 95 acres, all species of fish were considered. Census III represents the results of a bean-drawing experiment conducted by the author. The main features of these three groups of data will be apparent from the following table:

Census Number	I	II	III
Number of Samples	79	39	14
Total in All Samples	15,284	10,033	2,055
Total Recaptures	81	2,176	767
Total Tagged	10,986	4,567	1,288

The data of Census III were such that the value of χ^2 based on the deviations of the observed from the expected number of recaptures would be exceeded 80% of the time in random drawings. Therefore, the data could reasonably be assumed to fulfill the hypothesis of random sampling.

When the methods of this paper were applied to these sets of data, the following results were obtained:

Formula	Census I	Census II	Census III*
1	$N_3 = 1,300,542$	$N_3 = 15,084$	$N_5 = 1862$
1a		$N < 15,345$	$N < 1915$
1b	$N < 1,300,543$	$N < 15,347$	$N < 1932$
2	1,300,276	15,307	1895
3	1,252,822	17,579	1963
4	734,755	14,093	1886

* The actual number of beans was 1872.

It is seen that N_3 is an excellent approximation to the root N of (1) for Census I and that the results of (1) and (2) agree closely in this case; this is due to the fact that M_i/N is small. N_3 and the corresponding bound of N computed from (1b) prove sufficient for the data of Census II. The more refined methods are necessary for Census III, since drawing was continued until M_i/N exceeded 0.6. Formula (3) can be applied to only a small part of the data in each set (19 samples in Census I and 3 samples in Census II) so that the results do not agree closely with those of other methods. The low results of formula (4) when applied to I and II are due to the fact that the hypothesis of a fairly large number of recaptures in comparison with the total is not satisfied. It is interesting that the deviations of the estimates of N from the true value in Census III, which most nearly satisfies the hypotheses under which the formulas were developed, range from 0.55% to 5% of the total.

Since the assumptions of random sampling and constant population are only rough approximations to the actual situation in taking a sample fish census, small differences between the results of the various methods are not important. The maximum likelihood solution has certain theoretical advantages and is applicable to a wide variety of data, but the other formulas give useful estimates when the data are such as to warrant their use. It should be emphasized, however, that none of the solutions can be expected to provide more than an estimate of the general order of magnitude of the total population.

CONTINUED FRACTIONS AND MODIFIED CONTINUED FRACTIONS FOR CERTAIN SERIES

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1. Introduction. Important in the study of series whose general terms are rational functions of the index is that particular series of this sort whose terms are the reciprocals of positive integral powers of the terms of an arithmetical progression, $a, a+d, a+2d, \cdots, a+nd, \cdots$, the series $\sum_{n=0}^{\infty}(a+nd)^{-t}$ and the alternating series $\sum_{n=0}^{\infty}(-1)^n(a+nd)^{-t}$, where t is a positive integer [1].* Convergent series of this type converge but slowly, and the problem of increasing their rate of convergence is one which has interested mathematicians for a century. The method for transforming series set forth by Kummer in 1837 [2], now commonly known by his name, has been the subject of numerous memoirs since that time [*e.g.*, 3, 4].

In a number of articles written by Glaisher about 1903 [5], he was concerned with this same problem of improving the convergence of certain series, but the method that he used was quite different. By adding to a given series term by term another series whose sum was known he obtained an extensive collection of equivalent series. Even as recently as 1933 the problem has engaged the attention of Shohat [6].

* See references at the end of the paper.

In 1912 the present author, following the method suggested by Glaisher, showed that a sequence of more and more rapidly convergent series could be found for $\pi/4$, whose first terms approached the form of an infinite product [7]. The attempt to extend this result to more general series has led to certain continued fractions and to certain other forms which may be called modified continued fractions. The latter results appear as infinite continued fractions having an obvious and simple law of formation, but with this modification, that if it is desired to break off at any point, the last partial fraction is replaced by a different one.

2. The alternating series. We first consider the alternating series $\sum_{n=0}^{\infty}(-1)^n(a+nd)^{-t}$, a and d real, t a positive integer. To put this in simpler form we take out the factor $(2/d)^t$ and replace $2a/d$ by x. The series becomes $(2/d)^t\sum_{n=0}^{\infty}(-1)^n(2n+x)^{-t}$ and the problem is reduced to the consideration of the series $\sum_{n=0}^{\infty}(-1)^n(2n+x)^{-t}$, a function of the variable x, which we denote by $\phi_t(x)$. We shall assume for convenience that x lies in the interval $2\leqq x\leqq 4$. This will not prove a limitation, since any series of this form, by the addition or subtraction of a finite number of terms, may be made to depend on the value of $\phi_t(x)$ in this interval. Thus the series $1-2^{-t}+3^{-t}-\cdots$ and the series $1-3^{-t}+5^{-t}-\cdots$ appear, respectively, as $2^t\phi_t(2)$ and $1-\phi_t(3)$.

From the series

$$\phi_t(x) = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+x)^t} \tag{1}$$

we subtract term by term the series

$$\sum_{n=0}^{\infty}(-1)^n\left[\frac{g(2n+x-1)}{2f(2n+x-1)}+\frac{g(2n+x+1)}{2f(2n+x+1)}\right]=\frac{g(x-1)}{2f(x-1)}, \tag{2}$$

where $f(x)$ and $g(x)$ are polynomials of degree r and $r-t$, respectively, each containing either only odd powers or only even powers of x:

$$\begin{cases} f(x) = x^r + p_2x^{r-2} + p_4x^{r-4} + \cdots \begin{cases} +p_r, & \text{if } r \text{ is even,} \\ +p_{r-1}x, & \text{if } r \text{ is odd,} \end{cases} \\ g(x) = x^{r-t} + q_2x^{r-t-2} + q_4x^{r-t-4} + \cdots \begin{cases} +q_{r-t}, & \text{if } r-t \text{ is even,} \\ +q_{r-t-1}x, & \text{if } r-t \text{ is odd.} \end{cases} \end{cases} \tag{3}$$

The coefficients $p_2, p_4, \cdots$ and $q_2, q_4, \cdots$ are to be chosen so as to reduce as far as possible the degree in n of the resulting numerator,

$$\begin{aligned} F_t(2n+x) = {} & 2f(2n+x-1)f(2n+x+1) \\ & -(2n+x)^t[g(2n+x-1)f(2n+x+1) \\ & \qquad + g(2n+x+1)f(2n+x-1)]. \end{aligned} \tag{4}$$

Replacing $2n+x$ by s, it is readily seen that the coefficient of s^{2r} in

$$F_t(s) = 2f(s-1)f(s+1) - s^t[g(s-1)f(s+1) + g(s+1)f(s-1)] \tag{5}$$

vanishes identically, and that since $F_t(-s) \equiv F_t(s)$, $F_t(s)$ contains only even powers of s.

By equating to zero in succession the coefficients of s^{2r-2}, s^{2r-4}, s^{2r-6}, $\cdots$ in $F_t(s)$ the following equations are obtained, involving the so-called Eulerian numbers, which may be defined by the symbolic equation [8, pp. 25 and 458]

$$(E+1)^\nu + (E-1)^\nu = \begin{cases} 0, & \nu > 0, \\ 2, & \nu = 0, \end{cases} \qquad E_0 = 1.$$

The Eulerian numbers of odd index are zero, the first few of even index are $E_2 = -1$, $E_4 = 5$, $E_6 = -61$, $E_8 = 1{,}385$, $E_{10} = -50{,}521$, $E_{12} = 2{,}702{,}765$, $E_{14} = -199{,}360{,}981$, $E_{16} = 19{,}391{,}512{,}145$. Early calculation of these numbers was extended by Glaisher [9] as far as E_{54} and by Joffe to E_{100} [10].

$$(6) \qquad \begin{cases} p_2 - q_2 = -\binom{t+1}{2} E_2, \\ p_4 - q_4 = -\binom{t+1}{2} E_2 p_2 - \binom{t+3}{4} E_4, \\ p_6 - q_6 = -\binom{t+1}{2} E_2 p_4 - \binom{t+3}{4} E_4 p_2 - \binom{t+5}{6} E_6, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

Since these equations are independent of r they may be considered as continuing indefinitely. From this point on we shall have to treat separately the different cases for t.

3. The series $\phi_1(x)$. This function is $\frac{1}{2}\beta(x/2)$, where $\beta(x)$ is the so-called "small Beta function" [11].* We shall treat the case for which r is even; that for which r is odd may be handled in a similar fashion and the results for both cases can be combined in a single formula.

The $r-1$ equations obtained by equating to zero the coefficients of $s^{2r-2}, \cdots, s^2$ in $F_1(s)$ now assume for $F_1(s)$ the form

$$(7) \qquad \begin{cases} p_2 - q_2 = -E_2, \\ p_4 - q_4 = -E_2 p_2 - E_4, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ p_{r-2} - q_{r-2} = -E_2 p_{r-4} - E_4 p_{r-6} - \cdots - E_{r-2}, \\ p_r = -E_2 p_{r-2} - E_4 p_{r-4} - \cdots - E_r, \\ 0 = -E_2 p_r - E_4 p_{r-2} - \cdots - E_{r+2}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ 0 = -E_{r-2} p_r - E_r p_{r-2} - \cdots - E_{2r-2}. \end{cases}$$

* But practice is not uniform, Nörlund calls this function $\frac{1}{2}g(x)$ [8, p. 100].

If the coefficients $p_2, p_4, \cdots, p_r$ are chosen to satisfy the last $\frac{1}{2}r$ of these equations and the coefficients $q_2, q_4, \cdots, q_{r-2}$ to satisfy the remaining $\frac{1}{2}r-1$, $F_1(s)$ will be independent of s and hence equal to $2f(-1)f(1)$. This gives the result

$$(8)\quad \phi_1(x) - \frac{g(x-1)}{2f(x-1)} = \sum_{n=0}^{\infty}(-1)^n \frac{f(-1)f(1)}{(2n+x)f(2n+x-1)f(2n+x+1)}.$$

The following table gives the results of this calculation for $r \leqq 8$:

r	1	2	3	4	5	6	7	8
p_2		1	5	14	30	55	91	140
q_2			4	13	29	54	90	139
p_4				9	89	439	1,519	4,214
q_4					64	389	1,433	4,079
p_6						225	3,429	24,940
q_6							2,304	21,365
p_8								11,025

Thus, for example, using $r=4$ we have

$$\left\{\begin{aligned}\phi_1(x) &= \frac{(x-1)^3+13(x-1)}{2[(x-1)^4+14(x-1)^2+9]} \\ &\quad + \sum_{n=0}^{\infty}(-1)^n \frac{(4!)^2}{(2n+x)[(2n+x-1)^4+14(2n+x-1)+9]} \\ &\qquad \cdot[(2n+x+1)^4+14(2n+x+1)+9],\end{aligned}\right.$$

so that our original series is replaced by one whose general term has a constant numerator, while the denominator is a polynomial of ninth degree in n.

4. Rational approximations. It may be observed* that the polynomials $f(x)$ take on remarkable values for the odd integers 1, 3, 5, $\cdots$. Thus $f(1)=r!$, $f(3)=(2r+1)r!$, $f(5)=(2r^2+2r+1)r!$, $f(7)=1/3(4r^3+6r^2+8r+3)r!$, and $f(9)=1/3(2r^4+4r^3+10r^2+8r+3)r!$

From this it appears that the external term $\frac{1}{2}g(x-1)/f(x-1)$, taken without the series, constitutes an approximation to the function $\phi_1(x)$ in the interval $2 \leqq x \leqq 4$. For the first term of the series has its largest value in this interval for $x=2$, and for this value it takes on the form $\frac{1}{2}f(-1)f(1)/[f(1)f(3)]=1/[2(2r+1)]$, which approaches zero as r increases indefinitely.

The sequence of these external terms for successive values of r are successive convergents of an infinite continued fraction, which in the notation adopted by Perron [12, p. 3], leads to the equation

$$(9)\quad 2\phi_1(x) = \frac{1}{|x-1|} + \frac{1}{|x-1|} + \frac{4}{|x-1|} + \frac{9}{|x-1|} + \frac{16}{|x-1|} + \frac{25}{|x-1|} + \cdots.$$

* These conclusions, like most of the results in this article, have been obtained by induction: I have not yet obtained a rigorous proof.

This yields, for $x=2$,

$$\log 2 = 2\phi_1(2) = \cfrac{1}{\left|1\right.} + \cfrac{1}{\left|1\right.} + \cfrac{4}{\left|1\right.} + \cfrac{9}{\left|1\right.} + \cfrac{16}{\left|1\right.} + \cdots,$$

a relation known to Euler [**12,** p. 208], and, for $x=3$,

$$\frac{\pi}{4} = 1 - \phi_1(3) = 1 - \frac{1}{2}\left\{\cfrac{1}{\left|2\right.} + \cfrac{1}{\left|2\right.} + \cfrac{4}{\left|2\right.} + \cfrac{9}{\left|2\right.} + \cfrac{16}{\left|2\right.} + \cdots\right\}.$$

Sequences of rational functions which are better approximations to the function $\phi_1(x)$ may be found by combining with the external term one or more terms of the series in the sequence of equations (8). Using the first term only we have the approximation

$$\frac{g(x-1)}{2f(x-1)} + \frac{f(-1)f(1)}{xf(x-1)f(x+1)}.$$

However, since the first term in the series (8) was obtained by subtracting the first term of (2) from the first term of (1), this can be put in the simpler form

$$\frac{1}{x} - \frac{g(x+1)}{2f(x+1)}.$$

Setting aside the factor $1/2x$, this formula gives the sequence of rational fractions

$$\frac{x+2}{x+1}, \quad \frac{x^2+3x+4}{x^2+2x+2}, \quad \frac{x^3+4x^2+11x+12}{x^3+3x^2+8x+6}, \cdots.$$

An obvious and very simple law of formation of these is revealed when they are turned into continued fractions; they are all represented by the form

$$\left[1, x, \frac{x}{2}, \frac{x}{3}, \cdots, \frac{x}{r-1}, \frac{x+r}{r}\right], \tag{10}$$

using again a notation of Perron [**12,** p. 27], in this case the symbol $[b_0, b_1, b_2, \cdots]$, for the continued fraction whose partial numerators are all 1, and whose partial denominators are $b_1, b_2 \cdots$. This suggests an infinite continued fraction, but with this significant modification, that if one wishes to break off at any particular point and obtain an approximation, the last partial denominator x/r is to be replaced by $(x+r)/r$. The conclusion is that $\phi_1(x)$ can be represented approximately by

$$\frac{1}{2x}\left[1, x, \frac{x}{2}, \frac{x}{3}, \cdots, \frac{x}{r-1}, \frac{x+r}{r}\right]. \tag{11}$$

Euler's continued fraction for $\pi/2$,

$$1+\frac{1|}{|1}+\frac{1\cdot 2|}{|1}+\frac{2\cdot 3|}{|1}+\frac{3\cdot 4|}{|1}+\cdots,$$

is equivalent to the suggested infinite continued fraction for $x=1$ [**12**, p. 213].

If in the same way two terms of the series are used, the formula to be employed is

$$\frac{1}{x}-\frac{1}{x+2}+\frac{g(x+3)}{2f(x+3)}.$$

After setting aside the factor $1/2x$ the resulting rational fractions can be comprehensively represented by the formula

$$(12)\qquad \left[1, x, \frac{x}{2}, \frac{x}{3}, \cdots, \frac{x}{r-1}, \frac{x-r}{r}, \frac{x+4r+2}{8r+4}\right],$$

suggesting the same infinite continued fraction, but with a double modification.

The use of three terms of the series leads to the approximate representation of $\phi_1(x)$ by

$$(13)\qquad \frac{1}{2x}\left[1, x, \frac{x}{2}, \frac{x}{3}, \cdots, \frac{x}{r-1}, \frac{x+r}{r}, -\frac{x+4r+6}{8r+4}, -\frac{(2r+1)x}{(2r+1)^2+1}\right].$$

5. Estimating the approximation. The process by which the continued fractions were arrived at suggests an immediate means of estimating the accuracy of the approximation. If for a particular r and a number k of terms of the series used the resulting continued fraction C.F.(r, k) is determined, we shall have

$$(14)\qquad \phi_1(x)=\frac{1}{2x}\,\text{C.F.}(r,k)+\sum_{n=k}^{\infty}(-1)^n\frac{f(-1)f(1)}{(2n+x)f(2n+x-1)f(2n+x+1)}.$$

The error committed by using only the C.F. term is not greater than the first term of the series, that is

$$\frac{f(-1)f(1)}{(2k+x)f(2k+x-1)f(2k+x+1)}.$$

If x be limited to the range that we have been considering, $2\leqq x\leqq 4$, this has its largest value for $x=2$, *viz.*,

$$\frac{f(-1)f(1)}{(2k+2)f(2k+1)f(2k+3)}.$$

For example, if we take the largest values for which we have made the calculations, $r=8$ and $k=3$, the upper bound for the error is

$$\frac{f(-1)f(1)}{8f(7)f(9)}=\frac{1}{8(4r^3+6r^2+8r+3)(2r^4+4r^3+10r^2+8r+3)/9},$$

or about 4 in the eighth decimal place.

6. The series $\phi_2(x)$. This series $\sum_{n=0}^{\infty}(-1)^n(2n+x)^{-2}$ is the negative derivative of $\phi_1(x)$ with respect to x, that is $\phi_2(x) = -\frac{1}{4}\beta'(x/2)$. In this case r odd yields nothing, for then $f(x)$ and $g(x)$ have a common factor which can be removed.

The equations by means of which we calculate the coefficients are now

$$\begin{aligned}
p_2 - q_2 &= -3E_2, \\
p_4 - q_4 &= -3E_2p_2 - 5E_4, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
p_{r-2} - q_{r-2} &= -3E_2p_{r-4} - 5E_4p_{r-6} - \cdots - (r-1)E_{r-2}, \\
p_r &= -3E_2p_{r-2} - 5E_4p_{r-4} - \cdots - (r+1)E_r, \\
0 &= -3E_2p_r - 5E_4p_{r-2} - \cdots - (r+3)E_{r+2}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
0 &= -3E_{r-2}p_r - 5E_rp_{r-2} - \cdots - (2r-1)E_{2r-2}.
\end{aligned}$$

The coefficients of $s^{2r-2}, \cdots, s^2$ in $F_2(s)$ are thus made to vanish, leaving only the constant term. The resulting formula is

$$\phi_2(x) = \frac{g(x-1)}{2f(x-1)} + \sum_{n=0}^{\infty}(-1)^n \frac{f(-1)f(1)}{(2n+x)^2 f(2n+x-1)f(2n+x+1)}. \tag{15}$$

The following table gives the result of this calculation:

r	2	4	6	8	10
p_2	3	22	73	172	335
q_2		19	70	169	332
p_4		41	907	6,838	30,778
q_4			713	6,347	29,798
p_6			1,323	62,604	878,190
q_6				45,963	793,804
p_8				77,841	6,567,221
q_8					4,571,521
p_{10}					7,269,075

7. Rational approximations of $\phi_2(x)$. Since the first term of the series for $x=2$ is $f(-1)f(1)/[2^2f(1)f(3)]$ and $f(3)=(r+1)f(1)$, it again appears that the external term furnishes an approximation to $\phi_2(x)$ in the interval $2 \leqq x \leqq 4$. The continued fraction of which the successive external terms constitute the successive convergents is

$$\begin{aligned}
2\phi_2(x) = \cfrac{1}{(x-1)^2+3} + \cfrac{-16}{(x-1)^2+19} \\
+ \cfrac{-256}{(x-1)^2+51} + \cfrac{-1,296}{(x-1)^2+99} + \cdots.
\end{aligned} \tag{16}$$

Combining with the external term just the first term of the series that occurs

in (14), we obtain a sequence of rational functions which better approximate $\phi_2(x)$. The general form of these is

$$\frac{1}{x^2} - \frac{g(x+1)}{2f(x+1)},$$

and if, after setting aside the factor $1/2x^2$, they are expressed as continued fractions, they reveal a simple law of formation. We reach the conclusion that $\phi_2(x)$ may be represented approximately by

$$\frac{1}{2x^2}\left[1, \frac{x}{2}, \frac{x}{2}, \frac{x}{4}, \frac{x}{4}, \cdots, \frac{x}{r-2}, \frac{x}{r-2}, \frac{x}{r}, \frac{(x+r)}{r}\right] \tag{17}$$

which suggests an infinite continued fraction, modified when an approximation is desired.

If two terms of the series are used, it is necessary to compute

$$\frac{1}{x^2} - \frac{1}{(x+2)^2} + \frac{g(x+3)}{2f(x+3)}.$$

This yields a new modification of the same infinite continued fraction, that is to say

$$\frac{1}{2x^2}\left[1, \frac{x}{2}, \frac{x}{2}, \cdots, \frac{x}{r}, \frac{(x-r)}{r}, \frac{x+4r+3}{8r+8}, (8r+8)(x+1)\right] \tag{18}$$

is a still better approximation to $\phi_2(x)$.

8. The series $\phi_t(x)$, $t>2$. For the larger values of t the results are not so simple, since there are not enough coefficients $p_2, \cdots$ and $q_2, \cdots$ to enable us to make the numerator independent of n in the new series. It is a question whether the rational approximations obtained are simple enough to be valuable.

9. The non-alternating series. A series of integral powers of the reciprocals of the terms of an arithmetical progression, not involving the alternating factor $(-1)^n$, can be handled in similar fashion if the exponent $t \geqq 2$. Since

$$\sum_{n=0}^{\infty} \frac{1}{(a+nd)^t} = \left(\frac{2}{d}\right)^t \sum_{n=0}^{\infty} \frac{1}{(2n+x)^t},$$

where $x = 2a/d$, the problem is to investigate the function

$$\psi_t(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+x)^t}, \qquad t \geqq 2. \tag{19}$$

From this series we subtract term by term the series

$$\sum_{n=0}^{\infty}\left[\frac{g(2n+x-1)}{2(t-1)f(2n+x-1)} - \frac{g(2n+x+1)}{2(t-1)f(2n+x+1)}\right] = \frac{g(x-1)}{2(t-1)f(x-1)}, \tag{20}$$

in which $f(x)$ and $g(x)$ denote polynomials of degree r and $r-t+1$, respectively, each containing either only odd powers or only even powers of x;

$$\text{(21a)} \qquad f(x) = x^r + p_2x^{r-2} + p_4x^{r-4} + \cdots \begin{cases} + p_r, & \text{if } r \text{ is even,} \\ + p_{r-1}x, & \text{if } r \text{ is odd,} \end{cases}$$

$$\text{(21b)} \qquad g(x) = x^{r-t+1} + q_2x^{r-t-1} + \cdots \qquad \begin{cases} + q_{r-t+1}, & \text{if } r-t+1 \text{ is even,} \\ + q_{r-t}x, & \text{if } r-t+1 \text{ is odd.} \end{cases}$$

The coefficients are to be so chosen as to reduce as far as possible the degree of the numerator in n. Replacing $2n+x$ by s we find for this numerator,

$$\text{(22)} \qquad \begin{aligned} F_t(s) = {} & 2(t-1)f(s-1)f(s+1) \\ & - s^t[g(s-1)f(s+1) - g(s+1)f(s-1)], \end{aligned}$$

and the equations obtained by equating to zero the successive coefficients of powers of s are

$$\text{(23)} \qquad \begin{cases} p_2 - q_2 = -\binom{t}{2} D_2, \\ p_4 - q_4 = -\binom{t}{2} D_2p_2 - \binom{t+2}{4} D_4, \\ p_6 - q_6 = -\binom{t}{2} D_2p_4 - \binom{t+2}{4} D_4p_2 - \binom{t+4}{6} D_6, \\ \cdots\cdots\cdots\

r	2	3	4	5	6
p_2	1/3	7/5	26/7	70/9	155/11
q_2		16/15	71/21	67/9	454/33
p_4			27/35	407/63	329/11
q_4				4,096/945	4,237/165
p_6					375/77

11. Rational approximations to $\psi_2(x)$. It appears that the function $f(x)$ takes on for odd values of x the following values: $f(1)=2^r(r!)^3/(2r)!$, $f(3)=(r^2+r+1)f(1)$, $f(5)=\frac{1}{4}(r^4+2r^3+7r^2+6r+4)f(1)$, $\cdots$. From this we conclude that the external term furnishes an approximation to $\psi_2(x)$. These successive external terms, whose general form is $\frac{1}{2}g(x-1)/f(x-1)$, constitute the successive convergents of an infinite continued fraction. We have therefore a first approximation to $\psi_2(x)$ in the form

$$\frac{1}{2}\left\{\frac{1}{|x-1|}+\frac{1/3}{|x-1|}+\frac{16/15}{|x-1|}+\frac{81/35}{|x-1|}+\frac{256/63}{|x-1|}+\cdots\right\}. \tag{25}$$

A second approximation comes from combining with the external term the first term of the series in the sequence of equations (24). This requires the calculation of

$$\frac{1}{x^2}+\frac{g(x+1)}{2f(x+1)}$$

and its transformation into a continued fraction. The result is

$$\frac{1}{2x^2}\left[x+1,\ \frac{3x}{2},\ \frac{5x}{2\cdot 3},\ \frac{7x}{3\cdot 4},\ \cdots,\ \frac{(2r-1)x}{(r-1)r},\ \frac{x+r}{r}\right], \tag{26}$$

suggesting an infinite continued fraction modified to obtain an approximation. The same infinite continued fraction further modified appears as the result of combining two terms of the series with the external term; the result is

$$\left\{\frac{1}{2x^2}\left[x+1,\ \frac{3x}{2},\ \frac{5x}{2\cdot 3},\ \frac{7x}{3\cdot 4},\ \cdots,\ \frac{x+3r}{r},\ -\frac{x-(r^2+r-3)}{8},\ -\frac{8(x+r^2+r+1)}{(r^2+r+1)^2}\right]\right. . \tag{27}$$

12. The series $\psi_3(x)$. In applying the same process to the series $\sum_{n=0}^{\infty}(2n+x)^{-3}$, only even values of r can be used, since otherwise $f(x)$ and $g(x)$ would have a common factor which could be divided out. Here again all the coefficients of powers of s in $F_3(s)$ may be made to vanish, leaving just the constant term. The resulting sequence of series is given by

$$\psi_3(x)=\frac{g(x-1)}{4f(x-1)}+\sum_{n=0}^{\infty}\frac{f(-1)f(1)}{(2n+x)^3f(2n+x-1)f(2n+x+1)}. \tag{28}$$

The continued fraction furnished by the external term is

$$\frac{1}{4}\left\{\frac{1}{|(x-1)^2+1}+\frac{-4/3}{|(x-1)^2+5}+\frac{-256/15}{|(x-1)^2+13}+\cdots\right\}, \tag{29}$$

and the modified continued fraction when the first term of the series is combined with the external term is

$$\frac{1}{4x^3}\left\{x+2, \frac{x}{2}, \frac{3x}{2}, \frac{x}{4}, \frac{5x}{2\cdot 3}, \cdots, \frac{x}{r}, \frac{2(x+r)}{r}\right\}. \tag{30}$$

Whether the methods here employed are capable of further application or not, it is believed that the results set forth may throw light on the general problem of the approximate representation of functions by rational functions.

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APPLICATIONS OF THETA FUNCTIONS TO ARITHMETIC

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This paper deals with the application of the Jacobi theta functions to arithmetic. The first applications were made by Jacobi himself and since then by a long list of contributors. Further reference to these is made in the sequel.

The following discussion is written from an elementary point of view, since it is felt that the field deserves a wider acquaintance on the part of those possessing a mathematical background equivalent to the average undergraduate major. Either the results themselves or the proofs of old results are believed to be new.

The four Jacobi theta functions [1] are defined as follows:

$$\theta_0(x) = 1 + 2\sum_{n=1}^{\infty}(-1)^n q^{n^2}\cos 2nx = \sum_{n=-\infty}^{\infty}(-1)^n q^{n^2}e^{2nix},$$

$$\theta_1(x) = 2\sum_{n=0}^{\infty}(-1)^n q^{(2n+1)^2/4}\sin(2n+1)x = -i\sum_{n=-\infty}^{\infty}(-1)^n q^{(2n+1)^2/4}e^{(2n+1)ix},$$

$$\theta_2(x) = 2\sum_{n=1}^{\infty} q^{(2n+1)^2/4}\cos(2n+1)x = \sum_{n=-\infty}^{\infty} q^{(2n+1)^2/4}e^{(2n+1)ix},$$

$$\theta_3(x) = 1 + 2\sum_{n=1}^{\infty} q^{n^2}\cos 2nx = \sum_{n=-\infty}^{\infty} q^{n^2}e^{2nix},$$

where $q = e^{\pi i\tau}$, the imaginary part of τ being positive so as to insure convergence. These functions belong to a class named by Hermite [2] doubly periodic of the third kind, but we shall not be concerned here with their detailed properties.

There are in the literature a great many infinite series expansions for various products and quotients of the Jacobi theta functions together with their derivatives. These expansions proceed in powers of q and the coefficients are functions of the divisors of the exponent of q or, more generally, functions of the variables in some form representing the exponent of q. Arithmetical results are then obtained by equating like powers of q in an identity connecting such theta products and quotients.

We start with a simple illustrative example. The arithmetized expansion for $\theta_0'^2(x)/\theta_0^2(x)$, where the prime indicates differentiation, follows [3]:

$$\theta_0'^2(x)/\theta_0^2(x) = 4\sum q^n(\sum \delta) + 8\sum q^{n'}[\sum(\tau - t)\cos 2tx]. \tag{1}$$

The outer summation in the first term on the right is an infinite one extending over all positive, even integral values of n. The inner summation occurring as part of the coefficient of different powers of q is finite and extends over all positive, even integral divisors, δ, of n; thus the coefficient of q^6 in the first term is $4(2+6)=32$. In the second term the n' summation is infinite and is over all positive, integral values; while the inner sum is finite and extends over all positive, integral conjugate divisors t, τ of n', $(n'=t\tau)$, τ being restricted to odd val-

ues only. Thus the coefficient of q^6 is $8\,[(1-6)\cos 12x+(3-2)\cos 4x]$, which reduces to $-40\cos 12x+8\cos 4x$.

From the definition of $\theta_0(x)$ it is evident that $\theta_0'(0)=0$. Placing $x=0$ in (1) we obtain

$$\sum q^n(\sum \delta)+2\sum q^{n'}[\sum(\tau-t)]=0. \tag{2}$$

Since this is an identity in the q's, the coefficients of the different powers of q must be separately zero. Considering only even values of n' so that $n=n'$, we see that for any even number $\sum\delta+2\sum(\tau-t)=0$, which may be written $\sum(\delta+\tau)+\sum\tau=2\sum t$. This result may be interpreted as follows:

For any even number, the sum of all the divisors plus the sum of the odd divisors is equal to twice the sum of those divisors whose conjugates are odd.

We shall now give an example of the application of theta functions to the problem of the determination of the number of representations of any integer as a sum of squares. An extensive treatment has been given by Glaisher [4], and many other writers have written on the problem from various angles [5].

The distinctions between a representation, a composition, and a partition of a number as a sum of squares can perhaps best be made clear by an example. There are four representations of 4 as a sum of 2 squares, namely: 2^2+0^2, 0^2+2^2, $(-2)^2+0^2$, and $0^2+(-2)^2$. If positive numbers only had been used, the ways of expressing 4 as a sum of squares would be called compositions and there would be two of them. If the order is also neglected there would be but one way and it would be called a partition. If two or more of the numbers to be squared are alike, both in magnitude and sign, a different representation is not obtained by interchanging the positions of the two like numbers. Although the number of partitions or compositions would seem to be the more significant concept, the results can ordinarily be obtained more directly in terms of representations and are usually so given.

We start from the theta constant identity [6]:

$$\theta_2^4=\theta_0''/\theta_0-\theta_3''/\theta_3, \tag{3}$$

where $\theta_2=\theta_2(0)$, etc. If we take the exponential definition of $\theta_2(x)$ above, place $x=0$, and $2n+1=m$, an odd number, we have $\theta_2=\sum q^{m^2/4}$. Raising this equation to the fourth power gives

$$\theta_2^4=\sum\sum\sum\sum q^{(x^2+y^2+z^2+v^2)/4}, \tag{4}$$

and if we write this expression for θ_2^4 as a single summation by collecting the coefficients of like powers of q, thus

$$\theta_2^4=\sum A_n q^{n/4}, \tag{5}$$

it can be seen that A_n is the number of representations of n as a sum of four odd squares.

The expansions needed for the right side of (3) are [3]:

$$\theta_0'(x)/\theta_0(x) = 4\sum q^n(\sum \delta) + 8\sum q^{n'}(\sum \tau \cos 2tx),$$

$$\theta_3'(x)/\theta_3(x) = 4\sum q^n(\sum \delta) + 8\sum q^{n'}[\sum (-1)^t\tau \cos 2tx],$$

where the nature of the summations and values of n, n', δ, t, and τ are the same as in (1) above.

Putting $x=0$ in these expansions and substituting the results along with (5) in (3) gives

$$\sum A_n q^{n/4} = 8\sum q^n[\sum \tau(1-(-1)^t)]. \tag{6}$$

Note that if we are to equate like powers of q on the two sides of this equation, the n on the left must be taken equal to four times the value of the n on the right. Keeping this in mind and considering odd values of n only, so that n, t, τ, are all odd in $n=t\tau$ on the right, we may deduce from (6) a classical theorem of Jacobi [7]:

"The number of representations of the quadruple of any odd number, n, as a sum of four odd squares is equal to 16 times the sum of the divisors of n."

If n had been taken equal to an even number in (6), we could have concluded that there are no representations of the quadruple of any even number as the sum of 4 odd squares, a result which is otherwise fairly obvious.

If we now raise both sides of (6) to the mth power and rearrange by collecting the coefficients of like powers of q, there results

$$\sum B(4m, n)q^{n/4} = 8^m\sum q^N\sum \tau_1\tau_2\cdots\tau_m[1-(-1)^{t_1}][1-(-1)^{t_2}]\cdots[1-(-1)^{t_m}].$$

As can perhaps best be seen by referring to (4) and (5), $B(4m, n)$ is the number of representations of n as a sum of $4m$ odd squares. The inner summation on the right is the product of the original m finite summations and is over all combinations of t's and τ's which satisfy, for any particular N, the equation

$$N = n_1 + n_2 + \cdots + n_m = t_1\tau_1 + t_2\tau_2 + \cdots + t_m\tau_m.$$

Equating like powers of q, and remembering that $n=4N$, we get

$$B(4m, 4N) = 8\sum \tau_1\tau_2\cdots\tau_m[1-(-1)^{t_1}][1-(-1)^{t_2}]\cdots[1-(-1)^{t_m}].$$

If either N or m is odd while the other is even, $B(4m, 4N)$ is zero. This can be seen since the τ's are odd by assumption and all the t's must be odd in order for $B(4m, 4N)$ to be different from zero. When m and N are either both odd or even, however, there results

$$B(4m, 4N) = 16^m\sum \tau_1\tau_2\cdots\tau_m, \tag{7}$$

where in

$$N = t_1\tau_1 + t_2\tau_2 + \cdots + t_m\tau_m$$

all t's and τ's are odd.

It is possible by means of certain other theta function identities to deduce from (7) the classical theorem on the representation of the quadruple of any even number as the sum of eight odd squares [8], and certain other results for

12 and 16 odd squares, but we shall not take space here to develop this line of inquiry further.

Instead we shall take up the method of paraphrase [9] which adds a great deal to the generality of any results obtained. The functions of the divisors of n occurring as coefficients of q^n in the arithmetized series developments usually contain sine or cosine terms as in the above examples. Thus theorems concerning trigonometric functions are obtained by equating like powers of q. But, as is well known from the theory of Fourier series, the majority of analytic functions are, within certain limitations, representable by trigonometric series. Hence it is frequently possible by giving suitable particular values to the variables in these theorems and then summing up, to extend a theorem established for trigonometric functions to all functions which are representable by convergent series of the trigonometric function for which the theorem was first established. Thus, under certain conditions, a result concerning cosine terms can be extended to even functions, and a theorem concerning sine terms to odd functions.

An example will make these statements more definite. In a list of arithmetized expansions given by Bell [10] occurs

$$\theta_0'(x)/\theta_0(x) = 4\sum q^n(\sum \sin 2tx), \tag{8}$$

where in $n=t\tau$, t is an arbitrary positive integer, and τ is an odd positive integer. Throughout the sequel, τ will always represent a positive odd integer, and d, δ, t arbitrary positive integers unless otherwise indicated. We shall not point out where the latter are necessarily restricted, *e.g.*, t when n is odd in $n=t\tau$.

It is obvious that if we square (8) it may be equated to (1) above, thus obtaining

$$4\sum q^n(\sum \delta) + 8\sum q^{n'}\sum (\tau - t)\cos 2tx = 16\sum q^N(\sum \sin 2t_1x \sin 2t_2x),$$

where $N=n_1+n_2=t_1\tau_1+t_2\tau_2$. From trigonometry the right side can be written $8\sum q^N[\sum \cos 2(t_1-t_2)x - \cos 2(t_1+t_2)x]$. Equating like powers of q, there results

$$\sum_n \delta + 2\sum_{n'} (\tau' - t')\cos 2t'x = 2\sum_N [\cos 2(t_1 - t_2)x - \cos 2(t_1 + t_2)x], \tag{9}$$

where, to sum up, $n=d\delta$, $n'=t'\tau'$, $N=t_1\tau_1+t_2\tau_2$, and δ is given.

Let $f(y)$ be an even analytic function, well defined and finite everywhere between the limits $-h$ and h. Then $f(y)$ can be expanded in a Fourier series as follows:

$$f(y) = A_0/2 + A_1\cos \pi y/h + A_2\cos 2\pi y/h + \cdots + A_m\cos m\pi y/h + \cdots . \tag{10}$$

Next put $y=0$, $2t'$, $2(t_1-t_2)$, $2(t_1+t_2)$ successively in (10), and in (9) put x successively equal to π/h, $2\pi/h$, $\cdots$, $m\pi/h$, $\cdots$. Now multiply both sides of (9) by A_1 for $x=\pi/h$, by A_2 for $x=2\pi/h$, and in general by A_m for $x=m\pi/h$. If we add the resulting equalities and take into consideration the results of the substitution in (10), we have

$$\sum_n \delta f(0) + 2\sum_{n'} (\tau' - t')f(2t') = 2\sum_N f(2t_1 - 2t_2) - f(2t_1 + 2t_2). \tag{11}$$

(11) is called the paraphrase of (9) and is a much more general relationship. For a cosine series the paraphrase substitution is

$$\sum (a_i + b_i \cos c_i y) = \sum a_i f(0) + b_i f(c_i). \tag{12}$$

For an identity containing sine terms only it can be shown that the proper substitution is

$$\sum (a_i \sin b_i y) = \sum a_i F(b_i), \tag{13}$$

where $F(y)$ is an odd function.

If we place $f(y)$ equal to a constant in (11) we get the same result that was obtained from (2). If we place $f(y)=y^2$, we obtain

$$\sum_{n'} (t' - \tau')t'^2 = 4\sum_{N} t_1 t_2,$$

a result connecting the divisors of n' with two of the four variables in the form $N=t_1\tau_1+t_2\tau_2$. A check of this result for $N=n'=4$ gives $48=48$.

The even and odd functions considered above do not have to be continuous. It can be shown that it is only necessary for them to have definite, finite values for the integral values of the independent variable under consideration. For example, if in (11) we place $f(0)=1$, $f(y)=0$ for all other integral values of y, we may conclude that the sum of the even divisors of any number, n, is equal to twice the number of times in which $t_1=t_2$ in the compositions of n by means of $t_1\tau_1+t_2\tau_2$.

We give one further example for which we need the following expansion obtained by Basoco [11]:

$$\frac{\theta_1'^2\theta_0^2(x+y)}{\theta_0^2(x)\cdot\theta_1^2(y)} = \frac{2\theta_1'(y)}{\theta_1(y)}\left[\csc 2y + 4\sum q^n\left(\sum \sin 2(tx+\tau y)\right)\right] - 8\sum q^n \sum t \cos 2(tx+\tau y). \tag{14}$$

In the list of expansions given by Bell [5] occur:

$$\theta_1'^2\theta_0^2/\theta_0^2(x)\theta_1^2(x) = \csc^2 x - 8\sum q^{n_1}\left[\sum d_1 \cos 2(d_1-\delta_1)x\right], \tag{15}$$

$$\theta_1'(y)/\theta_1(y) = \cos y + 4\sum q^{n'}\left(\sum \sin 2d'y\right), \tag{16}$$

where δ' is even in (16) in $n'=d'\delta'$.

If we set $y=-x$ in (14) it is evident that it is identically equal to (15) and the two right-hand members may be equated. Terms free of q must be equal in such an identity. If we drop these terms out, multiply through by $\sin 2x$, use trigonometric formulas, and paraphrase by means of (13), there results after considerable detailed labor

$$\sum_{n'} 2F(2d') - \sum_{n}\{2F(2t-2\tau) + F(2t-2\tau+2) + F(2t-2\tau-2) + tF(2t-2\tau+2) - tF(2t-2\tau-2)\} + \sum_{n_1}\{d_1F(2d_1-2\delta_1+2) - d_1F(2d_1-2\delta_1-2)\} = R,$$

where

$$R = 2\sum_{N} \{F(2t - 2\tau - 2d' + 2) - F(2t - 2\tau - 2d' - 2) - F(2t - 2\tau + 2d' + 2) + F(2t - 2\tau + 2d' - 2)\},$$

and where $n' = d'\delta'$, δ' even, $n = t\tau$, $n_1 = d_1\delta_1$, $N = t\tau + d'\delta'$.

If we set $F(y) = y$, we obtain

$$\sum_{n'} d' - \sum_{n} (3t - 2\tau) + \sum_{n} d = 0,$$

a result which may be interpreted as follows:

For any even number, *the sum of the divisors whose conjugates are even plus the sum of all divisors is equal to three times the sum of all divisors whose conjugates are odd minus twice the sum of the odd divisors.*

These examples deal only with a few of the more elementary applications to number theory. Further applications may be found, among other places, in the works of Hermite, Jacobi, Legendre, Kronecker, Liouville, Humbert, and Glaisher. Particular attention is called to numerous applications made by E. T. Bell in papers occurring in many journals during the past twenty years, and to a book on the subject by Nazimoff [12].

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4. Glaisher, Quarterly Journal of Mathematics, vol. 36, pp. 305–359, vol. 37, pp. 36–49, vol. 38, pp. 1–63, 178–237, 289–352.

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6. J. D. Elder, Annals of Mathematics, vol. 31, 1930, p. 650.

7. Jacobi, Werke, vol. 1, p. 247.

8. H. J. S. Smith, Collected Mathematical Papers, vol. 1, p. 307.

9. An extensive and generalized account of the method is given by Bell (Transactions, vol. 22, 1921, pp. 1–30, 198–219). It is probable that Liouville was the first to use the method of paraphrase in obtaining certain results contained in a series of 18 memoirs appearing in the Journal de Mathématiques Pures et Appliquées, vols. 3, 4, 5, 9, and 10, although he did not give a proof or details as to how his results were obtained. In this connection see Hermite, Oeuvres, vol. II, pp. 109–124, also p. 220, and remarks by Bell in the Transactions article cited above.

10. E. T. Bell, Messenger of Mathematics, vol. 53, 1924, p. 166.

11. M. A. Basoco, Bulletin of the American Mathematical Society, vol. 38, 1932, pp. 560–68.

12. Nazimoff, Applications of the Theory of Elliptic Functions to the Theory of Numbers. There is now an English translation of the original which was published in Moscow in 1884. Résumé in Annales Scientifique de l'École Normale Supérieure, third series, vol. 5, 1888, p. 36.

ON CONGRUENCES INVOLVING SUMS OF PRODUCTS OF BINOMIAL COEFFICIENTS

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In some investigations the writer has been carrying out recently it has been found convenient to employ certain lemmas involving congruences of the type indicated in the title. These are generalizations of a result given by Glaisher* who found that

$$\text{(A)} \qquad \binom{a}{i} \equiv \binom{d}{i} + \binom{d}{i+(p-1)} + \binom{d}{i+2(p-1)} + \cdots, \pmod{p},$$

where $d \equiv a \pmod{\overline{p-1}}$; $0 < a \leqq p-1$; p is prime, $0 < i \leqq p-1$; and $\binom{v}{w} = 0$, when $w > v$.

We give here only the simplest of types of congruences we shall need elsewhere. There are extensive generalizations which we shall indicate how to obtain, without writing out the results explicitly, as some of the latter are quite complicated.

Let k and m be arbitrary integers and p a prime; then

$$\text{(1)} \qquad (m+k)^c[(m+k)^{p-1} - 1]^r \equiv 0 \pmod{p^r}, \qquad c \geqq r,$$

since if $(m+k)$ is prime to p, then $(m+k)^{p-1} \equiv 1 \pmod{p}$; and if $m+k \equiv 0 \pmod{p}$ then $(m+k)^c \equiv 0 \pmod{p^r}$, since $c \geqq r$. Expanding (1), we have

$$\text{(1a)} \qquad \sum_{h=0}^{r} (-1)^h \binom{r}{h} (m+k)^{h(p-1)+c} \equiv 0 \pmod{p^r}.$$

If we select k so that

$$\text{(2)} \qquad k^{p-1} \equiv 1 \pmod{p^r},$$

we may employ this in (1a), collect coefficients of powers of k, and obtain

$$\text{(3)} \qquad \sum_{h=0}^{r} \sum_{\alpha=0}^{p-2} \sum_{l=0,1,\cdots} (-1)^h \binom{r}{h} \binom{(p-1)h+c}{(p-1)l+\alpha} k^{\beta} m^{l(p-1)+\alpha} \equiv 0 \pmod{p^r},$$

where β is the integer in the set $0, 1, \cdots, p-2$, such that

$$\beta \equiv c - \alpha \pmod{\overline{p-1}}.$$

Hence as α ranges over the integers $0, 1, \cdots, p-2$, then β ranges over the same set in some order. Consequently (3) may be written in the form

$$\text{(4)} \qquad A_0 + A_1 k + \cdots + A_{p-2} k^{p-2} \equiv 0 \pmod{p^r}.$$

We may now select for k in (2) an integer k_a which also has the property that $k_a \equiv a \pmod{p}$; $0 < a < p$, by known methods. In (4) set $k = k_1, k_2, \cdots, k_{p-1}$ in

* Quarterly Journal of Mathematics, vol. 30, 1899, pp. 361–366. This was referred to by Dickson as a very interesting result.

turn, then the determinant of the coefficients of the A's in the resulting congruences is the alternant

$$\begin{vmatrix} 1 & k_1 & \cdots & k_1^{p-2} \\ 1 & k_2 & \cdots & k_2^{p-2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ 1 & k_{p-1} & \cdots & k_{p-1}^{p-2} \end{vmatrix} = \prod_{i,j} (k_i - k_j), \qquad i \neq j,$$

which is prime to p. Hence

$$A_0 \equiv A_1 \equiv \cdots \equiv A_{p-2} \equiv 0 \pmod{p^r};$$

or, using the values of the A's as obtained from (3) and dividing by m^α, assuming m prime to p, we find

$$(5) \qquad \sum_{h=0}^{r} \sum_{l=0,1,\cdots} m^{l(p-1)}(-1)^h \binom{r}{h} \binom{(p-1)h + c}{(p-1)l + \alpha} \equiv 0 \pmod{p^r}.$$

Now select m so that

$$m^{p-1} = m_g^{p-1} \equiv 1 + gp \pmod{p^r},$$

where g is a given integer, which is always possible; then

$$(6) \qquad m_g^{(p-1)l} = (1 + gp)^l.$$

Employing the expansion of (6) in (5) the result may be written in the form

$$B_0 + B_1 gp + \cdots + B_{p-2}(gp)^{p-2} \equiv 0 \pmod{p^r}.$$

Now assume $g = 1, 2, \cdots, p-2$, in turn; the above gives a set of congruences in which the determinant of the coefficients of $B_0, B_1 p, \cdots, B_{p-2} p^{p-2}$ is an alternant formed by powers of the g's which is prime to p since each $g < p$. Hence

$$B_i p^i \equiv 0 \pmod{p^r}, \qquad i = 0, 1, \cdots, p-2;$$

or using the values of the B's from (5) we have for $i < p-1$,

$$\sum_{h=0}^{r} \sum_{l=0,1,\cdots} (-1)^h p^i \binom{l}{i} \binom{r}{h} \binom{(p-1)h + c}{(p-1)l + \alpha} \equiv 0 \pmod{p^r},$$

which gives in particular for $i = 0$,

$$\sum_{h=0}^{r} \sum_{l=0,1,\cdots} (-1)^h \binom{r}{h} \binom{(p-1)h + c}{(p-1)l + \alpha} \equiv 0 \pmod{p^r}.$$

Hence we have the

THEOREM. *Let p be prime, α any integer in the set $0, 1, \cdots, p-2$; $c \geqq r$; $i < p-1$; $r > i$; then*

$$(7)\qquad \sum_{h=0}^{r}\sum_{l=0,1,\cdots}(-1)^h\binom{r}{h}\binom{(p-1)h+c}{(p-1)l+\alpha}\binom{l}{i}\equiv 0 \pmod{p^{r-i}}.$$

We now consider some generalizations. For example, instead of using (1) we can employ

$$(m+k)^c[(m+k)^{d(p-1)}-1]^r\equiv 0\ [\text{mod}\ (p^{rs},p^c)],$$

where $d=p^{s-1}d_1$; d_1 prime to p, where (x, y) is the greatest common divisor of x and y. This gives

$$\sum_{h=0}^{r}(-1)^h\binom{r}{h}(m+k)^{hd(p-1)+c}\equiv 0\ [\text{mod}\ (p^{rs},p^c)].$$

We may then select k so that

$$k^{p-1}\equiv 1 \pmod{p^{rs}},$$

which is an extension of (2), but instead we might select k so that

$$k^{p-1}\equiv 1+jp \pmod{p^{rs}},$$

where j is an arbitrary integer, and proceed as we did in the treatment following (6) with k in place of m. Then later we may treat m as we did in (6) and the relation which precedes. This will give in lieu of (7) a congruence which involves in general, in each term, the product of four binomial coefficients. For the particular case when $r=s=1$ we obtain Glaisher's relation (A).

A further generalization is obtained by considering

$$(m_1+m_2+\cdots+m_a)^c[(m_1+m_2+\cdots+m_a)^{d(p-1)}-1]^r\equiv 0\ [\text{mod}\ (p^{rs},p^c)],$$

and expanding the left-hand member by the multinomial and binomial expansions. For $r=s=1$ this reduces to a result concerning multinomial coefficients proved by Dickson.* In another paper† the writer obtained the result that

$$\binom{a}{i}\equiv\binom{d}{i}+\binom{d}{i+p^n-1}+\binom{d}{i+2(p^n-1)}+\cdots$$

modulo p, when $d\equiv a \pmod{p^n-1}$; $0<a\leqq p^n-1$; $0<i\leqq p^n-1$; which is a different type of generalization of (1). This was proved by the use of Galois (finite) field theory. To obtain results of this kind modulo p^α apparently it is necessary to employ finite rings instead of finite fields. We may start with the relatively simple type of finite ring formed by the residue classes with respect to the modulus $\mathfrak{p}^\alpha$ where $\mathfrak{p}$ is a prime ideal divisor of (p) in an algebraic field K, note that

$$(\rho_1+\rho_2+\cdots+\rho_a)^c[(\rho_1+\rho_2+\cdots+\rho_a)^{d(p^n-1)}-1]^r\equiv 0\ [\text{mod}\ (\mathfrak{p}^{rs},\mathfrak{p}^c)]$$

where now $d=p^{n(s-1)}d_1$; $\rho_1, \rho_2, \cdots, \rho_a$ are arbitrary integers in K, and then proceed as in the treatment of (1a).

* Quarterly Journal of Mathematics, vol. 33, 1902, pp. 381–384.

† Annals of Mathematics, vol. 28, 1927, p. 332.

QUESTIONS, DISCUSSIONS, AND NOTES

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The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

REMARKS ON PENCILS OF CIRCLES AND NETS OF SPHERES

N. A. COURT, University of Oklahoma

1. If the points P and P', Q and Q', R and R' divide the sides BC, CA, AB of the triangle ABC externally and internally in ratios such that the product of the three ratios is unity, the points P, Q, R are collinear, by Menelaus's theorem, and the points of division considered are three pairs of opposite vertices of a complete quadrilateral of which ABC is the diagonal triangle.

It is therefore possible to describe three circles (A), (B), (C) having A, B, C for centers so that the centers of similitude of these three circles shall coincide with the points P, P'; Q, Q'; R, R', as was shown in this MONTHLY, 1929, p. 56, problem 3295.

Thus the circles (PP'), (QQ'), (RR') described on PP', QQ', RR' as diameters are the circles of similitude of the circles (A), (B), (C), and the properties pointed out by J. H. Weaver (this MONTHLY, 1938, pp. 17 ff.) become immediately apparent.

When the points P, P', $\cdots$ are the traces of the bisectors of the angles of ABC, the radii of the circles (A), (B), (C) are proportional to the respective altitudes of the triangle ABC (*The Tôhoku Mathematical Journal*, vol. 39, Part 2, June, 1934, p. 264, art. 1).

2. If the points P and P', Q and Q', R and R', U and U', V and V', W and W' divide the edges BC, CA, AB, DA, DB, DC of the tetrahedron $DABC$ externally and internally in ratios such that the points P, Q, R, U, V, W are coplanar, the twelve points of division form a desmic system.*

It is possible to describe four spheres (A), (B), (C), (D) having A, B, C, D for centers such that the plane $PQRUVW$ shall be a plane of similitude of these spheres.† Thus the spheres (PP'), $\cdots$, (WW') described on PP', $\cdots$, WW' as diameters are the six spheres of similitude of the spheres (A), $\cdots$, (D) taken in pairs. These six spheres therefore form a coaxal net, are orthogonal to the circumsphere of the tetrahedron $ABCD$, *etc.*‡

When the points P, P', $\cdots$, W, W' are the traces of the bisecting planes of the dihedral angle of $ABCD$ upon the respectively opposite edges, the radii of the spheres (A), $\cdots$, (D) are proportional to the respective altitudes of the tetrahedron $ABCD$.§

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* Nathan Altshiller-Court, Modern Pure Solid Geometry, Macmillan, New York, 1935, p. 237, art. 728.

† Ibid., p. 206, art. 642. The Mathematics Student, vol. 3, 1935, p. 2.

‡ Ibid., p. 203, art. 634. § Ibid., p. 252, arts. 768 ff.

A SIMPLE APPROXIMATION FOR π

M. G. GABA, University of Nebraska

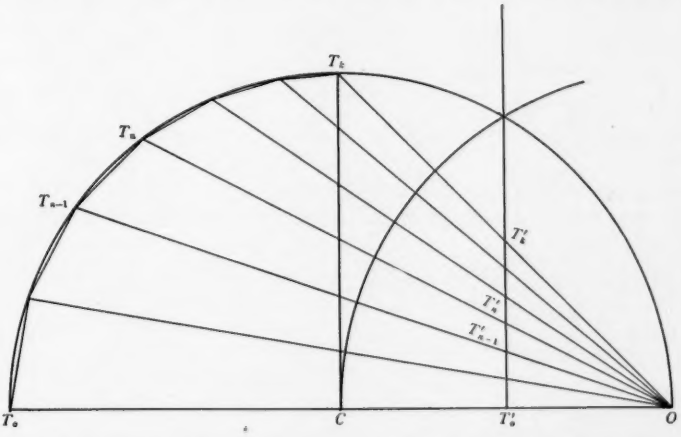
John Wallis is given credit for being the first mathematician to express π as the limit of a sequence of *rational* numbers. The earlier geometrical methods of inscribing and circumscribing regular polygons in or about a circle led to approximations involving successive extractions of square roots. It has been generally believed that the methods of the "Newer Analysis" introduced during the latter half of the seventeenth century are necessary in order to evaluate π as a limit of rational numbers. That this is not so will be shown by proving by elementary means that

$$\pi = \lim_{k\to\infty} \pi_k,$$

where

$$\pi_k = \frac{4k-2}{4k^2+1} + \frac{16k^3}{4k^2+1} \sum_{n=1}^{n=k} \frac{1}{n^2+k^2}.$$

Let T_0T_k be a quadrant of arc of a unit circle whose center is C and T_0O a



diameter. The inverse of this quadrant with respect to the unit circle whose center is at O will be the line segment $T_0'T_k'$. Obviously $OT_0' = T_0'T_k' = \frac{1}{2}$. Divide $T_0'T_k'$ into k equal parts and call the points of division $T_1', T_2', \cdots, T_{k-1}'$. Let the inverses of these points on the arc T_0T_k be $T_1, T_2, T_3, \cdots, T_{k-1}$.

The areas of triangles $T_{n-1}'OT_n'$ are all equal to $1/(8k)$. If we call this area Δ and that of the triangle $T_{n-1}OT_n$ is called Δ_n, then

$$\frac{\Delta_n}{\Delta} = \frac{OT_{n-1}\cdot OT_n}{OT_{n-1}'\cdot OT_n'}.$$

But since

$$(OT_n')^2 = \left(\frac{n}{2k}\right)^2 + \frac{1}{4}$$

and OT_n' and OT_n are reciprocals of each other, we have

$$\frac{\Delta_n}{\Delta} = \frac{1}{(OT_{n-1}')^2(OT_n')^2} = \frac{1}{\left[\left(\frac{n-1}{2k}\right)^2 + \frac{1}{4}\right]\left[\left(\frac{n}{2k}\right)^2 + \frac{1}{4}\right]},$$

$$\Delta_n = \frac{2k^3}{[(n-1)^2 + k^2][n^2 + k^2]}.$$

The area of the polygon $OT_0T_1T_2 \cdots T_k$ is

$$\sum_{n=1}^{n=k} \Delta_n = 2k^3 \sum_{n=1}^{n=k} \frac{1}{[(n-1)^2 + k^2][n^2 + k^2]}.$$

Since

$$\frac{1}{[(n-1)^2 + k^2][n^2 + k^2]} = \frac{1}{4k^2 + 1}\left[\frac{-2n+3}{(n-1)^2 + k^2} + \frac{2n+1}{n^2 + k^2}\right],$$

and

$$\sum_{n=1}^{n=k} \frac{-2n+3}{(n-1)^2 + k^2} = \frac{1+2k}{2k^2} + \sum_{n=1}^{n=k} \frac{-2n+1}{n^2 + k^2},$$

we have

$$\sum_{n=1}^{n=k} \Delta_n = \frac{4k^3}{4k^2 + 1} \sum_{n=1}^{n=k} \frac{1}{n^2 + k^2} + \frac{k(1+2k)}{4k^2 + 1}.$$

The area of the polygon $CT_0T_1T_2 \cdots T_k$ is that of the polygon $OT_0T_1T_2 \cdots T_k$ less the area of the triangle OCT_k. If we let $\pi_k/4$ be the area of $CT_0T_1 \cdots T_k$, then

$$\frac{\pi_k}{4} = \sum_{n=1}^{n=k} \Delta_n - \frac{1}{2} = \frac{4k^3}{4k^2 + 1} \sum_{n=1}^{n=k} \frac{1}{n^2 + k^2} + \frac{k - \frac{1}{2}}{4k^2 + 1}.$$

The area of the quadrant CT_0T_k is $\pi/4$. But we see that as k tends to infinity the area of the polygon $CT_0T_1 \cdots T_k$ approaches the area of the quadrant CT_0T_k as a limit, and therefore $\pi = \lim_{k\to\infty} \pi_k$, as we proposed to prove.

Computation shows that $\pi_{70} = 3.1413$; the sequence is therefore seen to be not very rapidly convergent.

Note by the Editor. It readily follows from the foregoing short article that

(a) $$\frac{\pi}{4} = \lim_{k\to\infty} \sum_{n=1}^{k} \frac{k}{n^2 + k^2},$$

which formula is seen to be closely related to

(b) $$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2}.$$

ITERATED AND DOUBLE LIMITS

ARTHUR SARD, Queens College

1. *Let $s(x, y)$ be a function with the property that the limits*

$$s_1(y) = \lim_{x\to\infty} s(x, y), \qquad s_2(x) = \lim_{y\to\infty} s(x, y) \tag{1}$$

exist for all positive y and x, respectively. We shall be concerned with the iterated and double limits of $s(x, y)$:

$$s_{12} = \lim_{y\to\infty} s_1(y), \qquad s_{21} = \lim_{x\to\infty} s_2(x), \qquad s = \lim_{(x,y)\to(\infty,\infty)} s(x, y).$$

It is evident that *if the double limit, s, exists, the iterated limits, s_{12} and s_{21}, exist; and all three limits are equal.* And it is well known that a sufficient condition for the existence of s is that either one of the limits (1) be uniformly approached over the entire range of the remaining variable.* We shall prove

THEOREM A. *A necessary and sufficient condition that the double limit, s, exists is the following: Given any sequence $\{y_j\}$ which tends to infinity, then $s(x, y_j) \to s_1(y_j)$ uniformly over $\{y_j\}$ as x tends to infinity.*

We may express the condition briefly thus: $s(x, y)\to s_1(y)$ uniformly over sequences $\{y\}$.

A. Pringsheim has made an extended investigation of the relation between the iterated limits and the double limit of a double sequence.† He introduces a generalization of the concept of uniform convergence which permits him to dispense with the assumption that the limits (1) exist. However, none of his results has the simplicity of Theorem A. Theorem A shows exactly the rôle played by uniform convergence in our problem.

2. To prove Theorem A we first consider

THEOREM B. *Let $\{x_i\}$, $\{y_j\}$ be two sequences, each of which tends to infinity. Then the double limit $s = \lim_{(i,j)\to(\infty,\infty)} s(x_i, y_j)$ exists if and only if $s(x_i, y_j \to s_1(y_j)$ uniformly.*

Proof: The sufficiency of the condition is well known.

To prove the necessity, observe that if s exists, s_{12} also exists and equals s. Let $\epsilon > 0$ be given. Then there is an integer, N, such that

* See, for example, W. F. Osgood, Funktionentheorie, Leipzig, 1928, I, pp. 617–623. The ideas of the present paper occurred to the author after study of this portion of the Funktionentheorie.

† A. Pringsheim, Zur Theorie der zweifach unendlichen Zahlenfolgen, Mathematische Annalen, vol. 53, 1900, pp. 289–321; and Vorlesungen über Zahlen und Funktionenlehre, vol. I, part 1, pp. 269–292.

$$|s - s_1(y_j)| < \epsilon/2 \quad \text{for all } j > N;$$

and an integer, N', such that

$$|s(x_i, y_j) - s| < \epsilon/2 \quad \text{for all } i, j > N'.$$

Hence

$$|s(x_i, y_j) - s_1(y_j)| < \epsilon \quad \text{for all } j > N, N'; i > N'.$$

On the other hand, since $s_1(y_j)$ exists, there is an integer, $M(j)$, dependent on j, such that

$$|s(x_i, y_j) - s_1(y_j)| < \epsilon \quad \text{for all } i > M(j).$$

Now let Q be the maximum of the finite set of numbers:

$$M(1), M(2), \cdots, M(N), \cdots, M(N'), N'.$$

Then the last two inequalities imply that

$$|s(x_i, y_j) - s_1(y_j)| < \epsilon \quad \text{for } i > Q \text{ and for all } j;$$

that is, $s(x_i, y_j)$ approaches $s_1(y_j)$ uniformly, as was to be shown.

Theorem A is an immediate consequence of Theorem B and the following lemma, the proof of which is left to the reader.

LEMMA. *The limit* $\lim_{x\to\infty} s(x, y_j)$ *is uniform if and only if, for each sequence* $\{x_i\}$ *that tends to infinity, the limit* $\lim_{i\to\infty} s(x_i, y_j)$ *is uniform.*

3. With obvious changes Theorems A and B apply equally well to the case in which x or y or both approach finite limits. The theorems apply also to the case in which x and y are points in any two topological spaces. This permits a certain elegance to be introduced in some practical applications. For example, in considering an iterated integral that is improper at both limits of integration of both variables, one need not resort to the common device of replacing the integral by the sum of other integrals, each of which is improper at only one limit of integration.

Uniform convergence over sequences $\{y\}$ is not equivalent to uniform convergence over a y interval, $a \leqq y < \infty$. Thus consider the function $s(x, y)$ defined for $x > 1$, $y > 1$ as follows:

$$\begin{cases} s(x, y) = 0 & \text{if } y - [y] \geqq 1/x,^* \\ s(x, y) = \sin\{\pi x(y - [y])\}/y & \text{if } y - [y] < 1/x. \end{cases}$$

For this function, the limits $s_1(y)$, $s_2(x)$, s exist. Hence $s(x, y) \to s_1(y)$ uniformly over sequences $\{y\}$. On the other hand it is clear that the convergence is not uniform over any y interval of length $\geqq 1$.

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* $[y]$ is the largest integer $\leqq y$.

RECENT PUBLICATIONS

EDITED BY TOMLINSON FORT, Lehigh University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

REVIEWS

On the Statistical Theory of Errors. By W. E. Deming and R. T. Birge. Washington, D. C., The Graduate School of the United States Department of Agriculture, 1937. 4+42 pages. $.35.

This work presents under a single cover a selection of topics which formerly were rather scattered in the mathematical literature.

A complete table of the symbolism employed in the pamphlet is followed by a brief introduction explaining in a general way the problems considered later. The frequency curve for the parent population is assumed to be normal. The authors justify this choice and answer some objections that have been advanced in the past for such a choice. Next, the distribution laws for $u=$(deviation of mean of sample from mean of parent population), $s=$(standard deviation of sample), and $z=u/s$ are derived. Following this is a section devoted to tests concerning hypotheses that a given sample was drawn from a normally distributed parent population with assumed parameters. These tests are five in number and are clarified with reference to the u, s frequency surface. The notion of fiducially related values of σ and s then follows. Consideration is now given to the problem of estimation from viewpoints of (1) the method of maximum likelihood, (2) empirical estimates, and (3) the posterior method.

Throughout the pamphlet the authors take considerable care in presenting the assumptions underlying the various tests and methods of estimation. Cautionary measures to be taken in applying the various tests are presented. The use of tables for calculation is described at some length and solved problems illustrating the applications of the theory are exhibited.

The typographical arrangement in a few instances leaves something to be desired in that at these places the content of the pamphlet is difficult to follow.

The booklet is especially recommended to those who desire a lucid presentation of the recent statistical viewpoint concerning the theory of errors.

F. S. BEALE

Lectures and Conferences on Mathematical Statistics. By J. Neyman. (Revised and supplemented by J. Neyman with the editorial assistance of W. E. Deming.) Washington, D. C., The Graduate School of the United States Department of Agriculture, 1938. 8+163 pages. $1.25.

This book consists of a report on a series of three lectures and six conferences dealing with various phases of mathematical statistics held at Washington, D. C., in April 1937.

Lecture I deals with the theory of probability in general. The author defines first a fundamental probability set and then proceeds to a definition of probability for each of two cases, (1) when the fundamental probability set is finite and (2) when this set is infinite. A characteristic feature of this definition consists in its expressed avoidance of the term "equally probable cases."

The second lecture contains a statement of the empirical law of big numbers. A large part of this lecture is devoted to a discussion of the construction of a bridge between the theory of probability and certain fields of application. This bridge is made possible through care in constructing a suitable mathematical model together with equal care in performing suitable experiments.

The final lecture, entitled "The testing of statistical hypotheses," is one of the most interesting in the entire book. The question of the retention or rejection of hypotheses together with tests for so doing are considered at some length. Former practices along these lines are mentioned in some detail and their general insufficiency exposed. Consideration is given in the lecture to possibilities of (1) rejecting a true hypothesis and (2) not rejecting a wrong hypothesis. Toward the end of the chapter the procedure of considering a set of n observed values of a property as defining a point in n-way space is introduced—a procedure which has been very helpful in recent times in the solution of certain statistical problems.

The first two conferences dealt mainly with certain applications of statistics to questions relating to agriculture. The third conference treated statistics as applied to social and economic research. It is devoted mainly to the question of procedure in selecting a suitable sample from a heterogeneous population. The main idea appears to be that such a population should be partitioned into homogeneous subdivisions or strata and the total sample should be selected from the members of these strata. A mathematical justification of this procedure is included in the chapter.

The fourth conference concerned itself with analysis of time series. The prediction of economic processes can be attacked in either one of two ways. First, the empirical method which simply concerns itself with final economic results and second, the *a priori* method which concerns itself with the inner workings of the economic machinery. Although the former of the two methods has been mainly employed in the past, the lecturer leans to the thesis that the *a priori* method has been too long neglected. He seems to feel that the best results probably are obtainable through a more prevalent use of the second method.

The final two conferences considered the problem of statistical estimation. Given the form of the distribution law of some property, how can we use a set of observed values of this property to estimate the unknown parameters of the distribution law. The principle of insufficient reason employed in the past was mentioned and a discussion of the "best unbiased" estimate which has been treated extensively by Markoff. Finally, the lecturer launched into a discussion of the problem of estimation from the standpoint of upper and lower estimates of a parameter, and the book closes with a general discussion of the theory of confidence intervals.

The lectures were delivered to an audience which already possessed considerable knowledge in mathematical statistics. It is to such a class of readers that the book has its greatest appeal. Frequent references are given throughout the book for more detailed study. A notable feature of the conferences consists in the inclusion of various questions put to the lecturer by his audience followed by the lecturer's response. The book is especially recommended for those who wish a general statement of modern mathematical statistics problems together with an up-to-date account of the procedure of attack on those problems employed mainly by Pearson and Neyman.

F. S. BEALE

Computation and Trigonometry. By H. J. Gay. New York, The Macmillan Company, 1938. 7+231 pp. $1.90.

This trigonometry, which I have used as a supplementary text for six weeks, is an unusually well written textbook. One of its many good points is the chapter having to do with accuracy, a subject many texts neglect. The various topics are clearly presented in an interesting manner. Approximately five per cent of the large number of well selected problems have been solved. There is a high degree of accuracy in the answers. Many of the problems provoked lively and interesting discussions in the class. The chapter, Solid Geometry and Trignometry, is something new in a trigonometry, and in many classes it should help correct a decided weakness. Other special features are a chapter on spherical trigonometry with applications to navigation and astronomy, and a chapter on the slide rule, both of which are well done.

WILLIAM BEVERLEY

First Year College Mathematics. By V. H. Wells. New York, D. Van Nostrand Company, 1937. Part I: *Trigonometry.* 7+133 pages. $1.25. Part II. *Mathematical Analysis.* 9+276 pages. $2.75.

These two books constitute a modification of the idea of unified mathematics for a first year college course. It is an interesting departure. We quote from the preface of Part II: "The author believes that the subject of trigonometry does not lend itself to a correlation with the subjects of algebra, analytical geometry, and elementary calculus, but that the latter subjects are closely related and are readily studied simultaneously." Acting on this belief the author has made trigonometry the subject matter of Part I. In this part there is nothing strikingly original in the selection and arrangement of material. Nevertheless, the presentation is fresh and stimulating. To quote from the preface: "One of the author's main objectives has been to write a text requiring as little memorization as possible, but rather, to develop an understanding of the underlying and fundamental principles of the subject." In the reviewer's opinion this is a commendable objective, and the author has reached it to a very gratifying degree. As to some of the details: The general definitions of the trigonometric functions of any angle are given at once by means of a rectangular coördinate system, and the functions of acute angles are considered as special cases. There are two chapters

on the solution of triangles, the first without logarithms. In dealing with the inverse functions the author starts with $y = \sin x$ and considers the relation $x = \arcsin y$. In the reviewer's opinion first year students will more quickly grasp the functional relations involved if (1) $y = \arcsin x$ is the starting point, and $x = \sin y$ is used only as an aid in plotting (1). No mention is made of the alternative notation $\sin^{-1} x$. It would have been well to call attention to this, since it is of such frequent occurrence in the mathematical literature. Tables are not included, except one page of natural functions to three places for instructional purposes. The ground covered is no more than what is essential for practical applications, and to serve as a background for further work in mathematics.

Coming to Part II, we find that the author adheres to his plan of unification, as the following chapter headings indicate: Cartesian coördinates, Equations of loci, Straight lines, Functions and their graphs, Solution of nth degree equations, Rate of change, Differentiation of polynomials, Parabolas, Circles, Differentiation of algebraic functions, Central conics, Curve tracing, Integration, Parametric equations, Transformation of coördinates, Polar coördinates, Complex numbers. These topics are dovetailed into a unified whole, and at the same time the treatment of each is adequate and comprehensive; *e.g.*, as sub-headings for the chapter on nth degree equations we find: Rational integral equations, solution of general quadratic equation, number of roots of an equation, relation between roots and coefficients, integral roots, Descartes's rule of signs, rational roots, irrational roots; and for the chapter on differentiation of a polynomial: Derivatives, differentiation formulas, geometric interpretation of a derivative, equations of tangents to curves, sign of the derivative, velocity, maxima and minima, applications of the theory of maxima and minima, Newton's method of approximating irrational roots. The remaining chapters are equally well rounded out.

A noteworthy feature of both Part I and Part II is the arrangement of exercises. These are divided into groups A, B, C, and D. Group A consists of easy problems calling for an obvious application of the theory; group B is similar to group A, but more difficult; group C calls for some ingenuity; group D, not present in every set of exercises, contains many problems whose solution constitutes an extension of the preceding theory.

The reviewer is anxious to avoid leaving with his readers the impression that *First Year College Mathematics* is just two more texts. It can come pretty close to the ideal solution of the problem of unified mathematics; for the student who takes only one course it provides substantial information in a wide range of useful topics; for the student who intends to go on to further work, it provides a rock-bottom foundation. Furthermore, the selection and arrangement of material; the concise and scholarly presentation; the attractive drawings; the sparing but effective use of display and bold-faced type; all these combine to form a set-up in textbook design and construction which is distinctly superior.

R. L. JEFFERY

Construction, Classification, and Census of Magic Squares of Order Five. By A. L. Candy, Author and Publisher. 1003 H Street, Lincoln, Nebraska, 1938. 5+141 pages. $1.00.

The purpose of this book is to analyze and classify completely the entire set of magic squares of order five. The term "magic square" is here used in the classical sense to mean a square array of the positive integers from 1 to 25 in which the sum of the numbers in each row, in each column, and in each of the two principal diagonals is the same, namely 65.

The author divides all magic squares of order five into seven general classes and then discusses certain sub-sets of these classes, called "types." He shows how magic squares of each of these types can be built up and succeeds in enumerating completely the squares belonging to some of the types. The labor involved in finding the number of squares in the remaining types proved to be prohibitive, so that their number was found by making a statistical estimate. Thus Professor Candy finds that there are 12,860,440 essentially different magic squares of order five. This number, although for the most part an estimate, is far from being a mere guess but seems to be based on fairly good statistical evidence.

Although much has been written on magic squares a great deal of the literature is difficult of access, so that an author may easily overlook some previous work. This has happened in some parts of the present book. For instance, the author's "Mixed squares Class II, Type I" are precisely the bordered squares discussed in detail by B. Violle in 1838. Both Violle and the author consider 26 cases but the number of squares corresponding to each case agree in 7 cases only. In 14 instances the author's number is greater than Violle's and in 5 cases smaller. However, in the example on page 80 the first pair of rows add to 63 and 67 instead of to 65 each. With this correction the number of squares for this case is reduced to 28 which agrees with Violle's number.

In the chapter on pandiagonal squares every example given is of the well known uniform step type. There are, however, other pandiagonal squares derivable from the uniform step squares by certain transformations. The total number of pandiagonal squares of order 5 is known to be 3600. Of these 3600 squares no one can be obtained from any other by rotations or reflections.

The author has succeeded in bringing to light a number of very interesting and unusual squares which may well be worth further investigation. Furthermore, the book offers a challenge to those interested in the subject to make a new attempt to determine the exact number of magic squares of order five. As the book amply illustrates, the solution of this problem is not easy, at least by the methods used there.

A number of misprints were noted, but they are of a rather obvious nature, so that it seems unnecessary to list them.

G. E. RAYNOR

The Mathematics of Investment. By P. R. Rider. New York, Farrar and Rinehart, Inc., 1938. 162 pages. $2.00.

This book is an excellent text for a one-semester first course in the mathematics of investment where the prerequisite work is at least one year of high school algebra. The text presents the important and fundamental conceptions in a clear, concise, and accurate fashion.

The first chapter contains a number of useful algebraic formulas and explanations of certain necessary processes and manipulations which are pertinent in the study of the mathematics of investment.

The second chapter is devoted to the consideration of interest and discount. It is pleasing to note a clearly stated and correct definition of interest and discount. Students should not have the usual confusion in understanding what discount is. The author, if not the only author, is among the very few who defines clearly and correctly an equation of value in terms of the "Principle of Equivalence."

The third chapter is devoted to the consideration of annuities. We do not find here the usual number of many unnecessary special case formulas. In fact, he very quickly arrives at the treatment of an annuity where the payments are made p times per period and the interest is converted m times per period. Such a concise and rather inclusive treatment is, in the reviewer's opinion, superior as it enables the student to consider problems as special cases of a single algebraic result. Capitalized cost is also presented in this chapter. In the reviewer's opinion, it would have been better to discuss this conception in chapter four where other applications of annuities are discussed.

The fourth chapter contains an adequate treatment of the important applications of annuities. Here we find a clear presentation of amortization, sinking funds, and the valuation of a mine. We also find here a discussion of certain aspects of the concept, depreciation. The author discusses a few of the common methods used for obtaining the periodic depreciation charge and building up the so-called depreciation fund. However, as is the case in many other texts on the mathematics of investments, depreciation itself is not defined. In the reviewer's opinion, it should have been emphasized that depreciation is in fact a statistical problem.

The fifth chapter contains a brief but adequate treatment of building and loan associations.

The sixth chapter contains a brief but adequate treatment of bonds and their valuation. The reviewer is of the opinion that stocks and their valuation should also have been discussed in this chapter.

The seventh chapter is devoted to an excellent brief presentation of the elementary principles of probability, the meaning of expectation, and the nature of a mortality table. Enough is said about the application and use of probability so that the student will have a clear understanding of the meaning of the various so-called rates of mortality as given in a mortality table.

The eighth chapter is devoted to a brief discussion of the meaning of an

endowment, a life annuity, deferred life annuity, temporary life annuity, and a forborne temporary life annuity. The author also defines the necessary commutation columns which are useful to simplify computation. Of course, much of the theory of life annuities has been omitted and justifiably so. However, the reviewer feels that the connection between a life annuity and an old age pension should have been specifically mentioned.

The ninth chapter is devoted to a brief discussion of the meaning of life insurance and a few of the common types of life insurance policies. It is shown how the annual and single premiums may be found by the use of commutation columns. Loading and one form of valuation is briefly discussed. The reviewer feels that one very important subject, the valuation of life insurance policies, is inadequately treated.

A very good set of tables is bound with the book. The clarity of the exposition is most pleasing. The text appears to be a very teachable one and in the reviewer's opinion is a valuable and worth while addition to the set of available texts.

F. M. WEIDA

Manual of Mathematics and Mechanics. By G. R. Clements and L. T. Wilson. First Edition. New York, McGraw-Hill Book Company, 1937. 5+263 pages. $2.50.

This compendium of mathematical information contains 85 pages of numerical tables, 145 pages of formulas, figures and curves pertaining to the usual topics in mathematics covered in the high school and the first two years of college, a little about vectors, and 44 pages of the principles and formulas of mechanics. There are 22 numerical tables including the usual common and Napierian logarithms; direct and inverse trigonometric, exponential, and hyperbolic functions, mostly to four figures after the decimal; squares, cubes, etc. of numbers; compound interest, annuity, and mortality tables; values of elliptic and probability integrals; conversion tables; and binomial coefficients.

The table of integrals contains 362 formulas for general or definite integrals. There are 15 short articles on series, 4 on the formulas of plane and solid geometry, 37 on algebra including such subjects as the cubic equation, probability, determinants, interest, annuities, and complex numbers. Plane and spherical trigonometry are given 17 articles and plane and solid analytic geometry 39 articles. There are 9 pages of figures of curves and surfaces followed by 44 articles on differential and integral calculus, consisting mostly of formulas including ordinary differential equations. Vectors are discussed in 9 articles and the formulas and principles of mechanics in 32. There are 15 pages of formulas giving the location of the centroid, moment and product of inertia, and radius of gyration of certain plane and solid figures. The book closes with 12 articles on the properties of materials followed by 4 pages of figures and formulas for shear, moment, and deflection of loaded beams. An index of 4 pages adds to the usefulness of the volume.

Certain minor omissions seem obvious to the reviewer, such as: the polar equations of the cissoid, the witch, and particularly the strophoid, which are so much simpler than the rectangular or parametric equations, the parabolic cylinder $y^2=ax+bz$, and especially the formula for area $\frac{1}{2}\int(xdy-ydx)$, since so many parametric equations are included.

It seems that the third form for $a_x=v_x(dv_x/dx)$ should be given in Art. 176 because it is so frequently useful, and that the special cases in which the formula $M=I\alpha$ can be used in addition to the one given in Art. 191 should be stated. The reviewer is inclined to question the use of the term constant density for uniform density and to ask for a definition of an element of a hoop.

These are all unimportant points, however, in view of the extent and detail of the book. It is certainly an excellent handbook for the student of college mathematics and mechanics. It will be useful, also, as a syllabus for college students majoring in mathematics.

J. B. REYNOLDS

Segmental Functions, Text and Tables. By C. K. Smoley. First Edition. Scranton, Pa., C. K. Smoley and Sons, 1937. 43+255 and 91+34 pages.

In this book are bound together in two parts, with non-consecutive paging, a number of tables some of which are new, while others are abstracted from previous publications by the author. The new tables, covering 255 pages, have to do with what are called segmental functions because of their relation to a segment of a circle.

If A, M, C, and R stand respectively for the lengths of arc, middle ordinate, chord, and radius of the circle of a circular segment, seven of these segmental functions are defined by

$$a = A/R, \qquad c = C/R, \qquad m = M/R, \qquad \alpha = A/C,$$
$$\beta = M/C, \qquad \gamma = M/A, \qquad \sigma = MR/A^2.$$

The logarithms of these functions are given for each minute to five figures after the decimal for values of the central angle, subtended by the segment, from 0° to 180°. There is also a table, for the same range, of the function $f=F/R^2$ in which F is the area of the segment. Separate tables give the logarithm of a for each 10″ over the first 10° of the central angle and the natural values of a over the wider range.

The first 43 pages are devoted to a discussion of segmental functions and explanation of their uses. The mathematician is surprised to read that the central angle, ϕ, is not included in the definition of segmental functions and then find that the very first one is the ratio of the arc to the radius. In fact all the eight functions listed are nothing more than trigonometric functions defined by $a=\phi$, $c=2\sin\frac{1}{2}\phi$, $m=\text{vers}\,\frac{1}{2}\phi$, $\alpha=\frac{1}{2}\phi/\sin\frac{1}{2}\phi$, $\beta=\frac{1}{2}\tan\frac{1}{4}\phi$, $\gamma=\text{vers}\,\frac{1}{2}\phi/\phi$, $\sigma=\text{vers}\,\frac{1}{2}\phi/\phi^2$, $f=\frac{1}{2}(\phi-\sin\phi)$. In view of this, it seems unlikely that these tables will be of service except to a few persons engaged in certain narrowly technical fields.

J. B. REYNOLDS

MATHEMATICS CLUBS

EDITED BY F. W. OWENS AND HELEN B. OWENS, State College, Pa.

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

Beginning with the next issue of the MONTHLY, the department of Mathematics Clubs will be under the editorship of Professor E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, New Jersey. All future correspondence and new material should be sent to him. Professor Hildebrandt brings to us vital interest and a fresh and stimulating viewpoint. The retiring editors bespeak for him the same sincere cooperation of the one hundred fifty mathematics clubs in the United States and Canada which has made the last three years such worth while ones for the editors.

It has been impossible to print all programs received, but all have been filed as a permanent record of undergraduate activities. These files, together with the Stunt Library, the modest beginnings of a File of Undergraduate Publications and Student Papers will be placed in the hands of the new editor.

Let every club make a sincere effort to send reports for 1937–38 to the editor before October first. Promptness in this matter is essential to the successful planning of the work for the new year. Particularly be sure that a correct list of officers for 1938–39 is sent direct to this department as soon as they are elected. The greater the cooperation of the many clubs, the better can this department serve its purpose.

BOOKS

34. *Excursions in Mathematics*, by Ernst R. Breslich. Chicago, The Orthovis Company, 1938. 48 pages. Illustrated. $1.20. A clever aid to visualization of the problems of area and volume.
35. *Poetry of mathematics and other essays*, by David Eugene Smith. New York, *Scripta Mathematica*, 1927. 96 pages. $1.00. Presents the subjects considered from a cultural viewpoint; quite worthy of consideration.
36. *Portraits of eminent mathematicians with their biographies*, by David Eugene Smith. New York, *Scripta Mathematica*. Portfolio I (12 mathematicians) $3.00. Portfolio II (13 mathematicians) $3.00. A real addition to any club library and interesting from the point of instruction in history of mathematics and in building visual familiarity with well known men in various fields.
37. *Mathematical Nuts*, by S. I. Jones, Nashville, Tenn. 1936. $3.50.
38. *Mathematical Wrinkles*, by S. I. Jones, Nashville, Tenn. 1930. $3.00. This and the preceding book by the same author are filled with interesting and unusual bits for the recreational side of club work.
39. *Mathematical Excursions and Essays*, by W. W. R. Ball. New York, Macmillan, 10th edition, 1922. $3.50. A favorite for many years as its frequent republication testifies. It is one of the books which will soon show wear if left on an open shelf.

CLUB REPORTS
1936–37

Kappa Mu Epsilon, Kansas State Teachers College, Pittsburg

President, Sylvia Smith; Vice-President, J. Shaw; Secretary, Aileene Kingsbury; Treasurer, Colleen Hawkins; Corresponding Secretary, Professor W. H. Hill. The chapter held open meeting each month with programs which always included musical numbers in addition to the talks on mathematical topics. Topics included: Mathematics of the American Aborigines; Number theory among the ancients; Philosophy of mathematics; Japanese mathematics; Algebraic and transcendental numbers; The development of geometry. In addition to these, several old favorites appeared which are always of interest to the new students. Among the social meetings the most noteworthy was the celebration of the fifth anniversary of the founding of the chapter. The original charter list bore twenty-seven names. Since that time one hundred and seventy-three names have been added. The celebration was attended by over seventy, many alumni returning for the event.

Mathematics Club, University of California at Los Angeles

This group which welcomes all students interested in mathematics holds open meetings each month. The topics are chosen with regard to their appeal to a large group. Paper folding, Trisection of an angle, and similar topics are favorites. Experiences in Canadian and English universities illustrated the different types of instruction in mathematics. Picnics took members from the sea side to the high mountains.

Kappa Mu Epsilon, Kansas State Teachers College, Emporia

The chapter held ten program meetings in addition to the social affairs which serve to increase interest among the members. Topics discussed included: Telescopes and their construction; Einstein, the man; Dependence of physics on mathematics; Educated man; Mathematics and art; Mathematics and chemistry. One meeting was devoted to reports of the Annual Meetings of the Kansas Association of Mathematics Teachers and the Kansas Section of the Mathematical Association of America. Special discussions concerned prophecies with reference to the future of mathematics in the secondary schools.

Delta x, University of Toledo

President, N. Farley; Vice-Presidents, A. Hiltner, E. Keefer; Secretary, Georgia M. Miller; Treasurer, E. Tom. The eighth year of this active group started off with a discussion of applications of conic sections. Each meeting is announced by a mimeographed sheet generally distributed. The first for this year closed with the modest statement: "Come and see why Delta x is the biggest club of its kind in the United States." The new feature of the year's program was a review of current mathematics magazines which pertain to undergraduates. Book reviews, problems, recreations, all received attention. The membership of over seventy made a high average of attendance. Membership is open to all students taking a course in calculus or who have completed such a course.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems. [By error problems for solution were omitted in this number of the MONTHLY. E. J. M.]

SOLUTIONS

E 303 [1937, 598]. *Proposed by J. E. Trevor, Cornell University.*

Given fixed rectangular axes, x and y, and a straight line T perpendicular to the x-axis but capable of displacement parallel to itself. One end of a straight line segment A of constant length a is pivoted to T at the point where T cuts the x-axis. The other end of A makes sliding contact with the y-axis. One end of a straight line segment B of constant length b is pivoted at the origin. The other end maintains sliding contact with T. The segments A and B, produced if necessary, intersect at P.

Find the rational algebraic equation of the locus of P. Describe the graph of this equation according as a is less than, equal to, or greater than b. Find the point at which the locus cuts the x-axis in the cases where a equals or exceeds b.

Solution by the proposer.

(a) Let T be the line $x=t$, and put

$$\alpha = \pm\sqrt{a^2 - t^2}, \qquad \beta = \pm\sqrt{b^2 - t^2}.$$

Then, by similar triangles, if x and y are the coördinates of P,

$$\frac{\alpha}{t} = \frac{y}{t-x}, \qquad \frac{\beta}{t} = \frac{y}{x};$$

whence, successively eliminating y and x,

$$x = \frac{\alpha t}{\alpha+\beta}, \quad y = \frac{\alpha\beta}{\alpha+\beta}, \tag{1}$$

which are parametric equations of the locus, with parameter t. Elimination of t yields the equation of the locus,

$$(a^2-b^2)^2(x^2+y^2)^4 - 2b^2(x^2+y^2)\left[(a^2-b^2)x^2+a^2y^2\right]\left[(a^2-b^2)x^2+(a^2+b^2)y^2\right] + b^4\left[(a^2-b^2)x^2+a^2y^2\right]^2 = 0. \tag{2}$$

This equation contains only terms of degrees eight, six, and four.

(b) From its equation it appears that the curve is symmetrical with regard

to the coördinate axes. Except for the case $a=b$, it lies in a finite part of the plane, with no asymptotes or parabolic branches. The origin is a four-fold point, isolated for $b<a$, but with two tangents, each counted twice, when $a<b$. The points $(\pm b, 0)$ are double points, isolated for $a<b$, but with two tangents for $b<a$. Aside from the four-fold point at the origin, the curve cuts the y axis in four real points,

$$\left(0, \frac{\pm ab}{a \pm b}\right).$$

Figures 1 and 2 sketch the curve for the cases $a=2$, $b=1$, and $a=1$, $b=2$. In

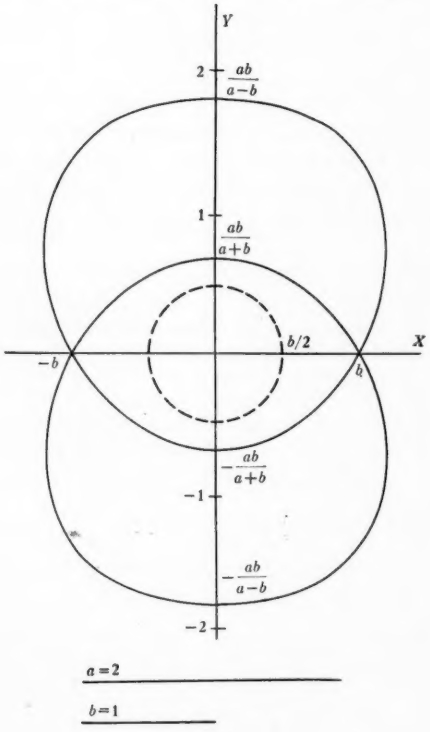

FIG. 1

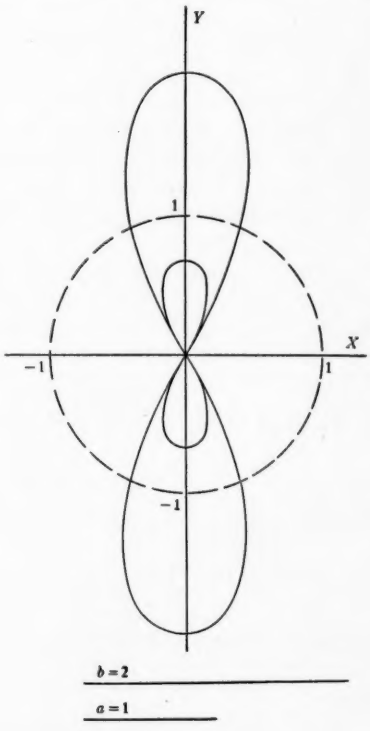

FIG. 2

each figure the dashed circle is the locus for $a=b$. In this case $\alpha=\beta$, and the equations (1) become

$$x = \frac{t}{2}, \qquad y = \frac{\pm\sqrt{a^2 - t^2}}{2},$$

whence, eliminating t,

$$x^2 + y^2 = \left(\frac{a}{2}\right)^2.$$

The locus is a circle of radius $a/2$. For this case (2) reduces to

$$-a^6(4x^2 + 4y^2 - a^2)y^4 = 0.$$

(c) When $a \geqq b$ and a approaches b, the curve cuts the x axis at $x = \pm b$. But when $a = b$ the point of intersection jumps to the intersection $x = \pm b/2$ of the axis with the circular graph for that case.

Also solved by Fred Discepoli.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch side.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3877. *Proposed by Louis Weisner, Hunter College, New York.*

Show that it is possible to construct a set S consisting of an infinite number of positive, and an infinite number of negative integers, which has the following properties: (1) If the integers of S are arranged in ascending order of magnitude, the difference between consecutive positive integers, and the difference between consecutive negative integers of S, tend to infinity. (2) If $f(x)$ is *any* non-constant polynomial with integral coefficients, S contains an infinite number of the integers represented by $f(x)$ for integral values of x. The problem becomes trivial if the first requirement is omitted: we could then take S as the set of all integers.

3878. *Proposed by V. Thébault, Le Mans, France.*

A convex quadrilateral is circumscribed about a circle. Show that there exists a straight line segment with ends on opposite sides dividing both the perimeter and the area into equal parts. Show that the straight line passes through the center of the inscribed circle. Consider the converse.

3879. *Proposed by V. Thébault, Le Mans, France.*

Show that it is possible to determine a plane section, limited by three faces, of a tetrahedron which divides both the surface and volume into equal parts. Show that the plane passes through the center of the inscribed sphere.

3880. *Proposed by W. R. Bennett, Bell Telephone Laboratories, New York.*

Show that the following identity involving continuants is true for any positive integer n. The notation is that used in Chrystal's *Algebra*, vol. II, p. 466:

$$K\{1, 2\lambda \sin \pi/2n, 2\lambda \sin 3\pi/2n, \cdots, 2\lambda \sin (2n-1)\pi/2n, 1\}$$
$$= 2[\lambda - ie^{\pi i/2n}][\lambda - ie^{3\pi i/2n}] \cdots [\lambda - ie^{(2n-1)\pi i/2n}].$$

3881. *Proposed by W. R. Bennett, Bell Telephone Laboratories, New York.*

Show that the following identity is true for any positive integer n, where the notation is that given in Chrystal's *Algebra*, vol. II, p. 466:

$$K\begin{Bmatrix} 1, & \cos^2 \pi/2n, & \cdots, & \cos^2 (n-1)\pi/2n \\ 1, & \lambda \sin \pi/2n, & \lambda \sin 3\pi/2n, & \cdots, \lambda \sin (2n-1)\pi/2n \end{Bmatrix}$$
$$= 2^{n-1}[\lambda - ie^{\pi i/2n}][\lambda - ie^{3\pi i/2n}] \cdots [\lambda - ie^{(2n-1)\pi i/2n}].$$

SOLUTIONS

3791 [1936, 378]. *Proposed by I. M. Sheffer, Pennsylvania State College.*

Let P_1, P_2, P_3 be three points on a circle C, so situated that P_2 and P_3 are on opposite sides of the diameter through P_1, and chords P_1P_2 and P_1P_3 make angles less than 45° with this diameter. Form a lattice-work in the plane with P_1P_2 and P_1P_3 as adjacent sides of a lattice parallelogram. Then of all the lattice points in the plane, only P_1, P_2, P_3 are in or on C.

Solution by E. P. Starke, Rutgers University.

The hypotheses on P_1, P_2, P_3 are equivalent to the statement that the inscribed triangle $P_1P_2P_3$ is acute-angled. Let L, M, N be the points P_1, P_2, P_3 in any order. Let Q be the vertex opposite L of the parallelogram $LMNQ$, and let Q' be the corresponding vertex of an inscriptible convex quadrilateral $LMNQ'$. Since angle Q' is the supplement of the acute angle Q (=angle L), angle Q' is greater than angle Q and point Q is outside the circle C.

All lattice points on the extended sides of triangle LMN are obviously outside C. All lattice points within the angle MLN (and hence within or on the sides of the vertical angle of angle Q) are outside C, for the line connecting any one of them with Q (outside the circle) when produced will intersect the chord MN (at a point in or on C). All lattice points within the vertical angle of angle MLN are outside C, for the line connecting any one of them with L (on C) when produced will intersect the chord MN (at a point in C).

Since L was any one of the points P_1, P_2, P_3, the proposition is established for all lattice points of the plane.

Solved also by the proposer.

Editorial Note. The proposer gave a proof by analytic geometry. It is simpler to use vector methods. Let P_1 be the origin of vectors, and let $\mathbf{b}_2$, $\mathbf{b}_3$, $\mathbf{c}$ be the vectors of P_2, P_3, C, where the last point is the center of circle (C) on which lie the three given distinct points. Any lattice point (n_2, n_3) is given by the vector $n_2\mathbf{b}_2+n_3\mathbf{b}_3$, where n_2 and n_3 are any two integers. A necessary and sufficient condition that all lattice points but the three given lie outside (C) is that the three points (1, 1), (−1, 1), (1, −1) lie outside (C). Denote by r the distance of a lattice point from C. Then

$$
\begin{aligned}
r^2 - \mathbf{c}^2 &= (n_2\mathbf{b}_2 + n_3\mathbf{b}_3 - \mathbf{c})^2 - \mathbf{c}^2 \\
&= n_2^2\mathbf{b}_2^2 + 2n_2n_3\mathbf{b}_2\cdot\mathbf{b}_3 + n_3^2\mathbf{b}_3^2 - 2\mathbf{c}\cdot(n_2\mathbf{b}_2 + n_3\mathbf{b}_3) \\
&= (n_2^2 - n_2)\mathbf{b}_2^2 + (n_3^2 - n_3)\mathbf{b}_3^2 + 2n_2n_3\mathbf{b}_2\cdot\mathbf{b}_3.
\end{aligned} \tag{1}
$$

The last result follows from

$$2c = \mathbf{b}_2^2\mathbf{b}_2' + \mathbf{b}_3^2\mathbf{b}_3', \qquad \mathbf{b}_i\cdot\mathbf{b}_i' = 1, \qquad \mathbf{b}_i\cdot\mathbf{b}_j' = 0, \qquad i \neq j,$$

see 3752 [1937, 405]. A necessary and sufficient condition that $(1, 1)$, $(-1, 1)$, $(1, -1)$ lie outside is given by (1) in the three inequalities

$$\mathbf{b}_2\cdot\mathbf{b}_3 > 0, \qquad \mathbf{b}_2^2 > \mathbf{b}_2\cdot\mathbf{b}_3, \qquad \mathbf{b}_3^2 > \mathbf{b}_2\cdot\mathbf{b}_3. \tag{2}$$

We now show that if (2) is true, every point but the given points $(0, 0)$, $(1, 0)$, $(0, 1)$ lies outside (C). For this purpose we may exclude the six points already mentioned; for the remaining points not both of the first two terms of the right member of the last equation in (1) can be zero. Hence by (2) we have from (1) for these points

$$r^2 - \mathbf{c}^2 > [(n_2 + n_3)^2 - (n_2 + n_3)]\mathbf{b}_2\cdot\mathbf{b}_3 \geqq 0, \tag{3}$$

where the final inequality results from the fact that n_2+n_3 is an integer and thus the bracket expression is never negative. This completes the proof. The inequalities (2) say that each angle of triangle $P_1P_2P_3$ is less than a right angle.

3792 [1936, 435]. *Proposed by F. Morley, Johns Hopkins University.*

A square is divided into n^2 unit squares, like a chess-board. Any two horizontal lines and any two vertical lines form a rectangle. We count a square as a rectangle. Each rectangle has a breadth b, less than or equal to its length. There is one rectangle of breadth n, namely the original square. Prove that there are 2^3 rectangles of breadth $n-1$, 3^3 of breadth $n-2$, $\cdots$, n^3 of breadth 1.

Deduce the formula

$$1^3 + 2^3 + \cdots + n^3 = [n(n + 1)/2]^2.$$

Solution by Roy MacKay, Eastern New Mexico Junior College.

There are only $k+1$ ways in which $n-k$ unit squares can be taken along one edge to form a rectangle of length $n-k$. Therefore the number of rectangles of width $n-k$ where the width is measured in an east-west (=EW) direction is $(k+1)[(k+1)+k+(k-1)+\cdots+2+1]=(k+1)(k+1)(k+2)/2$. Of this number $(k+1)^2$ are squares. Hence the total number of rectangles of width $n-k$ is $2(k+1)(k+1)(k+2)/2-(k+1)^2=(k+1)^3$.

To derive the desired formula we have but to consider the unit squares as formed by $(n+1)$ lines in the EW direction meeting $(n+1)$ lines in the NS direction. There are ${}_{n+1}C_2$ pairs of lines in each direction and hence $[{}_{n+1}C_2]^2 = [n(n+1)/2]^2$ rectangles in all.

Solved also by W. B. Campbell, G. N. Garrison, E. P. Starke, and C. W. Trigg.

3793 [1936, 435]. *Proposed by V. Thébault, Le Mans, France.*

Three parallels drawn from the midpoints of the sides BC, CA, AB of a triangle cut again the nine-point circle in α, β, γ. (a) Show that the circles with centers α, β, γ passing respectively through the feet A', B', C' of the altitudes AA', BB', CC' of the given triangle cut each other in pairs in three points of a straight line Δ and also pass through a point Q. (b) When the direction of the parallels varies, prove that the line Δ passes through a fixed point and that the point Q describes the nine-point circle of ABC. (c) Generalize the theorem by replacing the nine-point circle by the pedal circle of a point D in the plane of the triangle ABC and the points A', B', C' by the orthogonal projections of D on BC, CA, AB.

Solution by the proposer.

The solution of this problem will be deduced from the following theorem:

In the plane of the given triangle ABC the point D is arbitrarily selected. Denote by (ω) the circle with center ω through A', B', C', the orthogonal projections of D on the sides of ABC. Let A_2, B_2, C_2 be the projections of the vertices of ABC on the straight line Δ passing through D with an arbitrarily given direction; let α, β, γ be the center of circles (α), (β), (γ) circumscribing the triangles $A'B_2C_2$, $B'C_2A_2$, $C'A_2B_2$. Then (α), (β), (γ), (ω) intersect in a point Q on the circumcircle of the triangle $\alpha\beta\gamma$.

For the proof of this theorem we introduce three auxiliary circles (α'), (β'), (γ') with the diameters DA, DB, DC, and consequently with centers α', β', γ' at the midpoints of these segments. These circles circumscribe respectively the polygons $AA_2B'C'D$, $BB_2C'A'D$, $CC_2A'B'D$. With D as center of inversion and any convenient radius the inverses of the circles (α'), (β'), (γ') are three straight lines, while Δ inverts into itself. These four straight lines form a complete quadrilateral determining four triangles whose vertices are respectively the inverses of A', B', C'; B_2, A_2, C'; B', A_2, C_2; A', B_2, C_2. The circumcircles of these four triangles of the complete quadrilateral meet in the Miquel point M. These circumcircles are the inverses of (ω), (γ), (β), (α); hence these latter four circles must meet in the inverse Q of M.

We now make use of the following theorem:

The circumcenter of the triangle whose vertices are three of the intersections of three given circles, and the circumcenter of the triangle whose vertices are the remaining three intersections, are isogonal conjugate points with respect to the triangle whose vertices are the centers of the three given circles. Neuberg, *Nouvelle correspondence mathématique*, 1880–8.

Here we have the three circles (α), (β), (γ) which meet in the point Q while the other three intersections A_2, B_2, C_2 lie on Δ. Then one triangle of intersection degenerates to the point Q while the remaining triangle degenerates to the straight line Δ. Hence with respect to the triangle $\alpha\beta\gamma$ the point Q and the point at infinity on the perpendicular to Δ through Q are isogonal conjugate points. From this follows that the circumcircle of triangle $\alpha\beta\gamma$ is a fifth circle passing through Q and this completes the proof.

Editorial Note. The proof seems simpler without the use of the general theorem of Neuberg. Let U, V, W be the centers of circles (U), (V), (W) passing through the point M and intersecting in pairs in T, R, S on a straight line l, where the common chord of (U) and (V) is MT, etc. The side UV of triangle UVW is perpendicular to the common chord MT at its midpoint. Since the midpoints of chords MR, MS, MT lie on a straight line parallel to l, the feet of the perpendiculars from M to the sides of UVW must lie on this parallel to l. It now follows that the circumcircle of UVW passes through M. This simple theorem suffices for this part of the proof, but we shall note a slight extension. Let I be any point on l and let I_u, I_v, I_w, be the inverses of I with respect to (U), (V), (W). Then the circle through I_u, I_v, I_w, passes also through M, as is easily seen by inversion with respect to any suitable circle with I as center, noting that the inverse of I_u is the center of the inverse of (U). When I is at infinity on l we have merely the first theorem.

We now return to the complete quadrilateral with the Miquel point M. There are four circles which intersect in M any three of which also intersect in pairs on a side in the three vertices on that side. Hence by the above theorem the four centers of these circles are such that the circle through any three of these centers passes also through M. Thus we have eight circles through M. Now consider a point D on the side Δ of the quadrilateral and the inverses of D with respect to the three circles intersecting on Δ. The circle through these three inverse points also passes through M, and we now have nine circles through M. If for each position of D on Δ we invert this system with respect to D for the case in which we obtain the original figure of the problem, we have at once the theorem at the head of the above solution and also the following result:

Let the inverses of D with respect to (α), (β), (γ), (ω) be α_d, β_d, γ_d, ω_d. Then there are four circles each of which passes through three of these last four points and also through Q. We have now nine circles through Q. There are three other circles in the figure of the problem. The center γ lies on a circle (γ_1) through α', β', C_1, C'. For $\alpha'\gamma$, $C_1\gamma$ are perpendicular respectively to $C'A_2$, DA_2; and, as the direction of Δ varies with D fixed, A_2 describes the circle (α'). Hence γ describes a circle through α' and C_1. Similarly, we see that this circle passes through β' and C_1. When Δ has the position DC', the circle (γ) is the point circle C' and γ is at C', and this completes the proof. If D is at the orthocenter of ABC, the circles (α_1), (β_1), (γ_1), (ω) coincide in the nine-point circle of ABC; thus the circumcircle of $\alpha\beta\gamma$ and (ω) coincide.

3794 [1936, 435]. *Proposed by R. S. Underwood, Texas Tech. College, Lubbock, Tex.*

Prove that the series

$$\sum_{n=1}^{\infty} \left| \frac{\sin n\theta}{n} \right|$$

is divergent unless θ is an integral multiple of π.

Solution by H. Tate, McGill University, Montreal.

Abel's test states that, if $u_1+u_2+u_3+\cdots$ is a series such that the absolute value of the sum of the first n terms is less than a fixed number for all values of n, and, if $v_1, v_2, v_3, \cdots$ is a monotonically decreasing sequence of positive numbers which approach zero as n becomes infinite; then

$$u_1v_1 + u_2v_2 + u_3v_3 + \cdots$$

is a convergent series.

If $\theta \neq 2k\pi$, where k is an integer, then

$$\sum_{i=1}^{n} \cos i\theta = \frac{\sin n\theta/2}{\sin \theta/2} \cos (n+1)\theta/2, \tag{1}$$

and the absolute value of this sum is less than $1/|\sin \theta/2|$. Hence the series

$$\sum_{n=1}^{\infty} \frac{\cos n\theta}{n} \tag{2}$$

is convergent for all values of $\theta \neq 2k\pi$. Again

$$\left| \frac{\sin n\theta}{n} \right| \geqq \frac{\sin^2 n\theta}{n} = \frac{1}{2n}[1 - \cos 2n\theta],$$

and it has been proved that $\sum \cos 2n\theta/n$ is convergent if $\theta \neq k\pi$. Since $\sum n^{-1}$ is divergent, the series of the problem is divergent.

Solved also by J. H. Curtiss, M. Goldberg, E. P. Starke, and the proposer.

Editorial Note. Curtiss stated that the solution of the problem is contained in a theorem by P. Fatou, *Bulletin de la Société Mathématique de France*, vol. 41, 1913, pp. 47–53, which says that, if the series $\sum a_n \sin n\theta$ is absolutely convergent for a value $\theta_0 \neq 0$, mod π, then $\sum |a_n|$ is convergent provided that $|a_n|$ never increases. He also stated that a simple proof of Fatou's theorem is given in Zygmund's new book on trigonometric series p. 134, and pp. 3, 4.

A proof of Abel's test is given in Goursat-Hedrick, vol. 1, pp. 348, 349. In Tonelli's *Serie Trigonometriche* a proof of the Fatou theorem for $\sum a_n \cos n\theta$ is given on page 26, and the theorem of the problem is stated as an illustration of the theorems on pages 28 and 29.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

Dr. C. L. Alsberg of Stanford University has been appointed director of the Giannini Foundation of Agricultural Economics at the University of California.

Dr. M. T. Bird served as representative of the Mathematical Association at the Semicentennial Celebration of the Utah State Agricultural College June 5–7, 1938.

Garrett Birkhoff of Harvard University has been promoted to an assistant professorship in mathematics.

Dr. Julia W. Bower of Connecticut College has been promoted to an assistant professorship.

Dr. Melvin Dresher has been appointed to an instructorship at Michigan State College.

At Harvard University the following men have been appointed as instructors and tutors in the Division of Mathematics, on part time, for 1938–1939: R. F. Clippinger, A. D. Hestenes, D. T. McClay, Abraham Spitzbart. Dr. J. W. Green of the University of California and Dr. D. T. Perkins of Yale University have been appointed Benjamin Peirce instructors in mathematics for the academic year 1938–39.

Dr. E. J. Finan of the Catholic University of America has been promoted to the rank of assistant professor, effective October 1938.

The Rockefeller Foundation has extended the fellowship now held by Dr. A. S. Householder through the academic year 1938–39. He will continue his work in mathematical biophysics with Professor N. Rashevsky at the University of Chicago.

Assistant Professor R. R. McDaniel of the Virginia State College, Ettrick, Va., has been promoted to an associate professorship.

C. W. Trigg, formerly at Cumnock College, has been appointed an instructor at Los Angeles Junior College.

Dr. E. P. Wigner, who resigned a year ago from the faculty of Princeton University to become professor of theoretical physics at the University of Wisconsin, has been recalled to Princeton as Thomas D. Jones professor of mathematical physics.

The following seventy-one doctorates with mathematics as major subject were conferred during 1937 in the United States and Canada. The university, month in which the degree was conferred, minor subject (other than mathematics), and title of the dissertation are given in each case if available.

O. B. Ader, Kentucky, August. *Concerning affine invariants of convex regions.*

C. B. Allendoerfer, Princeton, January. *The embedding of Riemann spaces in the large.*

G. F. Alrich, Maryland, June. *Hyperconformal transformations.*

M. L. Bishop, New York, June, minor in physics. *On the algebra of finite square matrices with real quaternion elements.*

Archie Blake, Chicago, August. *Canonical expressions in Boolean algebra.*

R. P. Boas, Jr., Harvard, June. *Iterated Stiltjes transform.*

J. W. Calkin, Harvard, June. *Partial differential operators.*

Harold Chatland, Chicago, August. *The aysmptotic Waring problem for homogeneous polynomial summands.*

H. R. Cooley, New York, February, minor in physics. *Some systems of quasi-linear partial differential equations with identical principal parts.*

C. H. Denbow, Chicago, June. *A generalized form of the problem of Bolza.*

C. H. Dieckmann, California (Berkeley), May. *A study of the surfaces generated by a point that moves in certain relationships to two fixed lines of space.*

F. W. Dresch, California (Berkeley), August. *Applications of index numbers to the study of general economic equilibria.*

Melvin Dresher, Yale, June. *Multi-groups. A generalization of the notion of groups.*

W. A. Dwyer, Nebraska, June, minor in physics. *On certain fundamental identities due to Uspensky.*

F. G. Fisher, California (Berkeley), May. *The representation of quadric surfaces upon a space of nine dimensions.*

A. S. Galbraith, Harvard, June. *Second-order differential equations with irregular singular points of special types.*

F. C. Gentry, Illinois, June, minor in physics. *Groups of Cremona transformations in space of ternary type.*

M. J. Gottlieb, Washington University (St. Louis), June, minor in physics. *An investigation of polynomials orthogonal on a finite or enumerable set of points.*

Anne M. C. Grant, Bryn Mawr, June. *Asymptotic transitivity on surfaces of variable negative curvature.*

Louis Green, Chicago, August. *Systems of quadrics associated with a point of a surface.*

O. H. Hamilton, Texas, June, minor in economics. *Non-unique solutions of first order ordinary differential equations.*

B. A. Hausmann, Yale, June. *Quasi-groups.*

A. E. Heins, Massachusetts Institute of Technology, February. *The multidimensional operational calculus.*

A. S. Householder, Chicago, June. *The dependence of a focal point upon curvature in the calculus of variations.*

D. H. Hyers, California Institute of Technology, June, minor in physics. *Integrals and functional equations in linear topological spaces.*

J. E. Ikenberry, Cornell, June. *An involutorial transformation with a multiple correspondence on lines joining conjugate points.*

S. B. Jackson, Harvard, June. *Differential properties in the large of spherical curves.*

E. C. Kennedy, Rice, June. *Fuchsian groups of genus two.*

R. H. Kershner, Johns Hopkins, June. *The addition of convex curves and the Riemann zeta-function.*

L. A. Knowler, Iowa, May, minor in applied mathematics and physics. *Actuarial aspects of recent old age security legislation.*

D. B. Kohlmetz, Ohio State, December. *Certain problems of a special character in convex functions.*

O. E. Lancaster, Harvard, June. *Non-linear algebraic difference equations.*

Caroline A. Lester, Wisconsin, June. *A determination of the automorphisms of certain algebraic fields.*

L. L. Lowenstein, Cornell, September. *Linear equations with an infinity of unknowns.*

H. A. Luther, Iowa, July, minor in physics. *A third order irregular boundary value problem.*

Dorothy Manning, Stanford, June. *On simply transitive groups with transitive Abelian subgroups of the same degree.*

A. N. Milgram, Pennsylvania, June. *Decomposition and dimensions of closed sets in R''.*

Clemmer Mitchell, Cincinnati, June. *On Euler summability and one of its generalizations.*

Virginia Modesitt, Illinois, June, minor in astronomy. *Some singular properties of conformal transformations between Riemannian spaces.*

A. P. Morse, Brown, June. *Convergence in variation and related topics.*

N. A. Moscovitch, Chicago, March. *Studies of the inverse problem of the calculus of variations.*

Z. I. Mosesson, Harvard, June. *Maximal sequences of polynomials.*

D. C. Murdoch, Toronto, June. *Fundamental exponents in the theory of algebraic numbers.*

C. J. Nesbit, Toronto, June. *On the regular representations of algebras.*

A. C. Olshen, Iowa, May, minors in applied mathematics and commerce. *Transformations of the Pearson type III distribution.*

K. L. Palmquist, Kentucky, August. *Ideals in a quaternion algebra and Hermitian forms.*

E. W. Paxson, California Institute of Technology, June, minor in physics. *Analysis in linear topological spaces.*

Sallie E. Pence, Illinois, June, minor in astronomy. *The configuration of the double points of cubics of a pencil.*

Paul Pepper, Cincinnati, June. *An application of geometry of numbers to a generalization of a continued fraction.*

B. J. Pettis, Virginia, June. *On integration in vector spaces.*

M. H. L. Pryce, Princeton, June. major in mathematical physics. *On the theory of light quanta.*

W. T. Puckett, Jr., Virginia, June. *Some properties of cyclic elements of higher order.*

Harriet Rees, Chicago, August. *Ideals in cubic and certain quartic fields.*

Erwin Ritter, New York, June, minor in physics. *The problem of Plateau for a Schwarz chain.*

J. H. Schotland, New York, June. *An analysis of methods of plane curve fitting.*

Hyman Serbin, Pittsburgh, February. *Non-commutative polynomials.*

Ruth G. Simond, Michigan, August. *Relations between certain continuous transformations.*

M. F. Smiley, Chicago, August. *Discontinuous solutions for the problem of Bolza in parametric form.*

E. C. Stopher, Iowa, February, minor in applied mathematics. *Interrrelations of a family of operators on point sets and their canonical representation.*

W. C. Taylor, Wisconsin. *Asymptotic formulas for the Whittaker function.*

R. M. Thrall, Illinois, June, minor in physics. *Metabelian groups and trilinear forms.*

Annita Tuller, Bryn Mawr, June. *The measure of transitive geodesics on certain three-dimensional manifolds.*

F. A. Valentine, Chicago, March. *The problem of Lagrange with differential inequalities as added side conditions.*

R. H. Valentine, Massachusetts Institute of Technology, June, minor in physics. *Travel-time curves in oblique structures.*

R. W. Wagner, Michigan, June. *Multiple-valued functions in matrix space.*

Henry Wallman, Princeton, June. *Lattices and topological spaces.*

J. V. Wehausen, Michigan, August. *Studies in linear topological spaces.*

F. P. Welch, Illinois, June, minor in physics. *Singular non-linear first order differential equations containing a parameter.*

E. T. Welmers, Michigan, February. *Set functions and measurability conditions.*

L. S. Winton, Duke, June. *Compatible integro-differential systems.*

Shaun Wylie, Princeton, June. *Duality and intersection in general complexes.*

THE JOURNAL OF DOCUMENTARY REPRODUCTION

The following notice concerning a new quarterly may be important to mathematicians:

"The American Library Association, Chicago, announces the publication, beginning this month, of a quarterly *Journal of Documentary Reproduction.*"

The significance of this brief announcement is not apparent and, were it not amplified, would fail to reveal a forecast of the library of the future. The new *Journal* presents a glimpse of a library which will have at its readers' disposal the resources of the great libraries of the world—the rare and inaccessible books, manuscripts, and newspapers heretofore available alone to those few who could journey to distant museum or library.

Today the preservation of civilization's records are more certain because of the achievements of microphotography and related techniques in reproducing written and pictorial documents at low cost. The earliest printed books, unique manuscripts and folios, the crumbling files of news-

papers great and small, and other priceless historical records, are now being recorded. . . . Research manuscripts are reproduced for use of scholars for a few cents a page.

The *Journal of Documentary Reproduction* will keep librarians, archivists, scientists, and others abreast of ever-changing developments in the reproduction of documents. . . .

Outlining the scope of the new *Journal* in its first issue, Winter 1938, a thick 128-page periodical, Charles E. Rush, associate librarian of Yale University Library and chairman of the editorial board, points out that "attention will be given to the several allied processes of reproduction and duplication adapted to the dissemination of information and not at present within the range of the printing press." The *Journal* will supplement and keep up to date *Microphotography for Libraries*, the first two volumes on the subject ever published, which were edited by M. Llewellyn Raney, director of the University of Chicago Libraries, and published by the Association in 1936 and 1937.

Other features of the first issue of the *Journal* include a résumé of technical research by the Bureau of Standards; announcement of the newly organized American Documentation Institute, a central agency for microfilming service; a technical section with descriptions of new equipment, new services, and applications of new techniques; news of projects under way in libraries, museums, and elsewhere; and an extensive bibliography of photographic methods of documentary reproduction.

The *Journal*, published by the American Library Association, is edited by its Committee on Photographic Reproduction of Library Materials with Vernon D. Tate, chief of the Division of Photographic Reproduction of the National Archives at Washington as managing editor. The editorial board includes Mr. Rush as chairman; Mr. Fussler; Robert C. Binkley, professor of history, Columbia University; Keyes D. Metcalf, director, Harvard University Libraries; George A. Schwegmann, Jr., director, Union Catalogue, Library of Congress; and Paul Vanderbilt, librarian, Pennsylvania Museum of Art. The subscription price is $3 a year.

EXAMINATION QUESTIONS FOR THE FIRST WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION, APRIL 16, 1938

MORNING SESSION: 9:00 A.M. to 12:00 NOON.

1. A solid is bounded by two bases in the horizontal planes $z=h/2$ and $z=-h/2$, and by such a surface that the area of every section in a horizontal plane is given by a formula of the sort Area $= a_0z^3 + a_1z^2 + a_2z + a_3$ (where as special cases some of the coefficients may be 0). Show that the volume is given by the formula $V = 1/6\, h[B_1 + B_2 + 4M]$, where B_1 and B_2 are the areas of the bases, and M is the area of the middle horizontal section. Show that the formulas for the volume of a cone and of a sphere can be included in this formula when $a_0=0$.

2. A can buoy is to be made of three pieces, namely, a cylinder and two equal cones, the altitude of each cone being equal to the altitude of the cylinder. For a given area of surface, what shape will have the greatest volume?

3. If a particle moves in the plane, we may express its coordinates x and y as functions of the time t. If $x=t^3-t$ and $y=t^4+t$, show that the curve has a point of inflection at $t=0$, and that the velocity of the moving particle has a maximum at $t=0$.

4. A lumberman wishes to cut down a tree whose trunk is cylindrical and whose material is uniform. He will cut a notch, the two sides of which will be planes intersecting at a dihedral angle θ along a horizontal line through the axis of the cylinder. If θ is given, show that the least volume of material is cut out when the plane bisecting the dihedral angle is horizontal.

5. Evaluate the following limits:

(*a*) $\lim_{n\to\infty} \dfrac{n^2}{e^n}$.

(*b*) $\lim_{x\to 0} \dfrac{1}{x} \int_0^x (1 + \sin 2t)^{1/t}\, dt$.

6. A swimmer stands at one corner of a square swimming pool and wishes to reach the diago-

nally opposite corner. If w is his walking speed and s is his swimming speed ($s<w$), find his path for shortest time. [Consider two cases: (*a*) $w/s<\sqrt{2}$, and (*b*) $w/s>\sqrt{2}$.]

7. TAKE EITHER (*a*) OR (*b*).

(*a*) Show that the gravitational attraction exerted by a thin homogeneous spherical shell at an external point is the same as if the material of the shell were concentrated at its center.

(*b*) Determine all the straight lines which lie upon the surface $z=xy$, and draw a figure to illustrate your result.

AFTERNOON SESSION: 2:00 P.M. to 5:00 P.M.

8. TAKE EITHER (*a*) OR (*b*).

(*a*) Let A_{ik} be the cofactor of a_{ik} in the determinant

$$d = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}.$$

Let D be the corresponding determinant with a_{ik} replaced by A_{ik}. Prove $D=d^3$.

(*b*) Let $P(y)=Ay^2+By+C$ be a quadratic polynomial in y. If the roots of the quadratic equation $P(y)-y=0$ are a and b ($a\neq b$), show that a and b are roots of the biquadratic equation $P[P(y)]-y=0$. Hence write down a quadratic equation which will give the other two roots, c and d, of the biquadratic. Apply this result to solving the following biquadratic equation:

$$(y^2 - 3y + 2)^2 - 3(y^2 - 3y + 2) + 2 - y = 0.$$

9. Find all the solutions of the equation

$$y\,y'' - 2(y')^2 = 0$$

which pass through the point $x=1$, $y=1$.

10. A horizontal disc of diameter 3 inches is rotating at 4 revolutions per minute. A light is shining at a distant point in the plane of the disc. An insect is placed at the edge of the disc furthest from the light, facing the light. It at once starts crawling, and crawls so as always to face the light, at 1 inch per second. Set up the differential equation of motion, and find at what point the insect again reaches the edge of the disc.

11. Given the parabola $y^2=2mx$. What is the length of the shortest chord that is normal to the curve at one end?

12. From the center of a rectangular hyperbola a perpendicular is dropped upon a variable tangent. Find the locus of the foot of the perpendicular. Obtain the equation of the locus in polar coordinates, and sketch the curve.

13. Find the shortest distance between the plane $Ax+By+Cz+1=0$ and the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$. (For brevity, let

$$h = 1/\sqrt{A^2 + B^2 + C^2} \text{ and } m = \sqrt{a^2A^2 + b^2B^2 + c^2C^2}.)$$

State algebraically the condition that the plane shall lie outside the ellipsoid.

Note. Chairmen of mathematics departments may obtain copies of the examination questions, as long as the supply lasts, by writing for them to Professor W. D. Cairns, Oberlin, Ohio.

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THE MAY MEETING OF THE ILLINOIS SECTION

The nineteenth annual meeting of the Illinois Section of the Mathematical Association of America was held at the Southern Illinois State Normal University, Carbondale, on Friday and Saturday, May 13–14, 1938. The chairman of the Section, Professor W. B. Storm, presided.

The attendance was about sixty, including the following thirty members of the Association: Beulah M. Armstrong, Edith I. Atkin, H. G. Ayre, Walter Bartky, Laura E. Christman, C. E. Comstock, D. R. Curtiss, A. E. Gault, M. C. Hartley, E. W. Hellmich, E. C. Kiefer, Ruth G. Mason, J. R. Mayor, E. B. Miller, E. J. Moulton, Mary W. Newson, Echo D. Pepper, E. W. Ploenges, W. C. Randels, R. G. Sanger, H. A. Simmons, Norma K. Stelford, Eugene Stephens, Jessica Y. Stephens, R. C. Stephens, W. B. Storm, C. J. Stowell, M. E. Wescott, F. E. Wood, Alice K. Wright.

There were program sessions on Friday afternoon and Saturday morning. After a dinner Friday evening at beautiful Midland Hills Country Club the group adjourned to the Little Theatre on the campus and were entertained by two movies: "Parabola," a sound film designed and directed by Mr. Rutherford Boyd, showing mathematical design in art; and "Astronomy movies" planned and directed by Professor Walter Bartky of the University of Chicago. Professor Bartky was present and gave a short explanatory talk preceding the presentation of the film.

For the year 1938–39, Professor J. R. Mayor of Southern Illinois State Normal University was elected chairman; the department of mathematics of Knox College was asked to select one of its members for vice-chairman; and Professor C. N. Mills of Illinois State Normal University was elected secretary-treasurer. The next meeting will be held at Knox College, May 12–13, 1939.

The following papers were presented:

1. "Analytic geometry with complex coördinates and applications" by Professor F. E. Wood, Northwestern University.

2. "Some dissection problems" by Dr. Ruth G. Mason, University of Illinois.

3. "Applied operational mathematics" by Professor Eugene Stephens, Washington University.

4. "Teaching mathematical analysis in the junior college" by Dr. J. S. Georges, Wright Junior College.

5. "An application of the theory of summability" by Professor H. L. Garabedian, Northwestern University, introduced by Professor Curtiss.

6. "Some interesting problems in the study of porisms" by Dr. Josephine H. Chanler, University of Illinois, introduced by Dr. Echo D. Pepper.

7. "Relative claims of special education and general education upon the mathematics courses of the high school and junior college." A panel discussion by Dr. R. G. Sanger, University of Chicago, Professor Mabel Heren, Knox College, Professor E. W. Hellmich, Northern Illinois State Teachers College,

and E. B. Hexter, Belleville Township High School; Professor J. R. Mayor, Southern Illinois State Normal University, Chairman.

In the absence of the authors the papers by Dr. Georges and by Professor Heren were read by title only.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. The basic formulas of complex (or circular) coördinates were developed in a series of simple, logical steps by Professor Wood; also an attempt was made to show how these coördinates have been applied to metric theorems regarding triangles and their related points, lines, and circles.

2. Dr. Mason defined the solution of a dissection problem to be a construction which divides a given figure or figures into parts which by euclidean transformations can be converted into a pre-assigned figure or figures. Methods were shown for cutting up two squares so that the pieces would form one square, for transforming any rectangle into a square, and for separating a square into four pieces that can be assembled into an equilateral triangle. Dudeney's discussion of the Greek Cross was generalized for the stepagon. The step process was explained and the possibility of so-called paradoxes was illustrated by the geometrical "proof" that $441=442$.

3. Professor Stephens gave definitions for operations with permissible algebraic transformations and identities by the method of abstraction of operators. He included the operator D, the partial operator and its generalization, and the noncommutative operator XD. The uses and applications of these then followed, with references to the history and literature covering over one hundred years.

4. To meet the needs of students who seek a general education and those who prepare for professional schools—the chief group in a public Junior College—Dr. Georges reported that the freshman course in mathematics at Wright Junior College includes materials of college algebra, trigonometry, analytic geometry, and elements of calculus, stressing the idea of mathematical analysis, and presenting mathematics as a method of thinking.

5. Professor Garabedian discussed briefly applications of the theory of summability to a problem in the flow of heat in the domain of mathematical physics. The principal supporting theorems are classical ones due to Fejer and Bromwich. The application is due to C. N. Moore of the University of Cincinnati. The speaker has given the shortest and most recent proof of Bromwich's theorem and has published results on applications of this theorem.

6. In her paper Dr. Chanler defined a porism as a proposition affirming the possibility of finding such conditions as will render a certain problem indeterminate or capable of an infinite number of solutions. Illuminating examples were presented from elementary geometry, including the classic theorem of Poncelet in 1822. Modern illustrations and discussions of the Poncelet theory were taken from the literature, including some which relate the poristic polygons to the theory of the elliptic cubic. Algebraic generalization leads to the poristic double binary form $F_{k,K}\equiv(\alpha t)^k(aT)^K$, investigated by A. B. Coble. Intimately

related to certain poristic forms are the involution curves like the Lüroth quartic curve. Examples of poristic double binary forms and involution curves were given from the writer's papers on these subjects.

7. In this panel discussion Dr. Sanger emphasized the need of requiring mathematics in the high school, but not of all college students. He indicated how a plan of giving a choice of courses, based on the need for technical knowledge, was carried out at the University of Chicago.

Professor Hellmich defended required mathematics courses in high school and college and presented a plan for three years of required mathematics, two of which would come in the high school, and one in college. Rather than defending the traditional courses in mathematics in both high school and college, a broad outline of work was presented for the three years which he thought much better adapted to the claims of general education.

Mr. Hexter suggested that criticisms of high school teaching by college teachers are reflections of the type of work these professors are doing. He raised some questions about the preparation of college teachers for teaching and mentioned the possible desirability of high school experience. A defense was given for trigonometry and college algebra in the high school.

EDITH I. ATKIN, *Secretary*

THE APRIL MEETING OF THE OHIO SECTION

The twenty-third annual meeting of the Ohio Section of the Mathematical Association of America was held at the Ohio State University, Columbus, Ohio, on Thursday, March 31, 1938, with an afternoon session, a dinner, and an evening session. Professor O. E. Brown, chairman of the Section, presided at these sessions. The Section was happy to have Professor Karl Menger of the University of Notre Dame as guest-speaker.

Ninety-two persons registered attendance, fifty-four of whom were members of the Association, namely: W. E. Anderson, Max Astrachan, I. A. Barnett, P. E. Baur, H. M. Beatty, H. A. Bender, Henry Blumberg, M. G. Boyce, J. B. Brandeberry, O. E. Brown, R. S. Burington, I. W. Burr, W. D. Cairns, F. E. Carr, E. H. Clarke, Rufus Crane, O. L. Dustheimer, T. M. Focke, N. A. Gilbert, B. C. Glover, R. C. Hildner, E. J. Hirschler, E. M. Justin, L. C. Knight, H. W. Kuhn, A. C. Ladner, Lincoln LaPaz, Florentina Mathias, Karl Menger, C. C. Morris, Max Morris, J. R. Musselman, J. R. Overman, W. A. Patterson, Jesse Pierce, H. S. Pollard, S. E. Rasor, C. E. Rhodes, R. F. Rinehart, N. S. Risley, S. A. Rowland, W. G. Simon, Mary E. Sinclair, H. E. Stelson, Otto Szász, C. F. Thomas, Fern Welker, R. B. Wildermuth, F. B. Wiley, C. O. Williamson, J. B. Winslow, C. R. Wylie, Jr., B. F. Yanney, C. H. Yeaton.

The following officers were elected for the coming year: Chairman, C. O. Williamson, College of Wooster; Secretary-Treasurer, Rufus Crane, Ohio Wesleyan University; Member of Executive Committee, C. E. Rhodes, Ohio State University; Member of Program Committee, H. A. Bender, University of

Akron. It is expected that the next meeting will be held on Thursday, April 6, 1939, at the Ohio State University.

The following papers were read:

1. "The application of determinants and projective transformations to nomography" by the chairman of the Section, Professor O. E. Brown, Case School of Applied Science.

2. "The line of images" by Professor J. R. Musselman, Western Reserve University.

3. "A series of line involutions in S_3 defined as point transformations of a V_4^2 of S_5 into itself" by Dr. C. R. Wylie, Jr., Ohio State University.

4. "The standard deviation of the medians of small samples" by Professor H. S. Pollard, Miami University.

5. "A note on the elementary symmetric functions" by H. Reingold, University of Cincinnati, introduced by Professor Barnett.

6. "Solutions of $dx/dt=(1/2x)\sum_{h=0}^{\infty} f_h(t)x^h$, $f_0(t)\not\equiv 0$, in the vicinity of branch points, in terms of infinite series of definite integrals" by Professor Jesse Pierce, Heidelberg College.

7. "The foundations of projective and affine geometry" by Professor Karl Menger, University of Notre Dame, by invitation.

8. A symposium, "The mathematician in action" by Professors C. C. Morris, B. F. Yanney, and Karl Menger.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Brown stated the fundamental problem of alignment chart design, namely, given a relation, $F(u, v, w)=0$ in three variables u, v, w, to determine three curves C_u, C_v, C_w, and graduations upon them, such that the graduations corresponding to any three particular values of those variables will lie on a straight line if and only if those particular values satisfy the given relation. He illustrated by a number of examples, the role played by determinants of the third order in solving this problem, and the possible variation of the solution through projective transformations.

2. This paper appears in this issue of the MONTHLY.

3. A family of lines in five dimensions, with the property that a unique line of the system passes through any given point, can be used to map a quadratic primal into itself. If the primal be the one which represents the lines of three dimensions, such a mapping relation defines a line involution in three dimensions. Dr. Wylie considered five systems of lines with the above property; namely, the lines through a fixed point, the lines meeting a fixed line and a fixed S_3, the lines meeting two fixed planes, the lines meeting a fixed plane and two fixed S_3's, and the lines meeting four fixed S_3's. The associated line involutions in three dimensions range in order from one to seven, and include a number of well known transformations as special cases.

4. Professor Pollard showed that the formula which is ordinarily used to calculate the standard deviation of the median yields only an approximation to its

actual value, and that for samples which do not contain a fairly large number of items, and for distributions in which the median is located at a point of relatively small frequency, this approximation may be untrustworthy. He proposed another method of measuring the stability of the median, which involved the determination of the frequency function according to which the medians of samples drawn from a given distribution are distributed, and the calculation of the third quartile of this frequency distribution of medians. The deviation of this third quartile from the median of the distribution was used as a measure of the probable error of the median.

5. Mr. Reingold discussed an interesting property of the elementary symmetric functions $E_1, E_2, \cdots, E_n$ of the independent variables $x_1, x_2, \cdots, x_n$. He showed that the Jacobian of the elementary symmetric functions is, except possibly for sign, equal to the Vandermonde determinant of the variables $x_1, x_2, \cdots, x_n$; also that the Jacobian of the n coefficients of the characteristic equation of the nth order square matrix a_{ij}, with respect to any set of n elements $a_{r_1s_1}, a_{r_2s_2}, \cdots, a_{r_ns_n}$, of the matrix is, except possibly for sign, equal to a generalized determinant of Vandermonde $\left| a_{s_1r_1}^{(i-1)} a_{s_2r_2}^{(i-1)}, \cdots, a_{s_nr_n}^{(i-1)} \right|$, $(i=1, 2, \cdots, n)$, where a determinant of nth order is written by exhibiting its ith row.

6. Professor Pierce discussed solutions of a differential equation of the first order and first degree with a pole of order one with respect to the dependent variable. Two solutions with the initial conditions $x(t_0)=0$ are found in terms of infinite series of definite integrals. The method of finding the formal solution functions is similar to that used by the author in finding solutions of systems of linear differential equations in the vicinity of singular points (this MONTHLY, November, 1936) and in finding solutions of systems of differential equations in terms of infinite series of definite integrals (*Duke Mathematical Journal*, December, 1937).

7. Professor Menger gave a brief exposition of the foundation of geometry on a few simple assumptions concerning two operations in a class of undefined elements, the operations corresponding to joining and intersecting of the linear elements of space. The theory may be called *algebra of geometry* on account of its analogy to algebra of numbers, based in its modern abstract form on assumptions concerning two operations in a class of undefined elements, corresponding to addition and multiplication of numbers; and to algebra of classes which likewise has been developed as a theory of two operations, corresponding to joining and intersecting of classes. (This latter theory can be obtained as a special case of algebra of geometry.) From the above mentioned basis we can derive projective, affine, and non-euclidean geometry, each one by adding a specific assumption concerning parallelism.

8. Professor Morris spoke of the mathematician in business, and in particular analyzed the mathematical aspects of the recovery program of President Roosevelt, showing why he took the steps he did, their result, and prophesying what his future steps will be. Professor Yanney spoke of the mathematician as a teacher, touching lightly on his qualifications, his opportunities for adventure

and pleasure in every new class, the flexibility of his methods, and incidentally, his strategic position today in our educational system. Professor Menger spoke of the mathematician in research, pointing out the stimulating influence of teaching activity upon research. Questions which arise from reading, lecturing, or from intelligent students should be answered before one's attention weakens or shifts. Six months might well be a limit beyond which one must not postpone attack.

RUFUS CRANE, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The twenty-seventh regular meeting of the Iowa Section of the Mathematical Association of America was held at Morningside College, Sioux City, Iowa, on Friday and Saturday, April 15–16, 1938, in conjunction with the fifty-second regular meeting of the Iowa Academy of Science. The chairman, Professor L. E. Ward, presided.

The attendance was about twenty-seven, including the following fourteen members of the Association: W. E. Ekman, Cornelius Gouwens, M. E. Graber, Dora E. Kearney, R. B. McClenon, E. E. Moots, E. N. Oberg, H. L. Rietz, Fred Robertson, W. J. Rusk, E. R. Smith, L. E. Ward, C. W. Wester, Roscoe Woods.

On Friday evening the members and friends of the Association and the Iowa Academy of Science had a joint dinner. The officers of the Section elected for 1938–39 are as follows: Chairman, E. E. Moots, Cornell College; Vice-Chairman, A. T. Craig, University of Iowa; Secretary-Treasurer, Cornelius Gouwens, Iowa State College. A resolution expressing the appreciation of the members of the Section for the hospitality and courtesy extended to them by their host, Morningside College, was adopted at the business session. The invited address was given by Professor R. B. McClenon of Grinnell College. The following seventeen papers were read:

1. "Osculatory interpolation formulas for finding approximate values of definite integrals" by Professor J. F. Reilly, State University of Iowa, by title.

2. "Generalization of the formula of Leibniz" by Fred Robertson, Iowa State College.

3. "What is a number?" by Professor C. W. Wester, Iowa State Teachers College.

4. "Note on the Gamma function" by Professor W. J. Rusk, Grinnell College.

5. "Some properties of a class of polynomials associated with Bateman's k-function" by J. M. Bates, Iowa State College, introduced by Professor E. R. Smith.

6. "The semi-invariants of Thiele of a binomial distribution" by Professor A. T. Craig, State University of Iowa, by title.

7. "Lagrange and his influence on modern mathematics" by Professor R. B. McClenon, Grinnell College.

8. "Periodogram analysis of selected agricultural prices" by R. L. Anderson, Iowa State College, introduced by the Secretary.

9. "An approximate solution by a generalized method for a square clamped plate" by C. J. Thorne, Iowa State College, introduced by the Secretary.

10. "Some problems dealing with the flow in rivers" by Professor E. E. Moots, Cornell College.

11. "Some properties of the orthic triangle" by Professor Helen K. Milleson, Buena Vista College, introduced by Professor Roscoe Woods.

12. "On the 'Curve of Deaths' and the associated 'Curve of Lives' " by Professor H. L. Rietz, State University of Iowa.

13. "An application of trilinear coördinates to some simple boundary value problems" by Professor D. L. Holl, Iowa State College, introduced by the Secretary.

14. "A rule for solving integral equations having special kernel functions" by Professor E. N. Oberg, State University of Iowa.

15. "Hysteretic probability" by Professor E. S. Allen, Iowa State College, introduced by the Secretary.

16. "Mathematical rigor and validity" by Professor M. E. Graber, Morningside College.

17. "An asymptotic formula for the small non-vanishing zeros of confluent hypergeometric functions" by Professor E. R. Smith, Iowa State College.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

2. J. A. Joseph in the January 1938 issue of this MONTHLY shows that the theorem of Leibniz

$$D^n(u \cdot v) = \sum_i \binom{n}{i} D^i u D^{n-i} v,$$

known to be valid for positive integral values of n, holds for negative integral values. In the present paper Mr. Robertson showed by the Bourlet method that the theorem is valid for all real values of the index n.

3. To clear up the confusion and mysticism resulting from trying to think number as cardinal number, and of a cardinal number as a collection of objects or else as a symbol for the collection, Professor Wester stated that a simple definition of number is needed. It should be authoritative to serve as a guide to teachers, textbook makers, and school authorities generally. Such definitions are now possible though manifestly impossible before the researches of the last century.

4. First by using the trigonometrical form of the integral for $B(m, n)$ and its expression in terms of Γ-functions and transforming the integrand function by multiplication by $(\sin^2\theta+\cos^2\theta)^p$, and expanding the new integrand function, Professor Rusk was able to express it as a summation in B-functions or Γ-func-

tions which were identities in p for all positive integral values of p. Secondly by replacing $\sin^{2p}\theta$ by $(1-\cos^2\theta)^p$ and expanding he was able to get analogous formulas which hold so long as m, n, and p are greater than zero. Then by using the formula $(n+k)=(n+k-1)_k\Gamma(n)$ he deduced formulas which might be considered analogous to Vandermonde's Theorem when the p was a positive integer, and as generalizations when the m, n, and p are positive but not integers.

5. The polynomials discussed by Mr. Bates are associated with Bateman's k-function which is a special type of Whittaker's function $W_{k,m}(z)$, corresponding to the case $m=1/2$. The family of polynomials is defined by the generating function

$$e^{2sx/(1+s)} = \sum_{n=0}^{\infty} u_n(x)s^n,$$

where $u_n(x)$ is the nth polynomial. A differential form, an integral form, difference and differential equations, and orthogonal relations were derived. Also two expansions for the zeros of the polynomials were obtained in terms of the zeros of the Bessel functions.

6. In this note, Professor Craig showed that $\lambda_{k+1}=pqd\lambda_k/dp$ is a recursion formula for the semi-invariants of Thiele of a binomial distribution where $\lambda_1=N_p$, p being the probability of occurrence of an event in a single trial, $q=1-p$, and N the number of trials. An explicit formula for λ_k was also found.

7. Professor McClenon outlined the three periods in the life of Lagrange, the Turin, Berlin, and Paris periods, and quoted from letters and other sources to show something of the circumstances under which Lagrange did his work. He also outlined some of the most outstanding of Lagrange's contributions to mathematics, and showed how many fields are making use of ideas which go back to Lagrange.

8. The periodogram analysis of Professor Arthur Schuster was used by Mr. Anderson to study the periodicity of cyclical variation in wholesale commodity prices for the period 1890–1937. If there is a regular period, P, in the variation, this variation can be represented by a Fourier series, but only the first harmonic term, $R_1 \sin(2\pi x/p+\phi_1)$, is considered. Rye prices disclosed a marked period of 39 months in the cyclical variation, rubber disclosed no true period, and hides revealed a true period of 42 months for only the pre-war data.

9. The generalization may be illustrated by a boundary value problem in which the Ritz and Boussinesq methods are made special cases of a more general functional method. This involves the use of a functional in the determination of the coefficients of the expansion function. Mr. Thorne applied the method to a homogeneous, square, clamped plate with a center point load. The result obtained (1) yields a center deflection of .02245 Pa^2/N in close agreement with the work of Barta and Marcus; (2) satisfies the differential equation and boundary equations at all points; and (3) furnishes a line integral of boundary shear yielding a value .003% high.

10. If we introduce into a river flowing at low water stage an additional

volume of water, a swell or flood wave is produced and propagated in a certain manner. A simple flood wave is a long, positive wave superposed upon a river in steady flow. After developing the usual equations of continuity and motion Professor Moots showed that the velocity of a small flood wave when treated as the rate of movement of a constant flow, is equivalent to 1.5 times the mean channel velocity. The differential equation of the monoclinal or rising flood wave was next developed and integrated. Profiles, plotted for a 12-foot floodstage and an 8-foot normal flow, with river slopes ranging from .002 to .0004, show a variation of effective wave lengths ranging from 19.2 miles to 96 miles. This wave, rather often, closely characterizes the flow in large rivers during certain flood stages.

11. Professor Milleson presented an enumeration of triangles similar to the pedal triangle of the orthocenter, in an original arrangement of relations existing between the orthic triangle and various systems of circles, especially the Miquel circles and two coaxaloid systems: the Tucker circles, derived from coaxal circles through the Brocard points of the triangle of reference, and another system, of which the circumcircle is also a member, whose radii are changed by the constant multiplier 1/2 and whose radical axis is perpendicular to the Euler line.

12. In this paper Professor Rietz introduced a set of concepts in a brief and simple manner leading up to the use of continuous biometric functions in the characterization of mortality as a function of age. He followed this by an exposition of some of the interesting properties of the "curve of deaths" and the associated "curve of lives."

13. Employing the distances (p_1, p_2, p_3) from the sides of an equilateral triangle as the trilinear coördinates of an arbitrary point in the plane of the triangle, one can solve several boundary value problems involving the Laplacian operator. Professor Holl derived known results in polynomial form for the torsion and flexure problems of prisms whose cross sections are the equilateral triangle and circle, the circular annulus and semi-circle. A slight generalization was obtained by generalizing the reference triangle to be an isosceles triangle.

14. The purpose of Professor Oberg's paper was to exhibit a simple rule for solving integral equations of the type

$$\phi(x) = f(x) - \lambda \int_{-1}^{+1} K(x \pm t)\phi(t)dt$$

when λ is not a characteristic number, and the functions $f(x)$ and $K(x \pm t)$ are assumed to be continuous in the intervals $(-1 \leqq x \leqq 1)$ and $(-1 \leqq x,\ t \leqq 1)$, respectively. Previous writers have solved the same equation by reducing it to a differential equation or by the application of the general Fredholm method. In the present instance, the solution was found by means of least squares.

15. In this paper Professor Allen considered probabilities which depend on the outcome or on the probability of previous events. Certain classical problems

are of this type, but the Markoff chain was the first to be intensively studied. Onicescu and Mihoc have considered a much more general case, investigating particularly the existence of limiting probabilities. The present paper was concerned especially with Pólya's "contagion in probability," and Eggenberger's and Linder's developments of it. Tests applicable to empirical data, to determine the existence and magnitude of hysteretic influence, were discussed.

16. Little progress was made in unifying various mathematical domains until the dawn of the twentieth century. Professor Graber spoke of the work of E. H. Moore, of the Logistic Formalists exemplified by Russell, Whitehead, and Keyser, of the Intuitionist School represented by Brouwer, Weyl, and Bell. For applied mathematics or meta-mathematics all we need to ask is (with Kempe) whether, if given a certain class of objects and a certain class of relations, the ordered groups of these objects do or do not satisfy the relations. The acceptance of this definition permits us to employ deduction, induction, and experimentation in building up our mathematical system.

17. Professor Smith showed that under certain restrictions the small roots of the confluent hypergeometric functions, and the associated Whittaker functions $W_{k,m}$ could be given by a formula obtained from an expansion of the functions in powers of the parameter $1/4k^2$ which satisfies the differential equation which defines the functions.

CORNELIUS GOUWENS, *Secretary*

THE FIFTEENTH ANNUAL MEETING OF THE NEBRASKA SECTION

The fifteenth annual meeting of the Nebraska Section of the Mathematical Association of America was held in Hastings, Nebraska, on Friday afternoon, May 6, 1938, in conjunction with the annual meeting of the Nebraska Academy of Sciences. Professor R. M. McDill of Hastings College was chairman.

The attendance was thirty-four, including the following fifteen members of the Association: A. K. Bettinger, W. C. Brenke, A. L. Candy, W. A. Dwyer, J. M. Earl, M. G. Gaba, Emma E. Hanthorn, W. G. Leavitt, F. E. Marrin, R. M. McDill, U. G. Mitchell, J. D. Novak, T. A. Pierce, Lulu L. Runge, S. T. Sanders, Jr.

Officers for the ensuing year were elected as follows: Chairman, W. C. Brenke, University of Nebraska; Secretary-Treasurer, T. A. Pierce, University of Nebraska; Member of Executive Committee, J. M. Earl, Municipal University of Omaha.

The following program was presented:

1. "How may mathematics secure and hold in the curriculum the position which it deserves?" by Professor U. G. Mitchell, University of Kansas.

2. "A note on the problem of interpolation by polynomials" by Professor J. M. Earl, Municipal University of Omaha.

3. "Certain facts about trigonometric series" by Professor W. C. Brenke, University of Nebraska.

4. "A simple approximation for π" by Professor M. G. Gaba, University of Nebraska.

5. "Planetary orbits in relativity" by W. G. Leavitt, University of Nebraska.

6. "Orthogonal and Appell polynomials" by Dr. M. S. Webster, University of Nebraska, introduced by the Secretary.

7. "The mathematics of factor analysis" by J. D. Novak, University of Nebraska.

Abstracts of the papers follow, numbered in accordance with their place on the program:

1. Professor Mitchell traced briefly the changes in the position of mathematics in secondary education during the last fifty years, pointing out the difficulties encountered in the attempt to give all children the same kind of schooling. The present situation is one important phase of the problem of mass education. The tremendous increase in high school enrollment since 1900 has forced a reorganization of the mathematical curriculum and further differentiation of both courses of study and methods of teaching must be made to meet the needs of individual and group differences. The development of an adequate evaluation program, which shall determine scientifically and objectively the effects of schooling upon the entire personality of pupils, is the greatest need and such a program is already under way.

2. Professor Earl dealt with interpolation to a given function of x by means of polynomials of degree $n(n=0, 1, 2, 3, \cdots)$ on the interval $(-1, 1)$. These polynomials assume the value of the function at $(n+1)$ equally spaced points. For a sub-interval whose length is of the order of the negative one half power of n, an upper bound for the magnitude of the polynomial of degree n is obtained; this upper bound contains n and an upper bound for the magnitude of the given function.

3. In this paper Professor Brenke considered the expansion of some specific functions in trigonometric series summable $(C, 1)$.

4. This paper appeared in the June-July issue of this MONTHLY.

5. Mr. Leavitt gave a brief description of the general relativity theory, leading through a specialization of the general line element to the differential equation of the planetary orbit. An approximation yielded the observed advance in perihelion. The integration of the orbital equation in terms of elliptic functions was considered and the shape of the orbit under various initial conditions was described.

6. Systems of Appell polynomials and systems of orthogonal polynomials are well known. Dr. Webster gave a concise proof, using recurrence relations, that there is essentially only one system of Appell polynomials which are also orthogonal.

7. Recent investigations in psychological measurements have made use of an increasing amount of higher mathematics. In particular, Professor L. L. Thurstone of the University of Chicago has developed a theory of multiple

factor analysis in which matrices and the geometry of hyperspaces play a fundamental part. Mr. Novak presented an outline of factor analysis and discussed some of the mathematics encountered in this theory.

T. A. PIERCE, *Secretary*

THE SPRING MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at the U. S. Naval Academy, Annapolis, Maryland, on Saturday, May 7, 1938. The chairman, Professor Gillie A. Larew, presided over both sessions, morning and afternoon. The Section was happy to have Professor Hans Rademacher of the University of Pennsylvania as guest-speaker.

The attendance was fifty-four, including the following thirty-one members of the Association: O. S. Adams, N. H. Ball, G. A. Bingley, Archie Blake, C. C. Bramble, Paul Capron, Randolph Church, G. R. Clements, Abraham Cohen, A. E. Currier, L. S. Dederick, J. A. Duerksen, P. J. Federico, E. J. Finan, Michael Goldberg, L. M. Kells, W. D. Lambert, A. E. Landry, Gillie A. Larew, C. L. Leiper, C. M. Lennahan, S. B. Littauer, T. W. Moore, O. J. Ramler, C. H. Rawlins, Jr., J. N. Rice, R. E. Root, J. B. Scarborough, F. W. Sohon, John Tyler, John Williamson.

At a business meeting the following officers were elected for next year: Chairman, Michael Goldberg, Bureau of Ordnance, Navy Department; Secretary-Treasurer, S. B. Littauer, U. S. Naval Academy; additional members of the Executive Committee, Mildred E. Taylor, Mary Baldwin College, and Florence P. Lewis, Goucher College. An invitation to hold the next meeting at the University of Maryland, College Park, Maryland, on December 10, 1938, was accepted.

After an address of welcome by Admiral Wilson Brown, Superintendent of the Naval Academy, the following five papers were read:

1. "The relation of the range of a sample to the standard deviation of the population" by Dr. L. S. Dederick, Aberdeen Proving Ground.

2. "Some naval tactics in vector analysis" by Professor C. H. Rawlins, Jr., Postgraduate School, U. S. Naval Academy.

3. "Infinite continued radicals and iteration of polynomials" by Aaron Herschfeld, Social Security Board, Washington, D. C., introduced by the Secretary.

4. "Mathematical applications common to practical meteorology" by Lieut. H. B. Hutchinson, Postgraduate School, U. S. Naval Academy, introduced by Professor Root.

5. "Partitio numerorum" by Professor Hans Rademacher, University of Pennsylvania, by invitation.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Dr. Dederick gave a simple expression in terms of a definite integral for the probability that the range of a sample does not exceed any given multiple of the standard deviation of the population from which it is drawn.

2. Professor Rawlins applied methods of vector algebra and calculus to a few problems in tactical graphics in which certain aircraft maneuvers, relative to a given set of moving ships, were to be performed in minimum length of time. From the resulting vector equations there were deduced suitable geometrical constructions for solving the problems.

3. Mr. Herschfeld showed that the infinite set of possible representations of a number by an infinite continued radical employing only positive integers has the power of the continuum. He used some of these representations to generalize the results of Pincherle and Lucas on quadratic polynomials to obtain explicit expressions for the iterates of special polynomials of any degree.

4. Meteorology originally attracted the keenest scientific minds. It failed to hold them during the nineteenth century. Investigation ceased, practical weather forecasting seemed to be the primary objective. Recently the science has been rejuvenated through the availability of exhaustive data from the third dimension of the atmosphere. These data are handled in the works of V. and J. Bjerknes, C. G. Rossby, S. Petterssen, and others by use of the methods of higher mathematics. Some of these applications were discussed by Lieutenant Hutchinson who showed the substantiation of the theory by the observational facts and the extent of their practical value to the field meteorologist of to-day.

5. Professor Rademacher gave a historical account of the problems involved in the partition of numbers. These problems date back to Euler who, in his "Introductio in analysin infinitorum" (1748), included a chapter "De partitione numerorum." In recent times they have been classified as forming a part of the additive theory of numbers. Euler's methods and results were sketched, in particular his "pentagonal number theorem," in which appears, for the first time, a special case of a theta-function. The speaker then referred to the modern analytic methods introduced into this field by a paper of G. H. Hardy and S. Ramanujan (1917), which contains the asymptotic formula for $p(n)$, the number of unrestricted partitions of n. He finally mentioned the convergent series for $p(n)$ published by himself in 1937, and recent results by Dr. H. S. Zuckerman and himself concerning identities which Ramanujan had conjectured.

MICHAEL GOLDBERG, *Secretary*

THE HISTORY OF BLISSARD'S SYMBOLIC METHOD, WITH A SKETCH OF ITS INVENTOR'S LIFE*

E. T. BELL, California Institute of Technology

1. It sometimes happens in the history of mathematics that the credit for a particular method is commonly ascribed to another than its originator. In the interests of historical justice, such oversights should be corrected, where the facts are known. A conspicuous instance is the extremely useful symbolic method expounded and widely applied by Edouard Lucas [3, 5],† and usually attributed to him. The kernel of this symbolic method is described below, §2.

This representative notation [1], to use the designation of its inventor, or symbolic method, as Lucas [3, 5] called it, or umbral calculus [8, vol. 31], was fully developed in 1861 by John Blissard [1], in a mathematical journal with a wide circulation among mathematicians, fifteen years before Lucas published his first papers [3, 4] on the subject. Although J. W. L. Glaisher [6, p. 159] called attention to Blissard's indisputable priority forty years ago, it seems still to be not generally known to those who use or refer to the symbolic method. Thus, for example, the symbolic treatment of the Bernoulli numbers, first given by Blissard [1, vol. 4], is assigned to Lucas [3]. Regarding this treatment, L. E. Dickson [7] expressed the opinion of most students of the Bernoulli (and allied) numbers when he stated that, "*Bernouillian numbers* are most conveniently employed in the symbolic notation of Lucas. . . ."

There are also occasional misconceptions of the abstract nature of Blissard's method itself. It is sometimes thought to be non-elementary, or transcendental, because it may, but need not, make use of strictly formal operations on power series. Such a use of power series is algebraic, not analytic, as no limiting process is involved, and convergence is irrelevant. As pointed out elsewhere [8], the appropriate algebra of power series, and hence also Blissard's tacit use of this algebra, is no more 'transcendental' than mathematical induction is. In fact, both the 'elementary' method of N. Nielsen [10] for the Bernoulli numbers, and Blissard's symbolic method, contain precisely the same amount of transcendence: each must use mathematical induction (as must any algebraic algorithm on countable sequences of elements) for a complete proof of the results obtained by it, and neither requires any other appeal to the infinite. Hence Nielsen's claim [10, p. IX] that his "elementary theory of the Bernoulli Numbers . . . is much more fundamental than the symbolic method, developed notably by Lucas . . . ," is without foundation. Likewise for his justification of this claim, "because a symmetric polynomial [as defined by him] gives immediately results concerning all the numbers B_n, E_n, T_n." Blissard's method gives at once [9] the 15 possible most general formulas connecting any 1, 2, 3, or 4 of the sequences of Bernoulli, Euler, Genocchi, and Lucas; and as these formulas, in the cases where precisely one of the four sequences is concerned, are both necessary and

* Presented for Slaught Memorial Volume of the MONTHLY.

† Numbers in square brackets refer to the list of references at the end of the paper.

sufficient to define each sequence, it follows that the definition by symmetric polynomials contains nothing that is not implicit in the completely general formulas of Blissard's type. Moreover, the symbolic treatment of the Bernoulli and allied numbers is but one detail in the applications of the symbolic method.

2. For a statement of the essentials of Blissard's method, any of his papers [1], or Lucas [5], or Bell [11], may be consulted. A recapitulation of the fundamental definition and its immediate consequences will suffice here to recall the nature of Blissard's calculus.

Let $a_n, b_n, \cdots, x_n, \cdots$ $(n=0, 1, \cdots)$ be any sequences of real or complex numbers; such numbers will always be denoted by small Latin letters *with suffixes*. A letter, say x, *without a suffix*, is the *representative*, or *umbra*, of the sequence x_n $(n=0, 1, \cdots)$ derived (notationally) from x in an obvious manner. Two umbrae, x, y, are defined to be *equal*, $x=y$, if and only if $x_n=y_n$ $(n=0, 1, \cdots)$. It follows that umbral equality has the properties of equality in abstract algebra (rings, fields, etc.): for any x, y one of $x=y$, $x\neq y$ is true; umbral equality is symmetric, reflexive, and transitive.

Blissard [1] observed that if x_n is written as a symbolic, or umbral, power, x^n, such powers have some of the properties of powers as in abstract algebra. In particular, if $x=y$, then $x^n=y^n$, $n=0, 1, \cdots$; that is, $x_n=y_n$, $n=0, 1, \cdots$. This is merely half the definition of equality restated; nevertheless, it is the germ of the entire calculus. For if x, y, z are any umbrae such that $x\neq y$ and

$$z_n = \sum_{s=0}^{n} (n, s)x_s y_{n-s}, \qquad n = 0, 1, \cdots,$$

where (n, s) is the binomial coefficient $n!/s!(n-s)!$, $(0, 0)=1$, we may write

$$z = x + y,$$

understanding by this,

$$z_n \equiv z^n = (x + y)^n, \qquad n = 0, 1, \cdots,$$

in which the umbral binomial is to be expanded as if x, y were ordinaries (real or complex numbers), and, after expansion, all exponents are to be degraded to suffixes. It is emphasized that in all rational operations on umbrae, as in the expansion of $(x+y)^n$, the exponents 0, 1 must be treated precisely as any other non-negative integer exponents, $x^0\equiv x_0$, $x^1\equiv x_1$, $(x+y)^0\equiv x_0y_0$, $(x+y)^1\equiv x_0y_1+x_1y_0$.

The umbral binomial theorem just described can be extended in an obvious way to multinomials. Thus if $x, y, \cdots, u$ are all unequal,

$$z_n = \sum M_{\alpha,\beta,\cdots\gamma} x_\alpha y_\beta \cdots u_\gamma,$$

where the M's are the multinomial coefficients occurring in the expansion of $(X+Y+\cdots+U)^n$, is equivalent to the umbral equality

$$z = x + y + \cdots + u,$$

that is, to

$$z_n \equiv z^n = (x + y + \cdots + u)^n, \qquad n = 0, 1, \cdots .$$

Hence, if for any umbra t, and a real complex variable θ, $\exp t\theta$ denotes the formal MacLaurin series on the right of the next,

$$\exp t\theta \equiv \sum_{n=0}^{\infty} t^n\theta^n/n! \equiv \sum_{n=0}^{\infty} t_n\theta^n/n!,$$

it follows that the three statements

$$\exp z\theta = \exp x\theta \exp y\theta \cdots \exp u\theta,$$
$$\exp z\theta = \exp (x + y + \cdots + u)\theta,$$
$$z = x + y + \cdots + u,$$

are equivalent, when, as in the algebra of formal power series [**8,** vol. 25], equality of such series, $\exp t\theta = \exp v\theta$, is defined to be equivalent to $t = v$.

From here on the umbral calculus is developed in easy detail; it includes an umbral differential and integral calculus as useful implements. It is not necessary here to explain the simple modifications required when two or more of $x, y, \cdots, u$ above are equal. There are immediate (and at present trivial) extensions of the method to sequences of elements from any commutative ring.

3. Lucas [**5,** p. 205] introduces his account of the symbolic calculus with some historical remarks which are immaterial for the calculus, and hence irrelevant and possibly misleading. He says (translation), "This method is already old; it is found as a mnemonic procedure in the writings of Leibniz, *for the successive derivatives of a product of two or of several factors*; it is found again in Taylor's series extended to the case of several variables. · · · Developed subsequently by Laplace, by Vandermonde, and by Herschel, it has been considerably augmented by the works of Cayley and of Sylvester, in the theory of forms.

"In a work entitled [*A Treatise on the*] *Calculus of Operations*, [1855, Robert] Carmichael has expounded the general methods of this rapid calculus; his developments refer above all to the use of symbols of operation, such as those of the sum and difference calculus, Σ and Δ, and those of the differential and integral calculus. But the symbolic method, which we employ here, *differs from the preceding in this respect*: *the symbols which we consider denote quantities* [real or complex numbers] *and not operations* . . . . The application of this method has enabled us to simplify, in a very notable manner, the reasoning and the results in the calculus of sums and differences, in the numbers of Bernoulli and Euler, in the theory of recurring series and, consequently, in the general theory of functions. By *our* method, the developments assume a more concise and more condensed form, which leads to successive and endless generalizations of the properties which concern numbers, and of the formulas which contain them."

In the preceding extract all italics except the title of Carmichael's book (which contains nothing relating to Blissard's method) are the translator's. The second italicized passage cancels the first, so far as Blissard's method is concerned, and makes all the historical citations irrelevant. The separation of symbols, as some older writers called the symbolic method of Leibniz and his successors, to which Lucas is referring in his historical remarks, *differs* from what Lucas calls 'our' method, as he himself takes pains to point out. A check of the sources mentioned by Lucas, also the like for the history of Leibniz' method from the historical references given by H. T. Davis [12], shows conclusively that none of the sources has anything whatever to do with Blissard's symbolic method. None of the authors mentioned has an example of the method. The italicized 'our' should be replaced by 'Blissard's.' It is not clear why Lucas gave all this irrelevant history; the matters to which it refers are not considered in his chapter on the symbolic method.

However, it has recently been shown by L. I. Neikirk [13] that the method of Leibniz, Cayley's hyperdeterminant notation, Aronhold's symbolic method in algebraic invariants and covariants, and Blissard's calculus, have a common *mathematical* (as distinguished from historical) source in symbolic differentiation. But here again it must be emphasized that Blissard's advance consisted in taking the final step of expunging all symbols of operation except addition subtraction, multiplication, and raising to positive integral powers from the formulas of his calculus and, as noted by Lucas in his account quoted above, using in addition only symbols which denote numbers themselves (independently of their possible origin as coefficients obtainable operationally from power series). If the common source referred to was known before 1937, there is no trace of it in the published literature. It seems curious that Lucas, who was widely read in the historical literature of the theory of numbers, and who did much notable research in that literature, and who further contributed papers written in English to English mathematical periodicals, should have overlooked Blissard's papers in one of the most widely circulated of such periodicals. Sylvester, however, is said to have surpassed even this obliviousness to what was going on or had gone on in his own field: he refused to believe that he had ever made some of his own discoveries.

Lucas also (*loc. cit.*) advocates passage from the symbolic notation to the ordinary when "the chain of the reasoning leaves a certain obscurity in the mind." There is no obscurity when the simple bases of the method are explicitly stated, which Lucas neglected to do. The root of the method is umbral equality, as defined in §2. The current practice of explicitly stating the definitions and postulates for a given method or theory was not customary when Lucas wrote. Attention to this usual precaution of clarity and definiteness removes obscurities, especially those that enter with unstated definitions and ignored postulates which must be inferred from ambiguous contexts.

Before proceeding to the sketch of Blissard's life, it will be of interest to recall some remarks of Glaisher in his paper [6, p. 159] on the Bernoullian function.

"The results in the nine preceding sections [of the paper in question], in so far as they relate to the Bernoullian numbers, . . . , are due to the late Mr. John Blissard, who published them in 1861 in [1]. Fifteen years later they were rediscovered by the late M. Eduard [*sic*] Lucas, and published by him in [3] and elsewhere. . . .

"In subsequent papers in the *Quarterly Journal*, Blissard developed in a systematic manner his symbolic method of investigation under the name Representative Notation.

"Lucas's papers attracted considerable attention, and the symbolic treatment of formulae relating to the Bernoullian and Eulerian numbers formed the subject of various papers and notes by M. Césaro [*sic*] and others. It does not appear, however, to have been pointed out that he had been anticipated by Blissard."

Cesàro (from remarks in his papers) credited Lucas with the method. Blissard was fully aware that he had devised a powerful algorithm, and claimed it for himself [1, vol. 5, p. 75], as he had the historical right to do. He says: "The subject of generic equation[s] appears to furnish a mine of analytical research, in which, by aid of my notation, as a fitting tool to work with, whatever direction may be taken, new and highly general results appear capable of being turned up in considerable abundance." 'Generic equations' are now called symbolic or umbral equations; thus, a generic equation for the Bernoulli numbers is Blissard's $(B+1)^n - B^n = 0$, $n = 0, 1, \cdots$, or $B+1 = B$.

4. Mathematically, Blissard was 'a man of one book.' He did in fact project a book on his invention, to unite his papers and give further applications of his method; but the constant demands on his time as a clergyman in an English village of the mid-nineteenth century postponed the book indefinitely, and it was never even begun. His life takes us back to an era that now seems almost as remote as the seventeenth century.*

The name is pronounced Blissard', and has been spelt variously Bloissart, Blizard, Blezard since the Black Prince in 1356, after the battle of Poitiers, gave his knights, one of whom was a Blissard, their coats of arms. For about 300 years the eldest son of the eldest son was named John and was a minister of the Church of England; the mathematician was almost in this tradition. There was also a medical tradition in the family; a biography of the distinguished English surgeon of the eighteenth and early nineteenth centuries, Sir William Blizard, will be found in the Imperial Dictionary of National Biography. This Blizard was active in the founding of the Royal College of Surgeons of London, where he was professor of anatomy. Mathematical talent appeared in two generations; both the subject of this sketch and his son specialized in mathematics at Cambridge, and both for a time did mathematical coaching. The son also went in

* The genealogical data in the following condensed account were supplied by John Morton Blissard, M.D., and his wife, of La Jara, Colorado. The few details of Blissard's life given below are summarized from a letter to the writer from Mrs. Evelyn Blissard Hooper, of Berkshire, England, a grand-daughter of the mathematician Blissard. This correspondent had access to family records and letters.

strongly for chess, and problems attributed to him are still to be found in chess manuals.

The mathematician John Blissard, 1803–1875, was born in Northampton, England, where his father was a physician "in good practise" [2]. He died at Hampstead Norreys, or Norris (from which inconspicuous village his mathematical writings are dated), after having served the local church, first as curate, then as vicar, for forty-six years. It may seem strange to some today that an inventive mathematician could have been content to live out his days as the pastor in a drowsy village when good (perhaps too good) Queen Victoria was at her very best, with all that superlative excellence meant for rustic society in England. But such was the fact, and Blissard was never heard to express any discontent with his lot, although "he was considered buried in that little village by those who saw his powers."

Blissard's Cambridge career was only moderately distinguished. He went into residence at St. John's College in 1822, and in the same year became prizeman and Mount Stephen exhibitioner. In the mathematical tripos of 1826 he came out senior optime, and was graduated B.A. His rank in the tripos, however, was not a just measure of his mathematical ability, as he had already done what would now be called independent research as an undergraduate. In 1827, he married Martha Morton, of the same family as the Scotch Earl of Morton. To anticipate, there were twelve children.

Having been ordained in 1827, Blissard served two years as a curate in a Bedfordshire parish, moving on "to be shelved" in Hampstead Norreys where, for a spell of fourteen years, he was curate. On the death of the vicar, Blissard was promoted—if that is the right word—to vicar in 1843. "Mr. Blissard was so beloved by his parishioners, that they presented a petition in his favor to the Marquis of Downshire (then patron of the living), who graciously presented it to Mr. Blissard" [2]. We are back with George Eliot in an age that is farther from us than the pyramids of Egypt. According to the same source, "Mr. Blissard was a clever mathematician, and wrote some erudite works on the subject." This slightly fatuous comment is what might be expected from our own Middletowns today; so possibly the Victorian past of rural England is less remote than it seems at first glance.

Blissard kept his soul alive—primarily by his ministrations to the parish, of course—by coaching private pupils in mathematics for Cambridge examinations, in which some of them, including two sons of one of Queen Victoria's personal physicians, did remarkably well. He was fifty-eight when his symbolic method was published. His daughter has recorded how this late productivity under adverse conditions may have been possible. In addition to his professional preoccupation with "things of the spirit, another part of his mind dwelt deeply in things mathematical, and he always had some abstruse problems or 'inventions' in his thoughts which one day he would bring out in a book."

Naturally, such a schism in fundamental loyalties resulted in frequent absentmindedness, especially as Blissard aged. One anecdote of many may serve to give a picture of this thoroughly amiable Victorian clergyman who was

mathematician enough to decline fees from his promising pupils when they happened to be hard up, and who kept mathematically alive long after an age at which men in easier positions often lose their intellectual interests.

In Blissard's day, churchgoers thought they were treated shabbily if the sermons were less than epic length, and Blissard agreed with them, once unwittingly. One Sunday he took as his text the whole of the twenty-third psalm, an adequate yardstick for what was to follow. Each sentence of the entire psalm provided a sub-text for a sub-sermon of moderate length, which the conscientious minister duly preached. By the time he reached "He restoreth my soul," he was just getting thoroughly warmed up. The phrase furnished the text for a complete address. By the end of this address, the preacher had forgotten its text. For the next lap of his marathon of a sermon he again gave out, "He restoreth my soul," and delivered another address, completely different from the first, on the same topic. Again he lost his cue, gave out the same sub-text a third time, and "preached a compact and beautiful little sermon on the same words without impinging upon anything he had said before. All three addresses," according to his wife, who, by the end of the third lap had caught the preacher's mildly excited eye, "were quite different and all of them quite perfect each in its own way. (And the people didn't mind a bit!)." There were giants in those days, and not all of them filled pulpits.

In a way that our generation cannot recover, Blissard's life was idyllic. But even in his tranquil rural paradise, the old Adam—or the ever-young Eve—leers at us through the lush vegetation. With an unconscious irony that speaks volumes of repressed social conflict, the official obituary [2] records that, "Throughout his life he [Blissard] scarcely made an enemy, and although his house was once set on fire by an evil-minded servant he forgave her, when she admitted her guilt."

In its own time and in its own degree, the laborious life of this inventive man illustrates a recent observation of the late J. N. W. Sullivan [**14**, p. 271]: "It seems very odd to reflect that these Bernoullis and Gauss might never have been known as mathematicians at all. There are plenty of instances of a genuinely promising young mathematician becoming by what seems mere accident of circumstance a lawyer or a churchman. The spectacle of square pegs in round holes is not so very uncommon. . . . "

It would be socially interesting to know how many potential mathematicians and other possibly useful citizens are stuck at this moment in unsuitable jobs in uncongenial holes in the United States. With this information in hand, it might be possible to dissuade certain "evil-minded" servants of society from their intention to burn down the schoolhouse.

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ON THE LINE OF IMAGES*

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Introduction. If we take any point T on the circumcircle of a triangle $A_1A_2A_3$ and reflect this point in the sides of the triangle we obtain three points lying on a line—the line of images of the point T. With the aid of this line we shall show a relationship which exists among certain known theorems and state some consequences which are believed to be entirely new. The method is analytic. In the study of the triangle it is convenient to take its circumcircle as the base circle and to let the coördinates of the vertices A_i $(i=1, 2, 3)$ be turns t_i, *i.e.*, complex numbers whose moduli are unity. We consider the t_i as roots of the equation $t^3-\sigma_1t^2+\sigma_2t-\sigma_3=0$, *i.e.*,

$$\sigma_1 = t_1 + t_2 + t_3, \qquad \sigma_2 = t_1t_2 + t_2t_3 + t_3t_1, \qquad \sigma_3 = t_1t_2t_3.$$

The conjugates of these elementary symmetric functions of the t_i are

$$\bar{\sigma}_1 = \frac{\sigma_2}{\sigma_3}, \qquad \bar{\sigma}_2 = \frac{\sigma_1}{\sigma_3}, \qquad \bar{\sigma}_3 = \frac{1}{\sigma_3}.$$

With this notation the parametric equation of the nine-point circle, with center at $\sigma_1/2$, and radius half of the base circle, is

$$x = \frac{(\sigma_1 - t)}{2}.$$

* Presented at the meeting of the Ohio Section of the Association at Columbus, March 31, 1938.

1. The line of images. The equation of any line may be written in the self-conjugate form

$$\frac{x}{b} + \frac{\bar{x}}{\bar{b}} = 1. \tag{1.1}$$

The perpendicular to this line through a point P (with coördinate p) is

$$\frac{x}{b} - \frac{\bar{x}}{\bar{b}} = \frac{p}{b} - \frac{\bar{p}}{\bar{b}}.$$

The intersection Q of these two lines is

$$x = \frac{b}{2}\left(1 + \frac{p}{b} - \frac{\bar{p}}{\bar{b}}\right).$$

Now the image x_1 of P in the line (1.1) is such that Q is the midpoint of x_1P, whence

$$x_1 = b\left(1 - \frac{\bar{p}}{\bar{b}}\right).$$

We note that this could be written down immediately from (1.1) by substituting $\bar{p}$ for $\bar{x}$ and solving for x.

The equation of the side A_2A_3 of the triangle $A_1A_2A_3$ is

$$x + t_2t_3\bar{x} = t_2 + t_3,$$

whence, since $\overline{T} = 1/T$, the image of the point T (with coördinate T) is

$$x = t_2 + t_3 - \frac{t_2t_3}{T}$$

$$= \sigma_1 - t_1 - \frac{\sigma_3}{Tt_1}.$$

The three images of T in the sides of the triangle $A_1A_2A_3$ lie on the line, with parameter t,

$$x = \sigma_1 - t - \frac{\sigma_3}{Tt}.$$

Since

$$\bar{x} = \frac{\sigma_2}{\sigma_3} - \frac{1}{t} - \frac{Tt}{\sigma_3},$$

upon elimination of the parameter t between these equations we obtain as the self-conjugate equation of the line of images of the point T,

$$Tx - \sigma_3\bar{x} = T\sigma_1 - \sigma_2. \tag{1.2}$$

Since this equation is satisfied for $x=\sigma_1$, we see that *the line of images of any point on the circumcircle of $A_1A_2A_3$ passes through the orthocenter of $A_1A_2A_3$* [1].* We note also that the lines of images of any two diametrically opposite points on the circumcircle are perpendicular and that when $T=t_i$, the line of images is the altitude of the triangle. More generally the line of images of T is parallel to the Simson line of T and hence the angle between the lines of images of two points equals the angle inscribed in the arc of the circumcircle between the points.

Given any line on the orthocenter of $A_1A_2A_3$ to find the point T for which the given line is its line of images we proceed as follows: Draw through any vertex A_i a line parallel to the line of images intersecting the circumcircle at a point V. The perpendicular from V to the side A_jA_k of the triangle will intersect the circumcircle at the required point T.

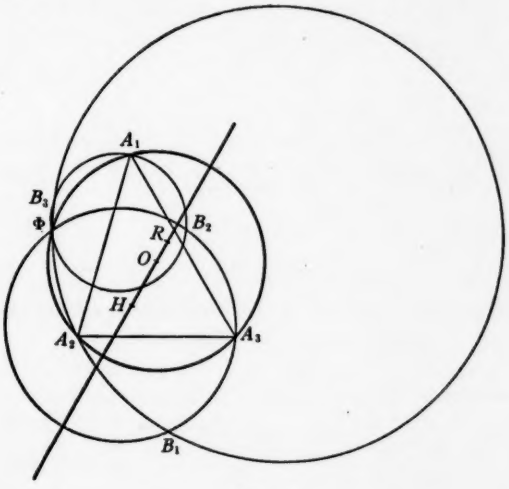

FIG. 1

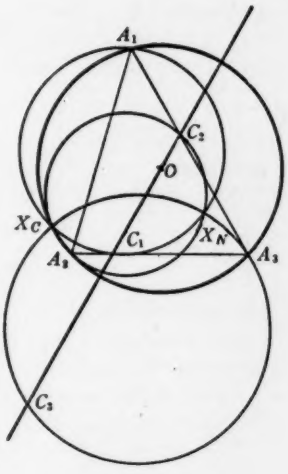

FIG. 2

Since the equation of the Euler line is

$$\sigma_2 x - \sigma_1\sigma_3\dot{x} = 0$$

we see that the point whose line of images is the Euler line has for its coördinate σ_2/σ_1. But this is the coördinate of the point of Feuerbach for the tangential triangle† of $A_1A_2A_3$ [2]. We shall designate this point by the symbol Φ.

2. A property of the line of images. We shall generalize a theorem of Canon. Given the images B_1, B_2, B_3 of the circumcenter O of the triangle $A_1A_2A_3$ in

* Numbers in brackets refer to references at the end of the paper.

† This is the point of tangency of the inscribed and nine-point circles for the triangle formed by the tangents to the circumcircle at the points A_1, A_2, and A_3.

the sides A_2A_3, A_3A_1, A_1A_2; show that the circles $B_1B_2A_3$, $B_2B_3A_1$, $B_3B_1A_2$ meet at the same point on the circumcircles [3]. Let P be any point in the plane with coördinate p. Its image B_1 in the side A_2A_3 has as coördinate, $t_2+t_3-t_2t_3\bar{p}$.

Consider the equation

$$(t_2 + t_3)x = \sigma_2 - \sigma_3\bar{p} + (t_2t_3 - \sigma_3\bar{p})t. \tag{2.1}$$

For $t=-1$, $t_3t_2^{-1}$, and $t_2t_3^{-1}$ respectively, this circle passes through the points A_1, B_2, and B_3. Hence this is the equation of the circumcircle of $B_2B_3A_1$. Also for the turn value

$$t = \frac{(\sigma_2 - \sigma_3\bar{p})(p - t_1)}{(\sigma_1 - p)(t_2t_3 - \sigma_3\bar{p})}, \qquad x = \frac{\sigma_2 - \sigma_3\bar{p}}{\sigma_1 - p},$$

a point M on the circumcircle of $A_1A_2A_3$. Since the coördinate of this point is symmetrical in t_i, the circumcircles $B_1B_2A_3$ and $B_3B_1A_2$ are likewise on M. Further, if R be any point on the line HP, where H is the orthocenter of $A_1A_2A_3$, and we take for coördinate of R,

$$r = \frac{H + \lambda P}{1 + \lambda} = \frac{\sigma_1 + \lambda p}{1 + \lambda},$$

the coördinate of M becomes $(\sigma_2-\sigma_3\bar{r})/(\sigma_1-r)$. Substituting

$$r = \frac{\sigma_1 + \lambda p}{1 + \lambda} \qquad \text{and} \qquad \bar{r} = \frac{\sigma_2 + \lambda\sigma_3\bar{p}}{\sigma_3(1 + \lambda)}$$

in this expression for M we obtain $(\sigma_2-\sigma_3\bar{p})/(\sigma_1-p)$. Since this is independent of the parameter λ we can state this generalization of Canon's theorem: *For every point R on the line HP, where H is the orthocenter of $A_1A_2A_3$ and P is any point in the plane, if we determine the images B_1, B_2, B_3 of R in the sides A_2A_3, A_3A_1, A_1A_2, the four circles $B_1B_2A_3$, $B_2B_3A_1$, $B_3B_1A_2$, and $A_1A_2A_3$ meet a fixed point M.* The circles $A_1A_2B_3$, $A_2A_3B_1$, $A_3A_1B_2$, and $B_1B_2B_3$ likewise meet at a point. A simple proof of this latter statement has been given by the author [4].

The equation of the line HP is

$$(\sigma_2 - \sigma_3\bar{p})x - \sigma_3(\sigma_1 - p)\bar{x} = \sigma_2p - \sigma_1\sigma_3\bar{p}. \tag{2.2}$$

Comparing this with (1.2) we see that HP is the *line of images of the point M*. Hence given HP, we have a simple method for constructing M and vice-versa. But we have proved the following: *For every point R on the line of images of a point T, if we determine the images B_1, B_2, B_3 of R in the sides of the triangle $A_1A_2A_3$, the circles $B_1B_2A_3$, $B_2B_3A_1$, and $B_3B_1A_2$ intersect at T.* This theorem is illustrated in Figure 1 for the point Φ.

3. A theorem of Blanc. If any transversal be drawn through O, the circumcenter of $A_1A_2A_3$, cutting the sides A_2A_3, A_3A_1, A_1A_2 in the points C_1, C_2, and C_3 respectively, the three circles with A_iC_i as diameters meet in two points, one

on the circumcircle and the other on the nine-point circle of $A_1A_2A_3$. Their common chord passes through H, the orthocenter of $A_1A_2A_3$ [5].

If we join the point T on the circumcircle to O, the equation of the transversal OT is

$$x - T^2\bar{x} = 0. \tag{3.1}$$

This cuts the side A_2A_3 in the point C_1, with coördinate

$$c_1 = \frac{T^2(t_2 + t_3)}{t_2t_3 + T^2}.$$

The equation of the circle on A_1C_1 as diameter is

$$2(t_2t_3 + T^2)x = T^2\sigma_1 + \sigma_3 + (T^2\sigma_1 - 2T^2t_1 - \sigma_3)t.$$

The three circles with A_iC_i as diameters meet at the two points

$$X_C = \frac{\sigma_3 + \sigma_1T^2}{\sigma_2 + T^2} \quad \text{and} \quad X_N = \frac{1}{2}\left(\sigma_1 - \frac{\sigma_3}{T^2}\right). \tag{3.2}$$

The first point is on the circumcircle of $A_1A_2A_3$, while the second point is on the nine-point circle (Figure 2). The equation of the common chord of the three circles is

$$T^2(\sigma_2 + T^2)x - \sigma_3(\sigma_3 + \sigma_1T^2)\bar{x} = \sigma_1T^4 - \sigma_2\sigma_3. \tag{3.3}$$

Since this equation is satisfied for $x=\sigma_1$, the chord (3.3) passes through the orthocenter of $A_1A_2A_3$. Now the image of the point X_C in the given transversal (3.1) is

$$\alpha = \frac{T^2(\sigma_2 + T^2)}{\sigma_3 + \sigma_1T^2}. \tag{3.4}$$

Comparing this with (3.3) we see that *the common chord of the three circles is the line of images of the point α, the image of X_C in the given transversal OT.* Consequently, it is possible, given any line on H, to construct the transversal through O which will cut the sides of $A_1A_2A_3$ in points C_i such that the common chord of the circles on A_iC_i will be the given line. We proceed as follows: Find the point α whose line of images is the given line. This given line intersects the circumcircle at two points R_1 and R_2. Now the perpendicular bisector of αR_i is the required transversal, if the common intersection of the three circles is to be R_i.

We state without proof the following properties of the points α and X_C. Let A_iV_i be parallels drawn through the vertices A_i to the transversal OT, cutting the circumcircle at the points V_i; then the three lines C_iV_i meet at α. If A_i' be the points on the circumcircle diametrically opposite to A_i, the lines $A_i'C_i$ meet at X_C.

4. The isogonal conjugate of OT. It is well known that the isogonal conjugate of any diameter of the circumcircle $A_1A_2A_3$ is an equilateral hyperbola cir-

cumscribing the triangle $A_1A_2A_3$ [6]. In our notation any two isogonal conjugate points x and y are connected by the relation [1, p. 196]

$$x + y + \sigma_3\bar{x}\bar{y} = \sigma_1$$

which, solved for y in terms of x, gives

$$y = \frac{\sigma_1 - x - \sigma_2\bar{x} + \sigma_3\bar{x}^2}{1 - x\bar{x}}.$$

If the point y travels over the transversal (3.1), the point x will describe the locus

$$(4.1) \qquad T^2x^2 + (\sigma_3 - \sigma_1T^2)x - \sigma_3^2\bar{x}^2 + (\sigma_2\sigma_3 - \sigma_3T^2)\bar{x} + \sigma_2T^2 - \sigma_1\sigma_3 = 0.$$

This is an equilateral hyperbola, also on the orthocenter H of $A_1A_2A_3$, with its center at

$$(4.2) \qquad C = \frac{1}{2}\left(\sigma_1 - \frac{\sigma_3}{T^2}\right).$$

The equation of the tangent line at H to the hyperbola is

$$(4.3) \qquad (\sigma_1T^2 + \sigma_3)x - \sigma_3(\sigma_2 + T^2)\bar{x} = (\sigma_1^2 - \sigma_2)T^2 + \sigma_1\sigma_3 - \sigma_3^2.$$

Now (4.2) is the point X_N, and (4.3) is the line of images of X_C of (3.2). Hence we can state the theorem *if any transversal be drawn through the circumcenter of $A_1A_2A_3$, cutting the sides A_2A_3, A_3A_1, A_1A_2 in the points C_1, C_2, and C_3 respectively, one point of intersection of the three circles on A_iC_i as diameters is the center of that equilateral hyperbola which is the isogonal conjugate of the given transversal, while the second point of intersection is the point on the circumcircle of $A_1A_2A_3$ whose line of images is the tangent at the orthocenter to the same hyperbola.*

The diametrically opposite point of H on the hyperbola (4.1), which lies also on the circumcircle of $A_1A_2A_3$, is

$$(4.4) \qquad x = -\frac{\sigma_3}{T^2}.$$

The line of images of (4.4) is

$$x + T^2\bar{x} = \sigma_1 + \frac{T^2\sigma_2}{\sigma_3},$$

which is perpendicular to (3.1). Hence *the fourth intersection of the equilateral hyperbola* (4.1) *with the circumcircle of $A_1A_2A_3$ is that point whose line of images is perpendicular to the given transversal.* We state without proof two other properties of the point (4.4). If we extend to the circumcircle, lines from the vertices of $A_1A_2A_3$ perpendicular to the transversal OT, and from these points draw perpendiculars to the opposite sides, the three perpendiculars meet at (4.4). The perpendiculars, from the two points of intersection of the transversal OT with the circumcircle of $A_1A_2A_3$, on their respective lines of images, intersect on the circumcircle at the point diametrically opposite to (4.4).

5. The Droz-Farny theorem. Any two perpendicular lines drawn through the orthocenter H of the triangle $A_1A_2A_3$ cut on the sides of the triangle three segments whose midpoints D_i lie on a line Δ. The perpendiculars through A_i to the lines HD_i intersect at a point P on the circumcircle; also PH is perpendicular to Δ.

We saw earlier that the lines of images of any two diametrically opposite points T and T' are perpendicular to each other at H. Using these as the transversals in the above theorem the coördinate of the point D_1 is found to be

$$d_1 = \frac{\sigma_3 + \sigma_1 T^2}{T^2 - t_1^2}. \tag{5.1}$$

The three points D_i lie on the line

$$T^2(\sigma_2 + T^2)x + \sigma_3(\sigma_3 + \sigma_1 T^2)\bar{x} = (\sigma_2 + T^2)(\sigma_3 + \sigma_1 T^2). \tag{5.2}$$

The coördinate of P is $-\sigma_3/T^2$, while the equation of PH is

$$T^2(\sigma_2 + T^2)x - \sigma_3(\sigma_3 + \sigma_1 T^2)\bar{x} = \sigma_1 T^4 - \sigma_2\sigma_3. \tag{5.3}$$

Hence, *the point P of the Droz-Farny theorem is the fourth intersection of the circumcircle with that equilateral hyperbola which is the isogonal conjugate of the diameter TT'. Also, the line PH is the chord of contact of the three circles on A_iC_i as diameters where C_i are the intersections of the sides of $A_1A_2A_3$ with the transversal TOT'.*

The point P is the second intersection of the line X_CX_N with the circumcircle. Its line of images is perpendicular to TOT'. Further the lines through the points V_i of section 3, perpendicular to the sides of the triangle $A_1A_2A_3$, will meet on the circumcircle at a point P', diametrically opposite to P. Also, the parallels through H to the lines A_1P', A_2P', A_3P' meet the sides A_2A_3, A_3A_1, A_1A_2 in the points D_i and hence lie on the line Δ.

6. The point of Miquel. From any four non-specialized lines we can form four triangles, whose circumcircles all meet at a point μ called the point of Miquel of the complete quadrilateral. If we consider the quadrilateral formed by the sides of the triangle $A_1A_2A_3$ and the transversal (3.1), its point of Miquel is

$$\mu_3 = \frac{T^2(\sigma_2 + T^2)}{\sigma_3 + \sigma_1 T^2}. \tag{6.1}$$

Similarly the coördinate of the point of Miquel for the quadrilateral formed by the sides of the triangle $A_1A_2A_3$ and the line (1.2) is

$$\mu_1 = \frac{\sigma_3 + \sigma_1 T^2}{\sigma_2 + T^2}. \tag{6.2}$$

The points μ_1 and μ_3 are the points X_C of (3.2) and α of (3.4). Hence we can state the following theorems: *The point of Miquel for the quadrilateral formed by the sides A_1A_2, A_2A_3, A_3A_1 and any transversal OT through the circumcenter of*

$A_1A_2A_3$ is the image in OT of the point of intersection on the circumcircle of the circles on A_iC_i as diameters, where the C_i are the points of intersection of the transversal with the sides of $A_1A_2A_3$. The point of Miquel for the quadrilateral formed by the sides A_1A_2, A_2A_3, A_3A_1 and the line of images of a point T on the circumcircle of $A_1A_2A_3$ is likewise the point of intersection on the circumcircle of the circles on A_iC_i as diameters, where the C_i are the points of intersection with the sides of $A_1A_2A_3$ by the transversal joining T to the circumcenter of $A_1A_2A_3$. These two points of Miquel are the images of each other in the transversal OT.

We note from (6.2) that the point of Miquel for the quadrilateral formed by A_1A_2, A_2A_3, A_3A_1 and the line of images of the point diametrically opposite to T on the circumcircle of $A_1A_2A_3$ will likewise be (6.2). Hence given any point μ on the circumcircle of $A_1A_2A_3$ to find the line which with A_1A_2, A_2A_3, A_3A_1 will have μ as its point of Miquel we proceed as follows: Join μ to the orthocenter H and find the point α whose line of images is μH. The perpendicular bisector of $\alpha\mu$ will cut the circumcircle in two points. The line of images of either point is the required line.

The line of images for the point of Miquel μ_1 is

$$(6.3) \qquad (\sigma_3 + \sigma_1 T^2)x - \sigma_3(\sigma_2 + T^2)\bar{x} = (\sigma_1^2 - \sigma_2)T^2 + \sigma_1\sigma_3 - \sigma_2^2 .$$

Let us find the points T for which the line (6.3) is parallel to OT. Equating the clinants of the lines we are led to $T^4 = \sigma_2\sigma_3/\sigma_1$. Let us find the points T for which the point of Miquel μ_3 will fall at Φ. Setting

$$\frac{T^2(\sigma_2 + T^2)}{\sigma_3 + \sigma_1 T^2} = \frac{\sigma_2}{\sigma_1},$$

we find $T^4 = \sigma_2\sigma_3/\sigma_1$. Since the line of images of μ_3 passes through μ_1, then μ_1 must be the intersection of the Euler line with the circumcircle and the points T are the intersections with the circumcircle of the perpendicular bisector of $\mu_1\mu_3$. Either point is such that the point of Miquel, for the quadrilateral A_1A_2, A_2A_3, A_3A_1 and the image line of T, has its image line parallel to OT, while the point of Miquel for the quadrilateral A_1A_2, A_2A_3, A_3A_1, and OT is Φ.

Let us find the points T for which (6.3) is the join of the orthocenter with the diametrically opposite point of T. This latter line has the equation

$$(6.4) \qquad (\sigma_3 + \sigma_2 T)x - \sigma_3 T(\sigma_1 + T)\bar{x} = \sigma_1\sigma_3 - \sigma_2 T^2.$$

For (6.3) and (6.4) to be identical we are led to

$$\sigma_1 T^4 + (\sigma_1^2 - \sigma_2)T^3 + (\sigma_1\sigma_3 - \sigma_2^2)T - \sigma_2\sigma_3 = 0.$$

whose four roots are $T = \sigma_2/\sigma_1$, $-t_1$, $-t_2$, $-t_3$. Hence *the four points, Φ and the diametrically opposite points of A_1, A_2, and A_3, are the only points whose lines of images form quadrilaterals with A_1A_2, A_2A_3, A_3A_1 such that the image line of their point of Miquel is the join of the orthocenter with their diametrically opposite point.*

7. The orthopole. If we drop perpendiculars from the vertices of a triangle $A_1A_2A_3$ upon any line, the perpendiculars from their feet are concurrent at a

point called the orthopole of the line as to $A_1A_2A_3$ [7]. Now the orthopole of the transversal OT as to $A_1A_2A_3$ is

$$x = \frac{1}{2}\left(\sigma_1 - \frac{\sigma_3}{T^2}\right) = X_N, \tag{7.1}$$

while the orthopole of (1.2), the image line of T, as to $A_1A_2A_3$ is the point

$$\beta = \sigma_1 - \frac{1}{2}\left(T + \frac{\sigma_2}{T}\right). \tag{7.2}$$

The locus of β as the point T traverses the circumcircle of $A_1A_2A_3$ is an ellipse with center at H, the orthocenter of $A_1A_2A_3$. It intersects the nine-point circle at the feet of the altitudes of $A_1A_2A_3$ and at the center of Jerebek's hyperbola. The equations of its axes are

$$\sqrt{\frac{\overline{\sigma_1\sigma_3}}{\sigma_2}}\,x \pm \sigma_3\bar{x} = \sigma_1\sqrt{\frac{\overline{\sigma_1\sigma_3}}{\sigma_2}} \pm \sigma_2.$$

These are the lines of images of the two intersections of the Euler line with the circumcircle, and hence can easily be constructed.

The image of the point β in the line (1.2) is

$$x = \frac{1}{2}\left(\sigma_1 - \frac{\sigma_3}{T^2}\right) = X_N.$$

We have proved the two theorems: *The orthopole of the line joining any point T on the circumcircle of $A_1A_2A_3$ to the circumcenter O is that point of intersection on the nine-point circle of the three circles on A_iC_i as diameters where C_i are the points of intersection on the sides of $A_1A_2A_3$ with the line joining T to the circumcenter of $A_1A_2A_3$. The orthopoles as to $A_1A_2A_3$ of the lines* (1.2) *and* (3.1) *are images of each other in the line* (1.2).

8. Some special cases. To illustrate the relationships shown in this paper, we state some particular cases, recalling that the isogonal conjugate of the Euler line is the hyperbola of Jerebek, while that of the Brocard line is Kiepert's hyperbola.

If the Euler line of the triangle $A_1A_2A_3$ cuts the sides A_2A_3, A_3A_1, and A_1A_2 in the points C_1, C_2, and C_3 respectively, the three circles on A_iC_i as diameters meet in two points—one the center of Jerebek's hyperbola, the other the image in the Euler line of the point of Miquel for the quadrilateral formed by A_1A_2, A_2A_3, A_3A_1, and the Euler line. If we reflect any point on the common chord of the three circles in the sides of the triangle $A_1A_2A_3$, obtaining the points B_1, B_2, and B_3 the three circles $B_1B_2A_3$, $B_2B_3A_1$, and $B_3B_1A_2$ meet at the point of Miquel for the quadrilateral formed by the lines A_1A_2, A_2A_3, A_3A_1, and the Euler line. The center of Jerebek's hyperbola is also the orthopole of the Euler line as to the triangle $A_1A_2A_3$.

If the Brocard line of the triangle $A_1A_2A_3$ cuts the sides of the triangle in

points C_1, C_2, and C_3, the three circles on A_iC_i as diameters meet at the center of the Kiepert hyperbola and at the point of Miquel for the quadrilateral formed by A_1A_2, A_2A_3, A_3A_1 and the line joining the orthocenter and the symmedian point of $A_1A_2A_3$. The orthopole of the Brocard line as to the triangle $A_1A_2A_3$ is the center of the Kiepert hyperbola.

If we use as the transversal the line joining the circumcenter to the point Φ, the three circles meet at the image in the Euler line of the center of Jerebek's hyperbola and at the point of Miquel for the quadrilateral formed by the lines A_1A_2, A_2A_3, A_3A_1, and the Euler line.

The fourth intersection of the hyperbola of Jerebek with the circumcircle is the diametrically opposite point on the circumcircle of Φ, the point of Feuerbach of the tangential triangle of $A_1A_2A_3$.

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THE NINE CIRCLE THEOREM AND THE ENLARGED GEOMETRY

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Here we state the Nine Circle Theorem in the following two forms:

(a). The *elementary* form: Let A_i $(i=1, 2, 3)$ be any three circles which have six external common tangents. If we construct the circles D_i $(i=1, 2, 3)$ such that D_i is tangent to the external common tangents of A_j and A_k $(i, j, k=1, 2, 3; j\neq k, k\neq i, i\neq j)$ at the points, through which the line is the polar of some point on the axis of the external centers of similitude of the given circles with respect to D_i, it may be proved that the six external common tangents of A_i and D_i $(i=1, 2, 3)$ will be tangent to a fixed circle. Further, let H be this circle, and construct the six circles E_i and F_i $(i=1, 2, 3)$ such that E_i is tangent to the external common tangents of A_j and A_k at their middle points and F_i to those of A_i and H at their middle points. Then each of the nine circles D_i, E_i and F_i $(i=1, 2, 3)$ will be tangent to two fixed circles.

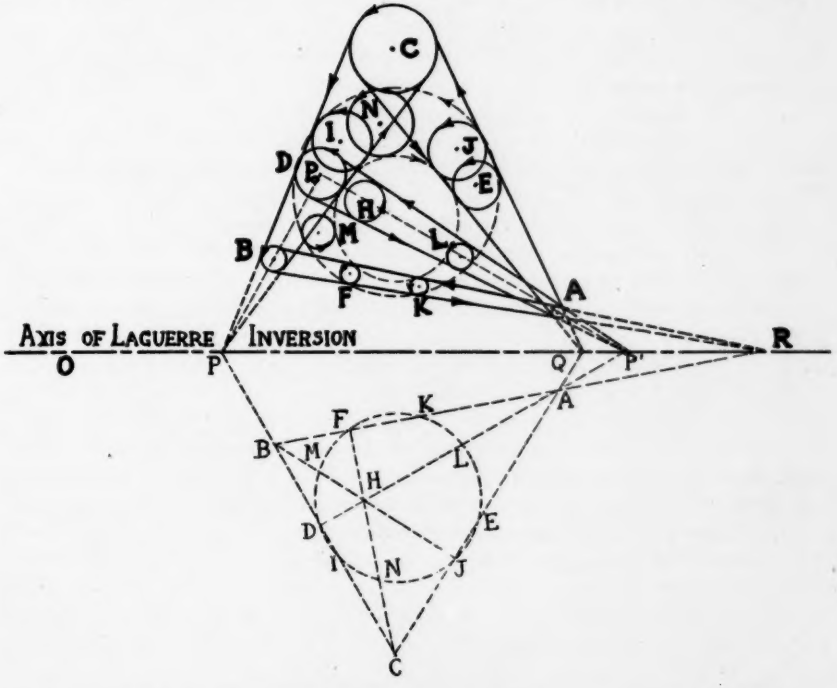
(b). The *oriented* form: Let **A**, **B**, and **C** be any three oriented circles, which have six common proper tangents. Construct the oriented circle **D**, such that **D** is properly tangent to the common proper tangents of **B** and **C**, and the centers of similitude of **B** and **C**, **A** and **D**, form a pair of conjugate points with respect to **D**. Similarly, we may construct the oriented circles **E** and **F** by replacing **ABCD** by **BCAE** and **CABF** respectively in the above construction. At this stage it may be proved that the six common proper tangents of **A** and **D**, **B** and

E, **C** and **F**, will be properly tangent to a fixed oriented circle. Call it **H**, and construct the six oriented circles **I**−**N** properly tangent to, and passing through the middle points of, the common proper tangents of the six pairs **B** and **C**, **C** and **A**, **A** and **B**, **A** and **H**, **B** and **H**, **C** and **H** respectively. Then each of the nine oriented circles **D**−**F** and **I**−**N** will be properly tangent to two fixed oriented circles.

It is evident that we have only to prove our theorem in the latter form. Accordingly we give the following:

Proof. For the sake of simplicity we prove our theorem by Laguerre inversion.*

Let O be the axis of similitude of the three given oriented circles, and take O as the axis of a Laguerre inversion. Let A be one of the two point circles, which



together with **A** have the same radical axis O. Take the constant of the Laguerre inversion such that **A** is inverted into A. Let P, Q, R be the centers of similitude of the pairs **B** and **C**, **C** and **A**, **A** and **B** respectively. Each of them is then inverted into itself. Since **B** is an oriented circle, which is properly tangent to the common proper tangents of **A** and R, **B** will be inverted into a unique oriented

* Laguerre, Oeuvres, Paris, 1905, vol. 2, pp. 592–619.

circle B, which is properly tangent to the common proper tangents of the point circles A and R, and is therefore a point circle. Similarly the oriented circles $\mathbf{C}-\mathbf{F}$ are inverted into unique point circles, which are denoted by $C-F$ respectively.

The center of similitude P' of the oriented circles **A** and **D** is then a point of O, for A and D, the Laguerre inverses of **A** and **D**, are point circles, so that P' will be inverted into itself and lie on O. By hypothesis the pair of points P and P' are conjugate with respect to **D**. Therefore P' is the point of intersection of O and the polar of P with respect to **D**. Now let P_0 be the center of **D**, d the radius of **D**, and t and t' the lengths of the tangents to **D** from P and P' respectively, if neither P nor P' is at infinity. Then since PP_0 is perpendicular to the polar of P with respect to **D**, we have

$$PP'^2 - P_0P'^2 = t^2 - d^2.$$

But $P_0P'^2 = t'^2 + d^2$, and therefore $t^2 + t'^2 = PP'^2$, which shows that in the inverted figure we must have

$$PD^2 + P'D^2 = PP'^2.$$

Hence D is the foot of the perpendicular from A to BC, which proposition may easily be proved also to be true when P or P' is at infinity. Similarly E and F are the feet of perpendiculars from B and C to CA and AB respectively.

Now by an elementary property of the triangle ABC, AD, BE, and CF are concurrent; *i.e.*, the six oriented lines AD, DA, BE, EB, CF, and FC are properly tangent to the same point circle H. Hence the six proper tangents of **A** and **D**, **B** and **E**, **C** and **F** will be properly tangent to the same oriented circle **H**, the Laguerre inverse of H. This completes the proof of the first part of our theorem.

Further, by hypothesis, it is evident that the oriented circles $\mathbf{I}-\mathbf{N}$ are inverted into the middle points $I-N$ of BC, CA, AB, AH, BH, and CH respectively. But from the ordinary property of the nine point circle of the triangle ABC we see that the nine points $D-F$ and $I-N$ are concyclic; *i.e.*, each of the nine point circles $D-F$ and $I-N$ is properly tangent to some two fixed oriented circles. Hence each of their Laguerre inverses, the nine oriented circles $\mathbf{D}-\mathbf{F}$ and $\mathbf{I}-\mathbf{N}$, will also be properly tangent to some other two fixed oriented circles. Q.E.D.

Note 1. If we assume that the triangle ABC is real, we have only to add the necessary and sufficient condition that none of **A**, **B**, and **C** contains a real point on their axis of similitude.

Note 2. The Nine Circle Theorem here and the Nine Point Circle Theorem in text books are so related that each of their figures is the Laguerre inverse of the other. Hence the truth of one theorem insures the truth of the other.

Note 3. In general each figure in a euclidean plane has a Laguerre inverse, so that a large number of new theorems concerning oriented circles can be deduced from euclidean theorems by Laguerre inversion.

Note 4. We may think of Laguerre inversion as an enlarger and define any

oriented circle, which is obtained from an ordinary point by a Laguerre inversion, as an enlarged point, which, not being ordinary, may be said to have not only position but also form and size. Then the length of any one of the common proper tangents of two oriented circles, the Laguerre inverse of the distance of two ordinary points, may be defined as the distance between the two corresponding enlarged points. A pair of oriented lines, being the Laguerre inverse of a line, is defined as an enlarged line, and any oriented circle which is properly tangent to them as an enlarged point of the enlarged line. A pair of oriented circles, being the Laguerre inverse of a circle, is defined as an enlarged circle, and any oriented circle which is properly tangent to them as an enlarged point of the enlarged circle. A pair of two enlarged lines is defined as parallel if the oriented lines of the first enlarged line are respectively parallel to those of the second. On the other hand a pair of two enlarged lines is defined as intersecting, if their oriented lines are all properly tangent to the same oriented circle. Lastly, a pair of two intersecting enlarged lines is defined as perpendicular, if the point of intersection of the oriented lines of the first enlarged line is conjugate to that of the second with respect to the enlarged point of intersection. The Nine Circle Theorem may then be stated in the following form (the enlarged form):

The three enlarged perpendiculars from the enlarged vertices of an enlarged triangle **ABC** to the opposite enlarged sides have an enlarged point in common. Call it **H**. Then the enlarged feet **D**, **E**, **F** of the enlarged perpendiculars and the enlarged middle points of **BC**, **CA**, **AB**, **AH**, **BH**, **CH** lie on the same enlarged circle.

Note 5. From the Notes 3 and 4 we obtain at once the following theorem, which may be called:

The Principle of Enlargement. From each euclidean theorem we can always deduce a new theorem by changing the ordinary conceptions, point, line, circle into the enlarged ones, enlarged point, enlarged line, enlarged circle respectively.

By means of this principle we obtain a new geometry, called the Enlarged Geometry, which contains the theorems of enlarged elements. The Nine Circle Theorem is one of such theorems if it is written in the enlarged form in Note 4. It becomes an application of this geometry if it is changed into the forms (a) and (b) by replacing the enlarged elements by their ordinary equivalents.

Note by the Editor. Clearly any transformation applied to a configuration in one theorem leads to a (new) theorem regarding a (new) configuration. The Laguerre inversion is a transformation known for over fifty years which leads to interesting, though somewhat complicated, results, as shown in this paper. This transformation seems to have received little attention, Laguerre receiving no mention, for example, in Morley's *Inversive geometry*.

THE SIMSON QUARTIC OF A TRIANGLE

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1. Introduction. The Simson line of any point on the circumcircle of a triangle is the line containing the feet of the perpendiculars from the point to the sides of the triangle. The existence of a line thus related to the circumcircle was discovered* by Robert Simson (1687–1768). It is also called the *pedal line* of the point with reference to the given triangle.

By the Simson quartic of a triangle ABC is meant the envelope of its Simson lines. The existence of this envelope as a curve of order four and class three was discovered by Steiner† in 1857, and its character as a three-cusped hypocycloid circumscribing the triangle demonstrated by Cremona‡ in 1865.

The properties that form the subject matter of this paper are the following: First it will be shown that the Simson quartic of a triangle is a three-cusped hypocycloid with its center at the nine-point center of the triangle,§ the hypocycloid being generated by a circle with a radius equal to that of the nine-point circle rolling on the inside of a circle with a radius which is three times that of the nine-point circle. This is Cremona's theorem, already referred to. The second property gives rise to a simple construction for determining the position of the cuspidal tangents of the Simson quartic of a triangle. The third property is the theorem that the points where the Simson quartic of a triangle touches its sides are the symmetrics of the feet of the altitudes relative to the midpoints of the respective sides. Finally the theorem is proved that the altitudes of a triangle touch its Simson quartic, each altitude touching the quartic at a point whose distance from the foot of the altitude is twice the projection on it of the circumradius to the vertex through which the altitude is drawn. The specifically detailed relationships of the Simson quartic of a triangle to that triangle given in the last three properties do not appear to have been explicitly stated before.

A simple construction of the cuspidal tangents of the Simson quartic of a

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* *Note by Editor.* This is debatable. See R. A. Johnson, Modern Geometry, p. 137. J. R. M·

† J. Steiner, Über eine besondere Curve dritter Klasse (und vierten Grades), Crelle, vol. 53, 1857, p. 231.

‡ Cremona, Sur l'hypocycloide à trois rebroussements, Crelle, vol. 64, 1865, p. 101.

§ The nine-point circle of a triangle is so named because it passes through the three midpoints of the sides of the triangle, the three feet of its altitudes, and the three midpoints of the segments of the altitudes joining the orthocenter of the triangle to its vertices. A theorem due to Euler states that the circumcenter, the centroid, the nine-point center, and the orthocenter of a triangle lie on one line, called *the Euler line* of the triangle in question. The radius of the nine-point circle is one half that of the circumcircle, and its center is the midpoint of the segment of the Euler line between the circumcenter and orthocenter.

If a circle of radius b rolls without slipping on the inside of a circle of radius a, $(a>b)$, a point of the first traces out a curve called a hypocycloid in the common plane of the two circles. If a equals $3b$, the resulting hypocycloid is the familiar three-cusped hypocycloid of elementary texts on the calculus, whose parametric equations are:

$$x = 2b\cos\theta + b\cos(2\theta - 3\alpha), \qquad y = 2b\sin\theta - b\sin(2\theta - 3\alpha). \tag{1}$$

triangle exists, based on a theorem in elementary geometry that seems to have escaped explicit statement.* It is stated most readily by reference to the accompanying figure; the proof, a simple exercise in elementary geometry, is omitted.

Let ABC be any triangle and let O be its circumcenter. Let D, E, F be the midpoints of the sides BC, AC, and AB respectively. Draw the side bisector DH of the side BC to cut the arc BAC of the circumcircle containing the opposite vertex A at the point H. The side bisectors EJ and FK are similarly drawn to cut the arcs ABC and BCA at the points J and K respectively. Choose the point L on the arc HA so that HL is one third of the arc HA. Select the points M and N in a similar manner on the arcs JB and KC respectively. The theorem in question is that whatever triangle ABC may be, the triangle LMN is equilateral. It will be referred to as *the equilateral derivative* of the triangle ABC.

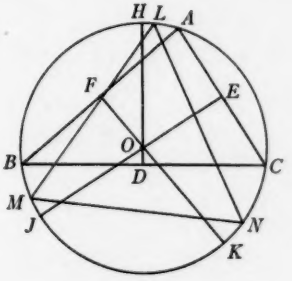

FIG. 1

2. The four theorems. Let ABC be any triangle (Figure 2), and let it be referred to its circumcenter O as origin, the side bisector of AC as axis of x and the line through O parallel to AC as axis of y. The radius of the circumcircle of ABC shall be the unit of length.

Let the vertices of the triangle ABC be the points: $A \equiv (\cos 2\alpha, \sin 2\alpha)$, $B \equiv (\cos 6\beta, \sin 6\beta)$, $C \equiv (\cos 2\alpha, -\sin 2\alpha)$. There is no loss of generality in this asymmetrical orientation of the triangle. The orthocenter R is the point $(2\cos 2\alpha + \cos 6\beta, \sin 6\beta)$. The nine-point center Q has the coördinates $(\cos 2\alpha + \frac{1}{2}\cos 6\beta, \frac{1}{2}\sin 6\beta)$. The radius of the nine-point circle of ABC is $1/2$. By assumption OEJ is the side bisector of AC and the axis of x. If the arc JM is $1/3$ of the arc JB, then M is a vertex of the equilateral derivative of ABC and has the coördinates $(\cos 2\beta, \sin 2\beta)$. The line OM with the equation $y = x \tan 2\beta$ is one of the altitudes of the equilateral derivative of ABC, since obviously the circumcenter of ABC is also the circumcenter of its equilateral derivative.

Let $P \equiv (\cos 2\phi, \sin 2\phi)$ be any point on the circumcircle of the triangle ABC. The equation of its Simson line STU relative to ABC is

* *Note by Editor.* The theorem was proved but not explicitly stated by Goormagtigh, Mathesis, vol. 31, 1911, p. 163. J.R.M.

$$y - \sin 2\phi = \tan(3\beta - \phi)(x - \cos 2\alpha). \tag{2}$$

If we differentiate (2) with respect to ϕ, and solve the resulting equation and (2) for x and y in terms of ϕ we obtain the parametric equations of the envelope of the Simson lines of the triangle ABC, namely,

$$\begin{cases} x = \cos 2\alpha + \frac{1}{2}\cos 6\beta + \cos 2\phi + \frac{1}{2}\cos(4\phi - 6\beta), \\ y = \frac{1}{2}\sin 6\beta + \sin 2\phi - \frac{1}{2}\sin(4\phi - 6\beta). \end{cases} \tag{3}$$

When referred to the nine-point center Q of the triangle and to axes parallel to the original axes the parametric equations of the envelope become

$$\begin{cases} x = \cos 2\phi + \frac{1}{2}\cos(4\phi - 6\beta), \\ y = \sin 2\phi - \frac{1}{2}\sin(4\phi - 6\beta). \end{cases} \tag{4}$$

A comparison of equations (4) with equations (1) yields the following result, remembering that the radius of the nine-point circle of the triangle ABC is $1/2$.

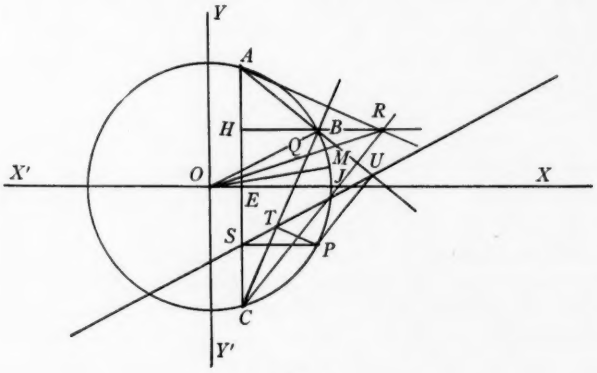

FIG. 2

THEOREM I. *The envelope of the Simson lines of the triangle ABC is a three-cusped hypocycloid generated by a circle equal in size to the nine-point circle of ABC rolling on the inside of a circle of radius three times the radius of, and with center at the center of, the nine-point circle of ABC.*

The line joining a cusp of a three-cusped hypocycloid to the center of its fixed circle is one of the cuspidal tangents possessed by the curve. Moreover since it makes the angle 2β with the x-axis it is parallel to OM of Figure 2. As OM is an altitude of the equilateral derivative of ABC, it follows that the cuspidal tangent in question is perpendicular to a side of the equilateral derivative of the triangle ABC. Since B may be any vertex and AC its opposite side, we have the result:

THEOREM II. *The cuspidal tangents of the Simson quartic of any triangle are perpendiculars through its nine-point center to the sides of its equilateral derivative.*

But this is not all. The slope of the cuspidal tangent in question is $\tan 2\beta$ and has the equation

$$y - \sin 2\beta = \tan 2\beta(x - \cos 2\alpha), \tag{5}$$

since it passes through the nine-point center. It is readily verified that it is the Simson line of the point $M \equiv (\cos 2\beta, \sin 2\beta)$, a vertex of the equilateral derivative of the triangle ABC. Hence it follows as a corollary to Theorem II that *the vertices of the equilateral derivative of any triangle have as Simson lines the cuspidal tangents of its Simson quartic.*

Obviously the location of the cuspidal tangents by means of the equilateral derivative of the triangle ABC locates also the points where its Simson quartic touches its nine-point circle.

The equation of the altitude through the vertex B to the side AC is $y = \sin 6\beta$. Its foot on AC is the point H with the coördinates $(\cos 2\alpha, \sin 6\beta)$. The side AC is the Simson line of the point $(-\cos 6\beta, -\sin 6\beta)$. The point where AC touches the Simson quartic of ABC is $(\cos 2\alpha, -\sin 6\beta)$, the symmetric of the foot of the altitude on AC relative to the midpoint of AC. As B may be any vertex and AC its opposite side, we have:

THEOREM III. *The Simson quartic of any triangle touches each of its sides at the symmetric of the foot of the altitude on that side relative to its midpoint.*

The Simson line of the vertex $B \equiv (\cos 6\beta, \sin 6\beta)$ is the altitude $y = \sin 6\beta$. Its point of contact with the Simson quartic of ABC is $(\cos 2\alpha + 2\cos 6\beta, \sin 6\beta)$. The distance of this point of contact from the foot of the altitude is $2\cos 6\beta$, twice the projection of the circumradius to B on the altitude from B to AC. Hence we have the result:

THEOREM IV. *The altitudes of any triangle touch its Simson quartic, each altitude touching the quartic at a point whose distance from its foot is twice the projection on it of the circumradius to the vertex through which the altitude is drawn.*

A NOTE ON FERMAT'S LAST THEOREM

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1. A theorem of Peter Barlow* states that in case

$$x^n + y^n + z^n = 0 \tag{1}$$

has a solution (x, y, z) in coprime integers prime to the odd prime n, then integers r, s, t exist such that

$$x + y = r^n, \tag{2}$$

$$y + z = s^n, \tag{3}$$

$$z + x = t^n, \tag{4}$$

In order to obtain lower limits for x, y, and z in (1) G. James† has had to show that

$$r + s + t \neq 0. \tag{5}$$

His proof of this interesting fact is a lengthy geometrical argument. The purpose of this note is to show how a number-theoretic proof of (5) may be given in a few lines.

In the first place since x^n+y^n is algebraically divisible by $x+y$ when n is odd, it follows from (1) and (2) that r^n divides z^n so that

$$z = ru. \tag{6}$$

Similarly

$$x = sv \tag{7}$$

and

$$y = tw. \tag{8}$$

Now suppose, if possible, that (5) is false so that

$$r + s + t = 0; \tag{9}$$

then it is seen that precisely one of (r, s, t) is even. In fact all three could not be even for in this case, in view of (6), (7), and (8), the same would be true of the supposedly coprime (x, y, z). Hence we may suppose by symmetry that r is even and s and t are odd.

Using (2), (3), (4), and (6) we may write

$$s^n + t^n = x + y + 2z = r^n + 2ru. \tag{10}$$

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* See for instance Dickson's History of the Theory of Numbers, vol. 2, p. 733, where the simple argument needed to establish the result is given. In the present note the notation has been slightly altered to render the problem completely symmetric.

† This MONTHLY, vol. 41, 1934, pp. 419–424

Dividing both members by r we have in view of (9)

$$-(s^n + t^n)/(s + t) = r^{n-1} + 2u. \tag{11}$$

Now the right member of (11) is even since r is even. But the left member is odd since it can be written as a sum

$$-(s^{n-1} - s^{n-2}t + - \cdots + t^{n-1})$$

of n odd numbers. This contradiction proves (5).

2. The limits set by James for x, y, and z can be raised very easily. A short proof can be given of the fact that each of r, s, and t must be greater than or equal to $2n^2+1$. If we suppose, for definiteness, that $x>y>z$, it follows from (1) that

$$(2n^2 + 1)^n < r^n = x + y < z.$$

Since n is known to be greater than 14000, we see that the smallest possible value of x, y, or z exceeds

$$(392000000)^{14000} > 10^{10^5}.$$

A HIGHER UPPER LIMIT TO THE PARAMETERS IN FERMAT'S EQUATION*

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1. Introduction. This paper enlarges the ranges of x, y, and z on which the Fermat equation with x, y, and z not divisible by n, the so-called "first case," has no solution. We consider the Fermat equation

$$x^n + y^n = z^n \tag{1}$$

where x, y, and z are positive integers, $n>2$, and x, y, $z\not\equiv 0 \pmod{n}$. As is well known it suffices to treat only those cases in which n is an odd prime and x, y, and z are relatively prime.

In a previous paper† I proved that this equation has no solution if the least of x, y, z is less than $n(2cn+1)^n$. Equation (7) of that paper states that if equation (1) has a solution, (x, y, z), then

$$(x/n) > z - y \tag{2}$$

where

$$x < y < z,$$

and Theorem II of that paper states that

* This is a combination of two papers, which were presented to the American Mathematical Society, November 28, 1936 and November 29, 1937.

† This MONTHLY, vol. 41, 1934, p. 421.

$$z - y \geqq (2cn + 1)^n \tag{3}$$

where

$$2cn = (z - y)^{1/n} + (z - x)^{1/n} - (x + y)^{1/n}, \tag{4}$$

and c is a positive integer.

The early part of the present paper obtains the identity, (4), by a method which shows that c contains n. Thereby the limit for $z-y$ given in inequality (3) is stepped up to $(2cn^2+1)^n$. And consequently the limit for x is, by relation (2), increased to $n\,(2cn^2+1)^n$.

The latter part of this paper raises this new limit to $(111/77)n(2cn^2+1)^n$, by means of a certain interpolation process.

2. Some basic theorems. The work that follows is based on a theorem by Peter Barlow* and a related theorem credited to Sophie Germain by Legendre.† We give brief proofs of both these important theorems.

The Barlow Theorem states that *if x, y, z satisfy the Fermat equation, are relatively prime, and x, y, $z\not\equiv 0 \pmod{n}$ then $z-x$, $z-y$, and $x+y$ are nth powers of integers.*

To prove this we start with the identity

$$z^n = (y + \overline{z - y})^n,$$

expand the right member, transpose the term, y^n, replace z^n-y^n by x^n and factor $z-y$ out of the terms on the right. This gives

$$\begin{aligned} x^n = (z - y)\big[ny^{n-1} + \tfrac{1}{2}n(n-1)y^{n-2}(z-y)^1 + \cdots \\ + ny(z-y)^{n-2} + (z-y)^{n-1}\big]. \end{aligned} \tag{5}$$

Now $z-y$ and the bracket are relatively prime, for $z-y$ and y are relatively prime since z and y are, by hypothesis, and also $z-y$ cannot contain n else x would. It follows that every prime factor of $z-y$ appears to the (kn)th power in $z-y$ since it appears to this power in x^n. Thus $z-y$ is the nth power of some integer. Similar arguments establish the same property for $z-x$ and $x+y$.

The Sophie Germain Theorem may be stated as follows.

If x, y, z satisfy the Fermat equation, are relatively prime, and x, y, $z\not\equiv 0 \pmod{n}$, then

$$\begin{aligned} &(a) \quad x = (z - y)^{1/n}(2c_1n^2 + 1), \\ &(b) \quad y = (z - x)^{1/n}(2c_2n^2 + 1), \\ &(c) \quad z = (x + y)^{1/n}(2c_3n^2 + 1). \end{aligned} \tag{6}$$

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* The Journal of Natural Philosophy, Chemistry and Arts, vol. 27, 1810, p. 193.

† Legendre, Théorie des Nombres, ed. 2, 1808, second supplement, September, 1825, pp. 1–40.

Proof: Equation (5) can be written

$$x^n = (z - y)(2d_1n + 1); \tag{7}$$

for by Fermat's Simple Theorem,

$$(z - y)^{n-1} \equiv 1 \pmod{n},$$

and the bracket in equation (5) is odd since two and two only of x, y, and z must be odd and the bracket and $z-y$ are relatively prime as we have just proven.

Again applying the Fermat Simple Theorem we have

$$x^n \equiv x \pmod{n},$$

and

$$[(z - y)^{1/n}]^n \equiv (z - y)^{1/n} \pmod{n}.$$

By means of these reductions, equation (7) can be written

$$\text{(a')} \quad x = (z - y)^{1/n}(2e_1n + 1). \tag{8}$$

And we obtain similarly the relations,

$$\text{(b')} \quad y = (z - x)^{1/n}(2e_2n + 1),$$

$$\text{(c')} \quad z = (x + y)^{1/n}(2e_3n + 1).$$

Now let α be any prime factor of $(2e_1n+1)$, that is, of $x/(z-y)^{1/n}$. Obviously α is odd and prime to n, $z-x$, $z-y$, and $x+y$.

Now we have

$$(x + y)^n - (z - x)^n \equiv 0 \pmod{\alpha} \tag{9}$$

since

$$(x + y)^n - (z - x)^n = x^n + y^n - z^n + \text{terms in } x. \tag{10}$$

In order to simplify congruence (9) we note that

$$[(z - x)^{1/n}]^{\alpha-1} \equiv 1 \pmod{\alpha}. \tag{11}$$

Raise this congruence to the power n^2 and write it in the form

$$(z - x)^n(z - x)^{n(\alpha-2)} \equiv 1 \pmod{\alpha}. \tag{12}$$

Now multiply congruence (9) by $(z-x)^{n(\alpha-2)}$ and apply congruence (12) to the second term. This leaves

$$(z - x)^{n(\alpha-2)}(x + y)^n \equiv 1 \pmod{\alpha},$$

which can be written

$$[(z - x)^{(\alpha-2)/n}(x + y)^{1/n}]^{n^2} \equiv 1 \pmod{\alpha}. \tag{13}$$

The number, n^2, is the least exponent which applied to this bracket gives a result congruent to unity, modulus α. For if there were a lesser one it must, by

elementary number theory, be a factor of all higher powers of the bracket for which the congruence holds, that is, it must be either unity or n. But if it were unity then

$$(z - x)^{(\alpha-2)/n}(x + y)^{1/n} \equiv 1 \pmod{\alpha},$$

which taken with congruence (11) would give

$$(x + y)^{1/n} \equiv (z - x)^{1/n} \pmod{\alpha}.$$

This congruence raised to the nth power gives

$$x + y \equiv z - x \pmod{\alpha}, \tag{14}$$

or

$$2x \equiv z - y \pmod{\alpha},$$

which is impossible since α is a factor of x but prime to $z-y$. Similarly if the least exponent possible in congruence (13) were n, we would have

$$(z - x)^{\alpha-2}(x + y) \equiv 1 \pmod{\alpha},$$

which taken with the nth power of congruence (11) would give us again the impossible relation (14).

Now, since α is prime to both $z-x$ and $x+y$, we have

$$[(z - x)^{(\alpha-2)/n}(x - y)^{1/n}]^{\alpha-1} \equiv 1 \pmod{\alpha}.$$

Hence $(\alpha-1)$ is a multiple of n^2, that is

$$\alpha - 1 = b_1 n^2,$$

or since α is odd

$$\alpha = 1 + 2c_1 n^2.$$

From this result, identity (a) of (6) follows. Relations (6), (b) and (c) are proved similarly.

3. Derivation of the limit, $n(2cn^2+1)^n$. We write, from equations (6), the congruences,

$$(a'') \quad (z - y)^{1/n} \equiv x \pmod{2n^2},$$
$$(b'') \quad (z - x)^{1/n} \equiv y \pmod{2n^2},$$
$$(c'') \quad -(x + y)^{1/n} \equiv -z \pmod{2n^2}.$$

It follows that

$$(z - y)^{1/n} + (z - x)^{1/n} - (x + y)^{1/n} \equiv x + y - z \pmod{2n^2}. \tag{15}$$

Also, raising each member of the congruences (a''), (b''), and (c'') to the nth power and adding, we obtain

$$x + y - z \equiv 0 \pmod{2n^3}, \tag{16}$$

since exactly two of x, y, z are odd.

Combining congruences (15) and (16) gives

$$(z-x)^{1/n}+(z-y)^{1/n}-(x+y)^{1/n}\equiv 0 \pmod{2n^2}. \tag{17}$$

By relation, (4), the left member of this congruence is positive. Hence we may write

$$(z-y)^{1/n}=2cn^2+(x+y)^{1/n}-(z-x)^{1/n},$$

where c is a positive integer. And since $(z-x)<(x+y)$ we have

$$(z-y)\geqq(2cn^2+1)^n. \tag{18}$$

Finally relation (18) taken with inequality (2) gives the important conclusion

$$x\geqq n(2cn^2+1)^n. \tag{19}$$

Since this, the so-called "first case" of the Fermat equation, is known to have no solution for $n\leqq 14000$, it follows, from the limit given by inequality (19), that it has no solution for any n if x, y, or z is less than 10^{120310}. The corresponding former limit was 10^{62200}. (Incidentally we are considering here only the highest degree terms in the expansions of the binomials $(2n^2+1)^n$ and $(2n+1)^n$.

4. A further increase in the lower limit of possible solutions of the Fermat equation.

The procedure here consists of setting up two inequalities between x, y, and $z-y$, then iterating between them to secure a larger, lower limit for x in terms of the known, lower limit of $z-y$.

The first equation is derived from the identity

$$y-x=(z-x)-(z-y). \tag{20}$$

Since by hypothesis

$$x<y<z,$$

we have

$$(z-y)^{1/n}+1\leqq(z-x)^{1/n}.$$

Applying this to (20) we may write

$$y-x\geqq[(z-y)^{1/n}+1]^n-(z-y)\geqq n(z-y)^{(n-1)/n}, \tag{21}$$

whence

$$y\geqq x+n(z-y)^{(n-1)/n}\geqq x+na^{n-1}, \tag{22}$$

where a^n is a constant for a given n and is equal to or less than $(z-y)$ for all z and y which could possibly satisfy equation (1) for the given n.

The second relation needed is obtained from the identity

$$z^n=(y+\overline{z-y})^n.$$

Subtracting y^n from both sides and replacing $(z^n - y^n)$ by x^n enables us to write

$$x = y[(1 + (z - y)/y)^n - 1]^{1/n} \geqq y[(1 + a^n/y)^n - 1]^{1/n}. \tag{23}$$

We now return to relation (22), replace x by na^n, in view of equation (2), and substitute the resulting right member for y in equation (23). This gives

$$x \geqq n(a^n + a^{n-1})[(1 + a/[n(a + 1)])^n - 1]^{1/n}. \tag{24}$$

This step is justified by the fact that the right member of equation (23) is an increasing function of y.

From inequality (24) we can conclude that

$$x \geqq n(a^n + a^{n-1}), \tag{25}$$

provided the bracket in relation (24) is equal to or greater than unity.* For the present we shall assume this to be true and proceed with our iterating. Substituting the right member of inequality (25) for x in relation (22) we obtain

$$y \geqq n(a^n + 2a^{n-1}). \tag{26}$$

Substituting the right member of this inequality in place of y in equation (23) gives

$$x \geqq n(a^n + 2a^{n-1})[(1 + a/[n(a + 2)])^n - 1]^{1/n}.$$

Assuming, as before, that the bracket is equal to or greater than unity we have

$$x \geqq n(a^n + 2a^{n-1}).$$

Carrying this process through h steps we obtain

$$x \geqq n(a^n + ha^{n-1}), \tag{27}$$

provided

$$[(1 + a/[n(a + h)])^n - 1]^{1/n} \geqq 1 \tag{28}$$

for $h = 1, 2, 3, \cdots, h$.

Obviously this bracket is, for a fixed n, a decreasing function of h. Hence we need only determine the largest integral value of h for which the equality sign holds. This is equivalent to solving for h the equation

$$(1 + a/[n(a + h)])^n = 2.$$

We obtain

$$h = a(1/[n(2^{1/n} - 1)] - 1).$$

When $n = 3$ this gives us

$$h = 11a/39.$$

* One could, of course, iterate without this restriction but this results in a seemingly unprofitable form.

When $n = 14000$, it gives

$$h = 34a/77 - ,$$

while

$$\lim_{n\to\infty} h = 34a/77 - .$$

(The difference in the two latter cases is beyond reasonable limits of computation.)

Hence for all values of n for which Fermat's Last Theorem has not been proved, namely those greater than 14000, we may take $h = 34a/77$.

But our process of iteration permits only integral values for h. However, we surmount this difficulty by the following device. Continue the iterating process until we reach $[34a/77]$, which denotes the largest integer in $34a/77$. Call this K. We have then by relation (27)

$$x \geqq n(a^n + Ka^{n-1}).$$

Substituting the right member of this inequality for x in equation (22) gives

$$y \geqq n[a^n + (K+1)a^{n-1}].$$

But

$$K + 1 \geqq 34a/77.$$

Whence

$$y \geqq n[a^n + (34a/77)a^{n-1}].$$

Substituting the right member for y in relation (23) we obtain, by virtue of inequality (28),

$$x \geqq n[a^n + (34/77)a^n]$$

$$\geqq (111/77)na^n, \tag{29}$$

which holds for all values of n greater than 14000.

Replacing a^n, $(\geqq z-y)$, by $(2cn^2+1)^n$ because of inequality (18), we have

$$x \geqq (111/77)n(2cn^2+1)^n, \qquad n \geqq 14000. \tag{30}$$

A similar procedure for $n = 3$ gives

$$x \geqq (50/39)n(2cn^2+1)^n \geqq 26388, \quad \text{for} \quad n \geqq 3.$$

A CONTRIBUTION TO THE MATHEMATICAL THEORY OF BIG GAME HUNTING

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This little known mathematical discipline has not, of recent years, received in the literature the attention which, in our opinion, it deserves. In the present paper we present some algorithms which, it is hoped, may be of interest to other workers in the field. Neglecting the more obviously trivial methods, we shall confine our attention to those which involve significant applications of ideas familiar to mathematicians and physicists.

The present time is particularly fitting for the preparation of an account of the subject, since recent advances both in pure mathematics and in theoretical physics have made available powerful tools whose very existence was unsuspected by earlier investigators. At the same time, some of the more elegant classical methods acquire new significance in the light of modern discoveries. Like many other branches of knowledge to which mathematical techniques have been applied in recent years, the Mathematical Theory of Big Game Hunting has a singularly happy unifying effect on the most diverse branches of the exact sciences.

For the sake of simplicity of statement, we shall confine our attention to Lions (*Felis leo*) whose habitat is the Sahara Desert. The methods which we shall enumerate will easily be seen to be applicable, with obvious formal modifications, to other carnivores and to other portions of the globe. The paper is divided into three parts, which draw their material respectively from mathematics, theoretical physics, and experimental physics.

The author desires to acknowledge his indebtedness to the Trivial Club of St. John's College, Cambridge, England; to the M.I.T. chapter of the Society for Useless Research; to the F. o. P., of Princeton University; and to numerous individual contributors, known and unknown, conscious and unconscious.

1. Mathematical methods

1. THE HILBERT, OR AXIOMATIC, METHOD. We place a locked cage at a given point of the desert. We then introduce the following logical system.

AXIOM I. *The class of lions in the Sahara Desert is non-void.*

AXIOM II. *If there is a lion in the Sahara Desert, there is a lion in the cage.*

RULE OF PROCEDURE. *If p is a theorem, and "p implies q" is a theorem, then q is a theorem.*

THEOREM I. *There is a lion in the cage.*

2. THE METHOD OF INVERSIVE GEOMETRY. We place a *spherical* cage in the desert, enter it, and lock it. We perform an inversion with respect to the cage. The lion is then in the interior of the cage, and we are outside.

3. THE METHOD OF PROJECTIVE GEOMETRY. Without loss of generality, we may regard the Sahara Desert as a plane. Project the plane into a line, and then project the line into an interior point of the cage. The lion is projected into the same point.

4. THE BOLZANO-WEIERSTRASS METHOD. Bisect the desert by a line running N-S. The lion is either in the E portion or in the W portion; let us suppose him to be in the W portion. Bisect this portion by a line running E-W. The lion is either in the N portion or in the S portion; let us suppose him to be in the N portion. We continue this process indefinitely, constructing a sufficiently strong fence about the chosen portion at each step. The diameter of the chosen portions approaches zero, so that the lion is ultimately surrounded by a fence of arbitrarily small perimeter.

5. THE "MENGENTHEORETISCH" METHOD. We observe that the desert is a separable space. It therefore contains an enumerable dense set of points, from which can be extracted a sequence having the lion as limit. We then approach the lion stealthily along this sequence, bearing with us suitable equipment.

6. THE PEANO METHOD. Construct, by standard methods, a continuous curve passing through every point of the desert. It has been remarked* that it is possible to traverse such a curve in

* By Hilbert. See E. W. Hobson, The Theory of Functions of a Real Variable and the Theory of Fourier's Series, 1927, vol. 1, pp. 456–457.

an arbitrarily short time. Armed with a spear, we traverse the curve in a time shorter than that in which a lion can move his own length.

7. A TOPOLOGICAL METHOD. We observe that a lion has at least the connectivity of the torus. We transport the desert into four-space. It is then possible* to carry out such a deformation that the lion can be returned to three-space in a knotted condition. He is then helpless.

8. THE CAUCHY, OR FUNCTIONTHEORETICAL, METHOD. We consider an analytic lion-valued function $f(z)$. Let ζ be the cage. Consider the integral

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{z-\zeta}\,dz,$$

where C is the boundary of the desert; its value is $f(\zeta)$, *i.e.*, a lion in the cage.†

9. THE WIENER TAUBERIAN METHOD. We procure a tame lion, L_0, of class $L(-\infty, \infty)$, whose Fourier transform nowhere vanishes, and release it in the desert. L_0 then converges to our cage. By Wiener's General Tauberian Theorem,‡ any other lion, L (say), will then converge to the same cage. Alternatively, we can approximate arbitrarily closely to L by translating L_0 about the desert.§

2. Methods from theoretical physics

10. THE DIRAC METHOD. We observe that wild lions are, *ipso facto*, not observable in the Sahara Desert. Consequently, if there are any lions in the Sahara, they are tame. The capture of a tame lion may be left as an exercise for the reader.

11. THE SCHRÖDINGER METHOD. At any given moment there is a positive probability that there is a lion in the cage. Sit down and wait.

12. THE METHOD OF NUCLEAR PHYSICS. Place a tame lion in the cage, and apply a Majorana exchange operator‖ between it and a wild lion.

As a variant, let us suppose, to fix ideas, that we require a male lion. We place a tame lioness in the cage, and apply a Heisenberg exchange operator¶ which exchanges the spins.

13. A RELATIVISTIC METHOD. We distribute about the desert lion bait containing large portions of the Companion of Sirius. When enough bait has been taken, we project a beam of light across the desert. This will bend right round the lion, who will then become so dizzy that he can be approached with impunity.

3. Methods from experimental physics

14. THE THERMODYNAMICAL METHOD. We construct a semi-permeable membrane, permeable to everything except lions, and sweep it across the desert.

15. THE ATOM-SPLITTING METHOD. We irradiate the desert with slow neutrons. The lion becomes radioactive, and a process of disintegration sets in. When the decay has proceeded sufficiently far, he will become incapable of showing fight.

16. THE MAGNETO-OPTICAL METHOD. We plant a large lenticular bed of catnip (*Nepeta cataria*), whose axis lies along the direction of the horizontal component of the earth's magnetic field, and place a cage at one of its foci. We distribute over the desert large quantities of magnetized spinach (*Spinacia oleracea*), which, as is well known, has a high ferric content. The spinach is eaten by the herbivorous denizens of the desert, which are in turn eaten by lions. The lions are then oriented parallel to the earth's magnetic field, and the resulting beam of lions is focussed by the catnip upon the cage.

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* H. Seifert and W. Threlfall, Lehrbuch der Topologie, 1934, pp. 2–3.

† *N.B.* By Picard's Theorem (W. F. Osgood, Lehrbuch der Funktionentheorie, vol. 1, 1928, p. 748), we can catch every lion with at most one exception.

‡ N. Wiener, The Fourier Integral and Certain of its Applications, 1933, pp. 73–74.

§ N. Wiener, *l. c.*, p. 89.

‖ See, for example, H. A. Bethe and R. F. Bacher, Reviews of Modern Physics, vol. 8, 1936, pp. 82–229; especially pp. 106–107.

¶ *Ibid.*

MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

This Department of the MONTHLY *has been created as an experiment to afford a place for the discussion of the place of mathematics in education. With this topic will naturally be associated other matters emphasizing the educational interests of those who teach mathematics. It is not intended to take up minute details of teaching technique. The columns are open to those who have thoughtful critical comment to make, be it favorable or adverse to the cause of mathematics. The success of this department obviously will depend upon the cooperation of the readers of the* MONTHLY. *Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

THE FUNCTION CONCEPT IN ELEMENTARY TEACHING AND IN ADVANCED MATHEMATICS*

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The ways in which the function concept occurs throughout mathematics have been dwelt upon by many writers. One of the first to emphasize its vital importance in all mathematical teaching was Felix Klein, who played such a prominent rôle in mathematics a generation ago in Germany. In this country, the Report of the National Committee on the Reorganization of Secondary-School Mathematics brought before teachers of mathematics the fundamental importance of the function concept and of functional thinking. Through this Report, and in other papers, I myself had a part in stressing the importance of these ideas.

In its elementary phases, the function concept agrees to all intents with the idea of the interdependence of quantities, though the more refined definitions of function do not *require* that there be dependence. Broadly speaking, then, the function concept in elementary work means the dependence of one quantity upon one or more others. The simple questions that arise are two-fold: (1) Upon what quantities does one in which we are interested depend? and (2) What is the precise manner in which the one depends upon the other or others? Both of these questions are normally present in practically every mathematical subject; they occur also very frequently in problems of living, and in the several sciences. Properly to be within the range of functional thinking, we need not concern ourselves necessarily with *both* of these questions: If either of the two questions is discussed, functional thinking is being done. Thus, if we think of the resistance of the air to the motion of an automobile, functional thought is involved when we say that the resistance depends upon the speed. To think functionally, it is not necessary to know the precise formula; indeed, I doubt very much whether any such formula can be given by anyone. Any observation, such as that an increase in speed causes an increase in resistance, falls under the head of functional thought. The search for experimental data is another phase of functional thinking, through which statistical information regarding

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resistances at various speeds is obtained. The drawing of a graph gives still more definite information about the nature of the relationship.

That the bare fact of dependence requires intelligent thought may be illustrated best by an example. For many centuries, the engineers of the Roman Empire constructed aqueducts, and they supplied water to cities for domestic purposes and to farmers for irrigation. The quantities of water furnished were determined solely by the size of the opening through which the water was delivered. Apparently the first man to notice that the amount of water that would pass through a given opening depended also on the *pressure* (or "head") was Leonardo da Vinci. He also stated the approximate formula that is still in use by engineers. In this country, even during the latter half of the nineteenth century, water rights for irrigation were stated in "inches," that is, per square inch of opening, without regard to the pressure; many legal battles have resulted, especially in California, as a result of this practice. I have stated this as a conspicuous instance of the failure, even by fairly intelligent persons vitally concerned, to note the dependence of one quantity upon another. We can not expect, then, that even fairly intelligent people will notice upon what quantities a given one depends, though they themselves may be deeply interested. Most certainly we can not expect children to know without training anything whatever about the ways in which quantities of very simple kinds are related.

In this paper, it is not my purpose to enter into detail regarding any one subject or any one school-age, more particularly because studies abound regarding such isolated instances; rather my purpose will be to show the persistence of the function idea through every stage of mathematical development. Finally, I shall touch upon the advanced studies which center around the function concept.

The emphasis placed upon functional thinking as the great central theme of all mathematics in the Report of the Committee mentioned above, was entirely sound and was very timely. The criticisms of mathematics as a school subject have been very severe and very bitter during the last twenty years. It has been said by many educators that mathematics has no relationship to life or to the affairs of the world. Harsh criticism has been leveled against the formalisms of the traditional courses in algebra and geometry in the schools; and these criticisms have had a considerable basis of truth. School authorities have insisted upon what is called "integration" of school subjects, which means in mathematics the constant attention to questions that arise in physics and in other sciences, and in the outside world. Often these educational leaders have concluded, from a superficial examination of traditional courses in mathematics, that mathematics does not lend itself to integration with physics, with the other sciences, or with affairs of life. If you will examine again the traditional courses, even in textbooks now in use, you will see that they are not greatly to be blamed for such a false conclusion. I do not blame them severely, though I do hold that expert advice on school subjects should have been based upon something more than a superficial examination of what has been done in the past.

Something more than a superficial examination will make clear that quantities and relationships between quantities do occur—in fact they abound—in physics and in all science. They do occur—in fact they abound—in the life of the world. Can we then integrate mathematical teaching with life and with science? We can indeed if we choose to give more than lip-service to the Report of the Committee mentioned above, and if we stress the function concept as the central theme in every course. Moreover, we shall not lack for examples and for illustrations if we do so. I propose to run through the whole gamut of the elementary mathematical subjects with you, to see where we may well alter the traditional presentations.

Even in elementary arithmetic, functional thinking may be carried on at every stage. The *word* "function" should not be used, of course, but the *word* is not the important thing. If one apple costs two cents, how much will five cost? Every such question in arithmetic associates two or more related quantities. The questions are childish, of course; they should be. What is lacking is appreciation of what quantities affect the answers. We should ask: "What do you need to know?" Why, of course, the price of one apple and the number of apples. In such childish ways the idea of relationship between quantities is born in the mind. We may keep it fresh, or we may let it die. Try to think, if you will, of some topic in arithmetic that does not involve related quantities. There are none. What is lacking is our own appreciation, and our insistence that the child see what the relationship is, what quantities are tied together, what must be known to find a desired answer. I can not detail them all. I defy you, however, to find any topic in arithmetic in which this question is not the paramount one. It is often buried beyond recall in a maze of formalisms.

The story in algebra is not far different. I am aware, of course, that traditional courses emphasize formalisms that are far from functional thinking. When they do, I challenge you to discover any thought whatever in what is being done. Thus overemphasis has been placed upon factoring; you will find that I do not go beyond the Report mentioned above when I condemn unreservedly any elaborate exercises in factoring. Solution of equations is made to appear the central theme in many courses in algebra, and the old-time texts went to absurd extremes in intricacies of simultaneous quadratics that are quite useless. What is really true is that the setting up of any equation does bring in functional thought. There must be some unknown quantity; its value is sought *always* by giving some other quantity or quantities which are related to the unknown one. We set up the equation *always* by expressing the connection that exists between the known and the unknown quantities. This is the prime purpose of the entire process. Merely to know how to solve the equation after it has been set up gives no fundamental power over real situations. This is emphatically proved by the inability of students to solve problems stated in English. They often say "If you will show me how to set up the equation, I can solve it." The truth is that the ability to recognize the relations between

the quantities mentioned in the problem is the vital thing, and the mechanical solution of the resulting equation is a minor matter.

In geometry, as it is traditionally taught, the principal functional thinking is in the mensuration formulas. That the area of a square is the square of the side is a typical instance. Other areas (of triangles, rectangles, circles, etc.) and volumes of solids are given by specific formulas. Often, however, little stress is placed upon real comprehension: thus that a circle twice as wide as another has four times the area, is often regarded as unimportant. But there are many other functional ideas in geometry, most of which are suppressed. Thus if a triangle is determined by two sides and the included angle, the other sides *must be* functions of the three given parts. The idea of trigonometric calculation is intrinsic in the congruence theorems. In fact, the better texts, both in geometry and in algebra, now give the elements of trigonometric calculation for right triangles. Throughout the geometry, in a similar manner, the question of dependence of parts of a figure upon other parts can be emphasized, and it is often far more important than some of the alleged logic that is more often too much emphasized. For example, in the relationships between arcs and angles the relationships which are really vital are too often made secondary, and such fallacious logic as that of the old treatments of incommensurables is called important. The result is that our strenuous efforts to give "mental discipline" miscarries, for there is real discipline to be had here in training the student to think functionally. It is not incidental, but quite vital, throughout geometry, that the student be trained to see the relationships between the various parts of each figure. This is, in fact, the real *geometry*.

Trigonometry is functional on its face. It is the first place in which we traditionally use the word "function." So be it. I have no brief for introduction of mere words into our teaching. Even in trigonometry, however, there has been too great a tendency to emphasize dreary and profitless formalisms that involve no real thought at all, as in the numerous problems in so-called trigonometric identities. Except for a very few simple ones, these might better be sacrificed to more thought about the way in which the trigonometric functions vary. Thus the graphs of the trigonometric functions, and some thought about quantities in physics and elsewhere which vary trigonometrically, is far more vital.

College mathematics has been discussed less than has secondary mathematics. The same questions arise, however, and the traditional courses dwell all too little upon functional thinking. In analytic geometry, for example, what is of primary importance is to know how to indicate graphically the behavior of relations which are given by equations. By contrast, it is not particularly important to discuss at length the general equation of the second degree, and the general conic. Far simpler questions that are often omitted entirely are to picture graphically the relations between quantities connected by such simple equations as $xy=k$, (inverse variation); $y=x^n$ for a variety of values of n;

$y=(ax+b)/(cx+d)$; $y=e^x$; $y=\sin x$; *etc.* If these are accompanied by "integrating" explanation of instances of such relations in science, great insight may be given into a variety of functional thinking. I should not omit to mention the manner of dealing with equations of tangents to curves. Here I may mention the traditional process of finding (by special devices) the equations of tangents to ellipses and hyperbolas, to the exclusion even of the tangent to the curve $y=x^n$ for integral values of n. These special devices leave the student wholly unaware that he is dealing with the rate of increase of a function. By comparison, the finding of tangents to ellipses and hyperbolas, which is really far more intricate, should be postponed to the calculus, and the far simpler cases of explicit functions, particularly the simple polynomial functions, should receive the great emphasis. Throughout the whole course, it is easy to see in any text whether or not the author himself realizes the importance of the function concept. In so far as he does, and in so far as the teacher emphasizes the functional thinking that is involved, the course will increase in value to the student, either in mathematics itself or in any of its applications.

Similar differences exist throughout the calculus. If the derivative is presented as a formalized thing, the rules can be learned and numerous examples can be handed in, without the slightest comprehension on the part of the student of the real meaning of the calculus. I know, because I did it just that way. The calculus begins to have vivid meaning when the student realizes that the derivative of a function represents the rate of increase. This should be held before his attention at all times. Thus the finding of equations of tangents to curves is not particularly important in itself; the important thing is the finding of the *slope* of the tangent, and the realization of its significance. It is easy to tell in any class or in any textbook whether or not the attention is upon functional thinking or upon the formal solution of set exercises. If the examples that are given are frequently or principally the familiar ellipses and hyperbolas, and such curves as $x^{2/3}+y^{2/3}=a^{2/3}$, you may be fairly sure that not much functional thinking is being done. If, on the other hand, there is quite a good proportion of examples that are based on $y=x^n$, $y=$a polynomial, $y=e^x$, $y=\sin x$, and so on, there is a higher probability that the student is getting some real comprehension. If there is much talk of geometrical properties of conjugate diameters and poles and polars, there is not much chance that the student is going to have any use for what he is learning; I know, for that is what I did. If there is a good deal of talk about rates of increase, and of speed and acceleration, and of relative rates, then there is more likelihood that functional thinking is going on. It does not take great practice to know whether or not the textbook, or the teacher, is in sympathy, or out of sympathy, with functional thinking. These things, both their lack and their presence, characterize every chapter of every text, and every lesson of every class.

I may call "advanced" the mathematics beyond the calculus. In this field, through the junior and senior years and in the graduate schools, mathematics normally divides into several branches. Time was when a student might pro-

ceed even to the doctorate without much more attention to the function concept than that which he got in his course in calculus, and his course in calculus may have been extremely weak in functional thinking, as was my own. I learned mainly the properties of many intricate curves, and much about such geometric entities as polars and subtangents. I did not know (though I was given high grades) that a derivative could mean a rate of increase, nor that an integral could be thought of as a limit of a sum. I knew much geometric lore, most of which I found to be useless, but I knew almost no calculus, that is, I knew nearly nothing about functional thought.

We still have, and we should have, specialists in geometry. No longer, however, can anyone be regarded as a mathematical graduate, or graduate student, if he knows only geometric things, and does not know functional thinking. The day of over-emphasis upon pure geometry has passed. Professor Julian Coolidge read before the Mathematical Association an excellent lecture in which he pointed out that the day of geometrical training, to the exclusion of all else, has passed. I believe that he—and perhaps some of you—acknowledged that with regret. It does not mean that any part of geometry is dead; that will never happen. It means, however, that fundamental training in functional thinking is insisted upon now, and will be throughout the future, for all who specialize in mathematics. In this, there is no cause for regret.

Courses in advanced calculus, or (what is the same thing) in mathematical analysis, are now part of every course for every student who seeks a degree in mathematics. In such a course, without too much display, the primary consideration is a more thorough training in functions and in functional thinking. Here for the first time an accurate idea of "function" becomes necessary. We say now, for the first time, that one quantity is a function of another if, when any value of the second is given (in a specified range), the value of the first one is determined. We proceed to examine more carefully all of the ideas of the calculus, such as limit, continuous function, derivative, integral, and the various theorems. We discuss more accurately the number system itself, and the associated ideas of sets of numbers. We deal with extensions of the elementary functional concepts, such as line-integrals, infinite series, and the like. In all, such a course should increase very greatly the student's comprehension of functional behavior and his power to think functionally. With such a course every teacher of a more elementary course becomes function-conscious, and he will normally give his own students more power to think functionally. Without such a course, in my opinion, no teacher should be permitted to teach any collegiate subject in mathematics.

For a moment, I would like to call your attention to some traditional aspects of the more advanced courses that bear the name "function." These are, specifically, the courses (usually graduate courses) entitled *Functions of a real variable*, and *Functions of a complex variable*. Studies in these two fields have been very intensive and the literature is now so formidable that I do not believe that any one man is master of it all. In the courses, naturally, only the beginning, and

only the fundamental ideas, are considered. I shall not attempt to outline even these first courses: either you have had them, in which case my explanations would be superfluous, or else a quick summary would be hard to grasp. What I do want to try to do is to contrast the very wide difference in attitude which characterizes the traditional courses in these subjects. When we speak of functions of a real variable, we allow ourselves every latitude: the definition of *function* is perhaps even more refined than that which I gave above, and no functions are excluded from discussions, except by hypothesis in each theorem. Thus we do not assume that a function must be continuous, even at any one of the points of its graph. Certainly we do not assume that the function has a derivative. We may discuss, to be sure, certain classes of functions: thus we may and we do derive theorems on the properties of continuous functions, or on functions of limited variation, or on functions that possess derivatives. When we do so, however, we recognize openly that we are segregating out a particular class of functions; nobody is deceived into thinking that *all* functions possess the properties that are enunciated for the functions of a particular class.

In the traditional course on functions of a complex variable, however, a very different procedure is traditional. It appears to me that it deceives many students. By tradition, we assume at the start that the function (of a complex variable) has a derivative, at least except for certain few values of the independent variable. Having done so, we tend traditionally to lose sight of the possible existence of other functions. This is all the more serious because the restriction imposed by the hypothesis that the derivative exists is far and away a more strenuous restriction than would be the similar assumption for functions of real variables. I myself was perplexed by this difference in traditional attitude, and I have been among those who have tried in recent years to extend to a base as broad as that for real variables the theory of functions for complex variables. Considerable progress has been made in this field, and I am quite sure that the future will see the treatments in the two fields placed openly upon the same basis. To do so will remove many of the perplexities which beset the beginning student. This will not prevent a treatment of functions that possess derivatives, as it does not prevent such a treatment now in real variables, but the student will not be deceived into thinking that the theorems derived are true for *all* functions.

Thus in the traditional theory for complex variables, *every* function (single-valued) is expansible in Taylor Series. It was not clear to me when I learned this that this theorem is true *only* for the restricted group of functions that possess derivatives. Hence the obvious fact that functions of real variables do not obey the same theorem seemed quite weird to me. I believe that this one case is responsible for many of the quite false statements that are made in texts on the calculus about expansion of functions in Taylor series.

Many other such apparent discrepancies exist between the two theories of functions, the one of real variables, and the other of complex variables. Thus a

function of a complex variable (in the restricted sense) is completely determined by its values in any small two-dimensional region; the apparently analogous theorem is wholly false for real variables. I shall not extend this discussion, though very numerous instances exist, some of which may have confused and mystified some of you, as they did me. Such confusion disappears entirely when we recognize openly that the traditional theory of functions of a complex variable is valid only for a special class of those functions. I hope that this bare statement may be of assistance to many of you who may have had difficulties similar to my own.

In closing, I may point out also that there are many theorems in the traditional theory of functions of a complex variable which happen to be valid for very broad classes of functions, so that the restriction traditionally made is not always needed. Thus the theorem that the absolute value of a single-valued function of a complex variable can not have a maximum happens to be true for very much wider classes of functions than those of the traditional theory. It is not necessary to assume that the derivative of the function exists; indeed the hypothesis that the ordinary jacobian exists and is different from zero, is sufficient. The more complicated theorems, usually called the Liouville theorems, also hold good under such a hypothesis. An interesting question is to discover how much the hypotheses of the traditional theorems may be reduced without affecting the validity of the conclusions.

I will not strain your patience further. My intention has been to review briefly the entry of the function concept into all mathematics, from the first steps of the arithmetic of children to the more advanced conscious treatment of the theory of functions as such. I trust that you will agree—as the Report of the Committee has it—that this is the one theme which tends to unify all of mathematics and to permit its integration with life and with science. Would that all teachers of mathematics, all students of mathematics, knew this truth! Would that all the professed leaders of education were aware of it!

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A NOTE ON CERTAIN FORMULAS USED IN SAMPLING THEORY

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In the October 1937 number of this MONTHLY, p. 491, C. N. Mills is reported to have given the formula*

$$\frac{1}{N}\left\{\sum (b-a)^2 - \frac{\sum (b-a)^2}{N}\right\}^{1/2}$$

as being better adapted to numerical computation than the usual formula for the standard error of the difference between the means of paired items a_i, b_i $(i=1, 2, \cdots, N)$. The formula printed is not correct. It should be

$$\frac{1}{N}\left\{\sum (b-a)^2 - \frac{(\sum b - \sum a)^2}{N}\right\}^{1/2},$$

and the derivation involves the assumption that N is large.

This suggests an instance where it seems that an incorrect, or inappropriate, formula is given in certain textbooks on sampling theory. Let $\bar{x}_1$, $\bar{x}_2$ be the means and s_1, s_2 the standard deviations of two independent samples consisting of N_1 and N_2 variates, respectively, from a normal universe with mean m and variance σ^2. Then the variance of the difference between the means is equal to $\sigma^2(N_1+N_2)/N_1N_2$, and, as is well known [1; p. 357], the variable

$$t = \frac{\bar{x}_1 - \bar{x}_2}{\sigma}\left(\frac{N_1N_2}{N_1+N_2}\right)^{1/2}$$

is normally distributed with mean zero and unit standard deviation. However, in practical problems σ^2 is seldom available and must be estimated from the samples. Let θ be a function of the variates supplied by a sample, or samples, for estimating a population parameter θ. If the expected value of θ is θ then θ is called an unbiased estimate of θ. An unbiased estimate of σ^2 is

$$\dot{\sigma}^2 = \frac{N_1s_1^2 + N_2s_2^2}{N_1+N_2-2}.$$

Fisher [2] and others [1 and 3] have shown that the variable

$$(1) \qquad t = \frac{\bar{x}_1 - \bar{x}_2}{\dot{\sigma}}\left(\frac{N_1N_2}{N_1+N_2}\right)^{1/2}$$

* *Note by Editor.* The formula submitted by Professor Mills was incorrectly printed. The numerator of the fraction was $[\sum(b-a)]^2$, and not $\sum(b-a)^2$. J.R.M.

is distributed in accord with the "Student" curve

$$F_n(t) = K_n(1 + t^2/n)^{-(n+1)/2}, \tag{2}$$

where $1/K_n = n^{1/2}B(n/2, 1/2)$, B being the Beta function, and $n = N_1 + N_2 - 2$ is the number of so-called "degrees of freedom."

Now as N_1 and N_2 become large, $(N_1+N_2)/(N_1+N_2-2)$ tends toward unity and (1) tends toward the value

$$t = \frac{\bar{x}_1 - \bar{x}_2}{(s_1^2/N_2 + s_2^2/N_1)^{1/2}}. \tag{3}$$

Since (1) is asymptotically normally distributed, as can be seen from (2), the procedure of referring (3) to a normal probability scale would not be invalid to any appreciable extent for large values of N_1 and N_2. But the procedure commonly given in textbooks for testing a null hypothesis [4] that two samples are from the same universe, using their means as a criterion of judgment, is to refer

$$t = \frac{\bar{x}_1 - \bar{x}_2}{(s_1^2/N_1 + s_2^2/N_2)^{1/2}}$$

instead of (3) to a normal probability scale.

It is interesting to observe that if one of the samples, say N_2, becomes infinitely large so that $\bar{x}_2$ becomes m and s_2 becomes σ, then (3) becomes

$$t = \frac{(\bar{x}_1 - m)(N_1)^{1/2}}{\sigma} \tag{4}$$

which, if the subscripts are dropped, is the formula used in testing a null hypothesis that a given sample comes from a universe with a proposed mean. If the universe is normal then (4) is normally distributed. When σ is unknown we may use in its place an estimate, $\dot{\sigma}$, defined by

$$\dot{\sigma}^2 = s^2N/(N - 1),$$

where s^2 is the variance of the sample. But if we substitute $\dot{\sigma}$ for σ in (4) and calculate

$$t = \frac{(\bar{x} - m)(N)^{1/2}}{\dot{\sigma}} = \frac{(\bar{x} - m)(N - 1)^{1/2}}{s}, \tag{5}$$

we are not justified in asserting that (5) is normally distributed unless N is large. As a matter of fact, (5) is distributed in accord with (2) for $n = N - 1$.

In 1925, "Student" published in Metron [5] an extensive table of the probability integral of (2). More recently, Fisher [6] has given a short table of the probability $P_n(t)$ for deviations outside $\pm t$, for values of t and n commonly met in practice, where

$$P_n(t) = 1 - 2\int_0^t F_n(t)dt. \tag{6}$$

If a sample, or samples, could not reasonably have arisen from the universe proposed, the null hypothesis is said to be refuted for the level of significance adopted. If the significance test yields a verdict of "not significant" at the probability level adopted, we say that the null hypothesis is not refuted or contradicted at that level. It is open to the investigator to be more or less exacting concerning the smallness of the probability he would require before he would be willing to admit that his test has demonstrated a significant result. However, the following rule is conventional among certain workers. If $P_n(t) \geqq .05$, t is not significant; if $P_n(t) \leqq .01$, t is significant; if $.05 > P_n(t) > .01$, any conclusion about t is doubtful until more information is available. The same rule is used when it is valid to replace $F_n(t)$ by the normal probability function in (6). Others may prefer a more conservative level of significance.

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4. R. A. Fisher, The Design of Experiments. Oliver and Boyd, London.

5. "Student," Metron, vol. 5, 1925, no. 3, pp. 114–120. "Student's" real name was W. S. Gosset. An interesting biographical sketch of Mr. Gosset has recently been given by J. Neyman in Journal of American Statistical Association, vol. 33, pp. 226–228.

6. R. A. Fisher, Statistical Methods for Research Workers. Oliver and Boyd, London.

A NOTE ON HEDGING

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A problem of considerable interest to elementary students and to others concerns the conditions under which a gambler can so hedge his bets that a profit is assured. Consider a contest in which there are three entries, A, B, and C, one of whom must win. Let their respective odds of winning be *quoted* as a to b, c to d, and e to f. Thus the contestants' "probabilities" of winning may be defined as

$$P_A = a/(a+b), \qquad P_B = c/(c+d), \qquad P_C = e/(e+f).$$

Suppose that we have a sum to wager on this contest, and that we distribute it over the three entries in the following proportion: x on A to win, y on B to win, and z on C to win, where $x+y+z=1$. Then if A wins, our gain (profit or loss) is

$$G_A = \frac{b}{a}x - (y+z) = \frac{b}{a}x - (1-x)$$

$$= x/P_A - 1.$$

Similarly,

$$G_B = y/P_B - 1,$$

and

$$G_C = z/P_C - 1.$$

Let us now see if we can so place our respective bets that the same profit K ensues no matter which entry wins. We wish that

$$G_A = G_B = G_C = K,$$

whence

$$x/P_A = y/P_B = z/P_C = K + 1, \tag{1}$$

i.e.,

$$x:y:z = P_A:P_B:P_C. \tag{2}$$

Thus, if we distribute our bets in proportion to the "probabilities" of winning we are assured of a gain K. In order that our gain be an actual profit and not a loss, a further condition is necessary. By (1) we have

$$x = (K + 1)P_A, \qquad y = (K + 1)P_B, \qquad z = (K + 1)P_C,$$

and, adding,

$$1 = (K + 1)(P_A + P_B + P_C),$$

whence

$$K = 1/(P_A + P_B + P_C) - 1,$$

which is positive if

$$P_A + P_B + P_C < 1. \tag{3}$$

An examination of (2) and (3) will show that they are both necessary and sufficient conditions for a constant profit K. These results can be generalized in an obvious manner. If (3) is not satisfied, but $P_A+P_B+P_C>1$, we need only bet *against* each entry to be assured of a profit. If $P_A+P_B+P_C=1$, it is not possible to hedge the bets in such a way that a profit is certain.

An interesting application of the above theory is found in the case of a contest in which there are but two entries. The following news item appeared last fall: "The betting odds listed the Tigers as a one to four choice $\cdots$. For those who liked the Bronchos, bookmakers were laying eleven to five." Thus the odds quoted for this game were (a) 11 to 5 that the Tigers win, 1 to 4 that the Bronchos win; or (b) 5 to 11 that the Bronchos win, 4 to 1 that the Tigers win. The condition (3) is satisfied by (a) only, in which case

$$P_A = \frac{11}{16}, \qquad P_B = \frac{1}{5}, \qquad P_A + P_B = \frac{71}{80} < 1.$$

To obtain the same profit no matter which team wins the game, we should distribute our bets by (2) in the proportion 11/16:1/5, or 55:16. Suppose we place \$550 on the Tigers at 11 to 5, and \$160 on the Bronchos at 1 to 4. Then if the Tigers win, we gain 5/11 of \$550, or \$250, and lose \$160. If the Bronchos win, we lose \$550 but gain 4×\$160, or \$640. In either case, our net profit is \$90.

APPROXIMATE EVALUATION OF CERTAIN ELLIPTIC INTEGRALS

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Consider the Elliptic Integral of the second kind:

$$E(k, \phi) = \int_0^{\phi} \sqrt{1 - k^2 \sin^2 \phi}\, d\phi = \int_0^{\phi} \sqrt{\cos^2 \phi + h \sin^2 \phi}\, d\phi,$$

where $1 - k^2 = h$. On expanding and integrating we get

$$E(k, \phi) = (1 - h/2) \sin \phi + \tfrac{1}{2} h \log \tan (45° + \phi/2) - \frac{h^2}{8} \int_0^{\phi} \tan^3 \phi \sin \phi\, d\phi + \cdots .$$

Thus we obtain the following simple approximation formula which holds for small values of ϕ and values of k^2 near one:

$$E(k, \phi) = (1 - h/2) \sin \phi + \tfrac{1}{2} h [\log \tan (45° + \phi/2)].$$

As an illustration of its applicability consider

Example 1. Given $\phi = 40°$, $k^2 = .8$. Find E.

$$E(k, \phi) = .9 \sin 40° + .1 \log \tan 65°.$$

$$E(k, \phi) = .5785 + .0763 = .6548 \text{ (Tables give .6551)}.$$

Example 2. Given $\phi = 22°36'$, $k^2 = .92$. Find E.

$$E(k, \phi) = .96 \sin 22°36' - .04 \log \tan 56°18'.$$

$$E(k, \phi) = .368923 + .016204 = .38513 \text{ (Tables give .3851)}.$$

In the latter example our formula probably gives the result quicker than does the tables. It would be impracticable to use formula 527, Peirce's Tables, on such a problem. The beauty of our formula lies in the fact that we can readily estimate the magnitude of our error and there is no increased labor involved when ϕ is not an even degree.

A correspondingly simple formula may be obtained for

$$F(k, \phi) = \int_0^{\phi} \frac{d\phi}{\sqrt{1 - k^2 \sin^2 \phi}}.$$

Thus

$$F(k, \phi) = (1 + h/4) \log \tan (45° + \phi/2) - h/4 \tan \phi \sec \phi + \cdots,$$

where $h = 1 - k^2$, ϕ is small, and k^2 is near one.

Example 3. Given $\phi = 30°$, $k^2 = .88$. Find F.

$$F(k, \phi) = (1.03) \log \tan 60° - .03 \tan 30° \sec 30°.$$

$$F(k, \phi) = .5658 - .0200 = .5458 \text{ (Tables give } .5456).$$

ON THE AVERAGE-SLOPE METHOD OF SOLVING DIFFERENTIAL EQUATIONS

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In the solution by successive approximations of differential equations of the first order which may be solved for y', a method sometimes employed is the following:

Given

$$y' = dy/dx = f(x, y), \tag{1}$$

we wish a solution which passes through the point (x_0, y_0). If y_0' is the slope at the point (x_0, y_0), a first approximation to the value of y_1 is

$$y_1 = y_0 + \Delta y_0', \tag{2}$$

where Δ is the increment of x. Substituting the value of y_1 from (2) in (1), together with the value $x_1 = x_0 + \Delta$, gives a first approximation to the true slope at (x_1, y_1). A second approximation to the true value of y_1 is then obtained by using in (2) the average of y_0' and the first approximation to y_1'. This second approximation to y_1, when substituted in (1), gives a second approximation to the slope at the point (x_1, y_1); and a third approximation to y_1 is obtained by using the average of y_0' and the second approximation to y_1', substituted again in (2). This method is continued until two successive values of y_1 differ by a sufficiently small amount. The simple example shown below, however, proves that, even though it be employed indefinitely, this method does not necessarily lead to the correct value of y_1'.

Consider the equation:

$$y' = y. \tag{3}$$

We shall seek the solution passing through the point (x_0, y_0). We have

$$y_0' = y_0, \tag{4}$$

and writing Δ for Δx

$$y_{1,1} = y_0 + \Delta y_0, \tag{5}$$

where the subscript (1, 1) means the first approximation to the value of y_1. Likewise, the subscript (1, 2) shall mean the second approximation to the value of y_1, *etc.*; and Δ is the increment of x.

Substituting (5) in (3), we obtain

$$y'_{1,1} = y_0 + \Delta y_0. \tag{6}$$

The average of (4) and (6) is

$$y_0 + y_0\Delta/2. \tag{7}$$

Substituting the average slope from (7) back into (5), we have

$$y_{1,2} = y_0 + y_0\Delta + y_0\Delta^2/2. \tag{8}$$

Substituting (8) in (3), we obtain

$$y'_{1,2} = y_0 + y_0\Delta + y_0\Delta^2/2. \tag{9}$$

The average of (4) and (9) is

$$y_0 + y_0\Delta/2 + y_0\Delta^2/4. \tag{10}$$

Substituting the average slope given by (10) back into (5), we get

$$y_{1,3} = y_0 + y_0\Delta + y_0\Delta^2/2 + y_0\Delta^3/4. \tag{11}$$

The continuation of this method will give as the nth approximation to the value of y_1:

$$y_{1,n} = y_0(1 + \Delta + \Delta^2/2 + \Delta^3/3 + \cdots + \Delta^n/2^{(n-1)}). \tag{12}$$

Letting n approach infinity in (12), we have

$$\lim_{n\to\infty} y_{1,n} = y_0 \frac{1 + \Delta/2}{1 - \Delta/2}; \tag{13}$$

(13) is then the value of y_1 obtained through the average-slope method described above. However, integration of (3) and substitution of the given point (x_0, y_0) gives $y = y_0 e^{(x-x_0)}$. Substituting $x_1 = x_0 + \Delta$, we get $y_1 = y_0 e^{\Delta}$.

CONCERNING NAPIER'S RULES, A CORRECTION

W. R. RANSOM, Tufts College

In a note in the January MONTHLY, I remarked that Napier's own proof seems not to have been reproduced in any American textbook. It has now been called to my attention that it does appear in R. E. Moritz's *Plane and Spherical Trigonometry* (John Wiley and Sons, New York, 1913) on page 18 of the part on spherical trigonometry. I regret that I did not have this book at hand when the previous note was written.

NOTE CONCERNING PROFESSOR DODD'S PAPER

Professor E. L. Dodd has reported that after it was too late to make a change in his recent article (this MONTHLY May, 1938) he discovered that the results given in his Sections 2 and 3, pp. 302–304, had been published by Professor Dunham Jackson in: "Note on quartiles and allied measures," *Bulletin of the American Mathematical Society*, vol. 29, 1923, pp. 17–20. E. J. M.

RECENT PUBLICATIONS

EDITED BY TOMLINSON FORT, Lehigh University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

A Text-Book of Convergence. By W. L. Ferrar. London and New York, Oxford University Press, 1938. 7+192 pages. $3.50.

Portraits of Eminent Mathematicians. By D. E. Smith. Portfolio II. New York, Scripta Mathematica, 1938. $3.00.

Principles of Mathematics. By B. Russell. Second Edition. New York, W. W. Norton and Company, Inc., 1938. 39+534 pages. $5.00.

Duodecimal Arithmetic. By G. S. Terry. New York, Longmans, Green and Company, 1938. 407 pages. $7.50.

Magic Squares of $(2n+1)^2$ *Cells.* By M. J. van Driel. London, Rider and Company, 1936. 90 pages. $2.75.

Trigonometry. By A. R. Crathorne and E. B. Lytle. Revised Edition. New York, Henry Holt and Company, 1938. 9+191 pages. Logarithmic and Trigonometric Tables. 16+95 pages. $2.00 with tables. $1.70 without tables.

Freshman Mathematics. By H. L. Slobin and W. E. Wilbur. Revised Edition. New York, Farrar and Rinehart, Inc., 1938. 20+584 pages. $3.50.

Functions of Real Variables. By W. F. Osgood. Reprint of 1936 edition published in Peking. New York, G. E. Stechert and Company, 1938. 399 pages. $4.00.

Functions of a Complex Variable. By W. F. Osgood. Reprint of 1936 edition published in Peking. New York, G. E. Stechert and Company, 1938. 257 pages. $3.00.

A Course of Pure Mathematics. By G. H. Hardy. Seventh Edition. Cambridge, University Press, and New York, The Macmillan Company, 1938. 12+498 pages. $3.75.

Essentials of Engineering Mathematics. By J. P. Ballantine. New York, Prentice-Hall, Inc., 1938. 11+502+76 pages. $3.75.

Calculus. By E. S. Smith, M. Salkover and H. K. Justice. New York, John Wiley and Sons, 1938. 12+558 pages. $3.25.

Construction, Classification and Census of Magic Squares of Order Five. By A. L. Candy. Lincoln, Nebraska, A. L. Candy, 1938. $1.00.

Aspects of Science. By T. Dantzig. New York, The Macmillan Company, 1938. $3.00.

The Queen of the Sciences. By E. T. Bell. New York, G. E. Stechert and Company, 1938. 138 pages. Reprint. $1.50.

An Introduction to Business Statistics. By J. R. Stockton. New York, D. C. Heath and Company, 1938. 5+378 pages. $3.00.

Erbmathematik. Theorie der Vererbung in Bevölkerung und Sippe. By H. Geppert and S. Koller. Leipzig, Quelle and Meyer, 1938. 8+228 pages. RM 16.

Actualités Scientifiques et Industrielles. Paris, Herman et Cie.

502. *Nécessaire mathématique.* By M. Curie and M. Prost. 1937. 20 fr.

542. VII. *Recherches sur la théorie cinétique des liquides (première partie): Fluctuations en densité.* By J. Yvon. Théories Mécaniques (Hydrodynamique—Acoustique). Published under the direction of Y. Rocard, 1937. 63 pages. 18 fr.

543. VIII. *Recherches sur la théorie cinétique des liquides (deuxième partie): La propagation et la diffusion de la lumière.* By J. Yvon. Théories Mécaniques. (Hydrodynamique—Acoustique). Published under the direction of Y. Rocard, 1937. 133 pages. 18 fr.

544. IX. *Les Phénomènes d'auto-oscillation dans les installations hydrauliques.* By Y. Rocard. Théories Mécaniques. (Hydrodynamique—Acoustique). Published under the direction of Y. Rocard, 1937. 69 pages. 18 fr.

551. *Sur les espaces à structure uniforme et sur la topologie générale.* By A. Weil. Publications of l'Institut Mathématique de l'Université de Strasbourg, 1938. 40 pages. 15 fr.

Enciclopedia della Matematiche Elementari. Volume II, Parte II. A cura di L. Berzolari, G. Vivanti e D. Gigli. (Biblioteca Matematica.) Milano, Ulrico Hoepli, Editore Librairo della Real Casa, 1938. 11+573 pages. 75 lire.

Introduction to College Mathematics. By M. A. Hill, Jr. and J. B. Linker. New York, Henry Holt and Company, 1938. 12+373+93 pages. $2.40.

Leitfäden der Algebra. By H. Stohler. Dritter Teil. (Mathematisches Unterrichtswerk für höhere Mittelschulen. Leitfäden und Aufgabensammlungen. Herausgegeben vom Verein Schweizerischer Mathematiklehrer.) Tabellen zu Stohler, Algebra Leitfäden, Dritter Teil, und Stähli und Meyer, Algebra Aufgabensammlung, Dritter Teil. (Unterrichtswerk des Vereins Schweizerischer Mathematiklehrer.) Zurich and Leipzig, Orell Füssli, 1938. 158 pages. Halbleinen Fr. 3.60. RM 2.20.

The Kelley Statistical Tables. By T. L. Kelley. New York, The Macmillan Company, 1938. 136 pages. $4.50.

The Reverse Notation. By J. H. Johnston. Introducing negative digits with twelve as base. London and Glasgow, Blackie and Son Limited, 1937. 10+74 pages. 3s. 6d.

Statistical Methods Applied to Economics and Business. By F. C. Mills. Revised Edition. New York, Henry Holt and Company, 1938. 19+746 pages. $3.75.

Corporations and their Financing. By Hastings Lyon. New York, D. C. Heath and Company, 1938. 7+946 pages. $4.25.

College Algebra. By W. L. Hart. Revised Edition. New York, D. C. Heath and Company, 1938. 8+408+30 pages. $2.24.

REVIEWS

Les Géométries. By Lucien Godeaux, Paris. (Collection Armand Colin No. 206) 1937. 216 pages. Bound 17.50 fr. Unbound 15 fr.

The little volume by an author well known for his studies in algebraic geometry, admirably achieves its modest aim of giving a brief general view of the history and nature of elementary geometry in language that may be significant to the non-professional reader. The trained mathematician need not expect to find here new theorems or unfamiliar philosophy. The work falls into six chapters: I. Elementary geometry, II. Analytic geometry, III. Projective geometry, IV. Principles of geometry, V. Geometry and the theory of groups, VI. Topology. Chapter IV discusses postulates, starting with the parallel postulate, and then taking up the postulate of Archimedes, the "fundamental theorem" of projective geometry, and duality. In Chapter V euclidean affine and projective transformations are considered in the light of group theory. An Arguesian space is defined as obtained from cartesian space by adjoining points at infinity. This chapter gives the theorem: The ruled projective geometry of ordinary space is equivalent to the geometry of a hyperquadric of five-dimensional projective space having as principal group the group of homographies of this space transforming this hyperquadric into itself. In the chapter on topology such topics as knots, the Peano curve, infinitesimal geometry, and the polyhedral formula are discussed. This book is not a textbook. It offers much not usually given in courses. It reads fluently, and avoids vague allusions to more difficult fields. This work should prove of service and inspiration to the inquiring student, particularly if he is seeking material for reporting in a mathematics club meeting and in any case if he is planning to teach in the elementary schools.

A. A. BENNETT

Aspects of Science. By Tobias Dantzig, New York, The Macmillan Company, 1937. 280 pages. $3.00.

The recent appearance of *Aspects of Science* was no doubt greeted with gladness by all those who had had the good fortune to read Dantzig's earlier volume, *Number, the Language of Science.* To say that the new book is on the same high level as the older one is high praise but well deserved. Both of the books are enlightened mathematically, scientifically, and philosophically. Both are wide in their appeal, being charmingly written in a style distinguished for its clarity, force, and magnanimity. Technical symbols are not present in sufficient force to frighten or repel the general reader. Neither can the narrowest specialist reasonably complain. For, though Dantzig neither eschews generalities nor disdains the graces of literary expression, he yet attains so high a degree of precision that not even the illiberal taste of a sheer technician is likely to be offended. I know of no teacher or mathematician or scientist or philosopher or student of letters who could not improve his attainments and outlook by an attentive reading of these books.

The point of view maintained throughout *Aspects of Science* is large, revealing, and thoroughly humane. It "views the evolution of scientific thought" as the outcome "of man's efforts to resolve the perplexities of human existence," to find himself and his place in the boundless complex of Nature. While the work is, therefore, not primarily devoted to the history of science and is not offered as a contribution to that field, yet a good deal of science history has been necessarily included. And if the book contained nothing but Dantzig's superb account of Galileo, it would be worth far more than the selling price. Not elsewhere, I believe, can one find a better portrait of Galileo the man, intense lover of life; or a better portrait of Galileo the genius, chief among the founders of experimental science; or a better portrait of Galileo the victim of the Inquisition. Many a reader will wish to thank the author for reminding him so vividly of what no devotee of science should ever forget—the Dread Antinomy between the authority of Reason and that of organized official Superstition.

But Galileo is only one among a host of creative pioneers, ancient, medieval, and modern, whose contributions are happily signalized in telling the fascinating tales of the growth and developments of geometry, of the number concept and analysis, of experimental science, and of the progressive mathematicization of experience. In this connection it is a pleasure to note that the roll of Dantzig's heroes is far from being a mere list of great scientists and mathematicians. Philosophers, too, are represented, and poets with other men of letters, and the cultural significance of the book is due in no small measure to the wisdom which these have contributed to its pages.

In a few lines it is, of course, quite impossible to analyze the work or to examine its abstract discussions with respect to questions of thoroughness, validity, and conclusiveness or to evaluate the wealth of its meditations. Even if there were room for it, the trial here would only be preliminary, for judgment in such cases belongs ultimately to the jurisdiction of the reader.

I cannot refrain, however, from pointing out that among the book's dozen chapters there are three which have impressed me as especially noteworthy for their lucidity, their solidity, and their penetration. They are respectively entitled "The Infinite," "The Crisis," and "In Quest of the Absolute."

Dantzig's discussion of the vexed question of mathematical infinity is acute and, in my opinion, sound. Its conclusion may be virtually summarized by the dictum: No infinity, no mathematics.

When he says that "the future historian may call our period *the great crisis*," he is not thinking exclusively or even mainly of the much talked-of revolution in physics. He is thinking of something much vaster, something of which that revolution is, properly speaking, not the cause but only one among many symptoms. For what he is thinking of is the deep, swift, multitudinous, uncontrollable changes occurring everywhere in the realm of human life—in the whole world of human thought, human emotion, human sentiment, and human will. Of *that* crisis what are the causes and what the implications? Such are the questions daringly grappled with in chapter six.

The cardinal enterprises of man—art, religion, philosophy, science—are but differing forms or aspects of the human spirit's age-old quest for the eternal, for what abides unchanged. The answers have been innumerable, and all of them false. From time immemorial the wake of the great ship of Thought has been strewn with abandoned absolutes, and still the ship sails on.

Why not? Koheleth may declare that "all is vanity and chasing after wind" and yet admonish: "Fear God and keep his commandments, for this is the whole duty of man."

In like spirit concludes Dantzig: "Read your instruments, and obey mathematics, for this is the whole duty of the scientist."

C. J. KEYSER

Portraits of Eminent Mathematicians, with Brief Biographical Sketches. By David Eugene Smith. New York, Scripta Mathematica, Portfolio Number Two, 1938. 13 portraits. $3.00.

Persons who have seen the first portfolio of the present series published by Scripta Mathematica need no recommendation of these portraits. Those who have not seen it can find a description on page 39 of the present volume of this MONTHLY. In both portfolios portraits are reproductions of pictures in the David Eugene Smith collections and the biographies are by him. Portraits of the following men make up Portfolio Number Two: Euclid, Cardan, Kepler, Fermat, Pascal, Euler, Laplace, Cauchy, Jacobi, Hamilton, Cayley, Chebishef, Poincaré.

I have only praise for the present work, as for its predecessor published in 1936. Portraits published by Professor Smith some years ago are now a classic in American mathematics. There is some duplication with the present series. All the pictures are suitable for framing and will make a handsome addition to any library or classroom.

TOMLINSON FORT

Magic Squares of $(2n+1)^2$ *Cells.* By M. J. van Driel. London, Rider and Company, 1936. 90 pages. $2.75.

As the title indicates, the book is concerned with magic squares of odd order only. The principal method used for constructing magic squares is essentially that of combining two latin squares, a method adopted by Bachet as long ago as 1624. However, by a skillful arrangement of the work the author is able to obtain some very interesting results. One of these, although not new, is the enumeration of all uniform step squares of order five by very elementary means. It should be pointed out in this connection that the meaning of the term "uniform step square" as used in the book is more general than the sense in which it is commonly used. The author includes under that term not only the usual uniform step squares but also those derivable from them by the method of substitutions.

The construction of pandiagonal squares of order 15 in Chapter 2 is note-

worthy since the uniform step process breaks down for pandiagonal squares whose order is divisible by 3. Chapter 3 contains a brief discussion of symmetric magic squares and Chapter 4 is devoted to the so-called rectangular, and square ply magic squares. A magic square is said to be $a \times b$ ply, where a and b are factors of the order of the square, if the sum of the numbers in every rectangular compartment of dimensions a cells by b cells is the same no matter from what part of the original square the compartment be chosen. If $a \neq b$ the long axis must be taken either horizontally or vertically throughout; it cannot in general be taken both ways in the same square. The author points out the interesting fact that uniform step squares of order 9 which are 3×3 ply are necessarily pandiagonal. In Chapter 6 it is shown that non-uniform step squares of order 9 and 3×3 ply may or may not be pandiagonal.

The fifth chapter contains a further discussion of symmetric squares and treats bordered magic squares in some detail. It is unfortunate that in the references to the literature on the latter type of squares the fundamental work by B. Violle published in 1838 is not mentioned.

In the sixth and last chapter some interesting squares of orders 9 and 15 are constructed by combining certain auxiliary squares having magic properties. The method may be used to construct simple squares or squares having the additional properties of being pandiagonal, symmetric, or $a \times b$ ply. The analysis is presented in such a way as to make clear how several of those properties may be obtained simultaneously in the same square.

The book concludes with a summary of four pages, written in French.

The author's limited command of English idiom occasionally leads to some confusion. In the paragraph just below the magic squares on page 10 the phrase "rows of numbers" occurs and is frequently used throughout the remainder of the book. Since in this connection the word "rows" does not refer to the rows of a magic square the reviewer suggests that this phrase be replaced by the expression, "sets of numbers." Also the meaning of the term "headlines" of sections 50 and 51 is perhaps better expressed by the phrase "principal lines."

The casual reader, looking merely for some diversion, is apt to find the book hard going. However, a reader willing to dig will turn up a lot of interesting material; sufficient to keep him occupied for some time.

G. E. RAYNOR

The Elements of Analytic Geometry. By H. E. Buchanan and G. E. Wahlin. New York, Farrar and Rinehart, 1937. 9+256 pages. $2.25.

This new textbook is a thorough course in the topics ordinarily considered essential in a study of plane analytic geometry (166 pages) and solid analytic geometry (64 pages). The type is fairly small so that it seems probable that the text could be used for as much as a five- or six-hour course if desired. Although this is the first edition of the text, it is singularly free from typographical errors.

In common with many analytic geometry texts, the conic sections (they are proved to be such) are treated in great detail. The general conic is first defined,

in terms of eccentricity, and particular properties of the specific types of conics are developed later. The authors give an excellent discussion of the general equation of the second degree. The transformed equations, following a translation or rotation, are calculated from the necessary invariants; and the usual method of determining the equation by direct substitution is also given.

Various topics, including the determination of the type of conic from the discriminant of its equation, are given as starred exercises rather than as part of the main body of the text. There is an abundance of well graded exercises, with the starred ones to provide additional stimulus for the better students. Included in these latter are suggestions for class reports on historical topics. Answers are given in the back of the book for all problems.

Some of the following topics which appear in this text are not always included in others:

(1) Emphasis upon directed line segments.

(2) A brief discussion of limits and the definition of a derivative.

(3) A chapter on empirical equations.

(4) A discussion of cylindrical and spherical coördinates.

(5) A mention of curves used to trisect angles and to duplicate the cube.

Some instructors will regret that:

(6) There is almost no use made of determinants.

(7) There is no attempt made to make the presentation inductive.

(8) Emphasis appears (in the reviewer's opinion) to be upon the results and derived formulas rather than upon the methods of procedure. For example, on page 118, after obtaining six forms of equations of tangents to conics by using derivatives, the authors state: "The above formulas should be memorized."

(9) Some definite statements are made which are matters of opinion. On page 90, we read, "If the real intercept of the hyperbola is on the y-axis, it is better to take the equation in the form

$$\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1, \cdots."$$

(10) In discussing geometry of three dimensions, there is almost no attempt at correlation with the analogous procedures and concepts of two dimensions.

In the preface the authors state; "Every mathematics instructor is aware of the urgent need . . . to impart to the average student some feeling for mathematics as a living subject, one that has not only had a tremendous influence on the development of our civilization, but is vitally important in present-day affairs." Whether or not *any* text in analytic geometry, including this one, has satisfactorily filled this need is still questionable. Professors Buchanan and Wahlin have, however, written a text including fairly standard material but with enough variations in method to stimulate, possibly, new viewpoints for the instructor who adopts it.

E. R. Ott

Advanced Calculus. By W. B. Fite. New York, The Macmillan Company, 1938. 12+399 pages. $5.00.

As the author states in his preface, "this book has been written to supply an introductory course in mathematical analysis for those who are looking forward to specializing in mathematics." To the reviewer it seems that the stated purpose has been attained in a satisfactory manner. Assuming familiarity with the working rules and simpler applications of the calculus, the first part of the book discusses the meanings of the fundamental concepts of the derivative and integral together with proofs of the fundamental theorems. The selection of subject matter in the book conforms with that which has now become somewhat standardized for a course in advanced calculus.

The book opens with a chapter on "the system of real numbers," giving a brief treatment of real numbers on the Dedekind basis; in the last four articles of the chapter the complex numbers are introduced as number-pairs. The second and third chapters treat functions of one and of several variables respectively, and ordinary and partial derivatives, including the directional derivative. The following chapter is on Taylor's expansion with a remainder, for functions of one and of several variables, with applications to indeterminate forms and maxima and minima; it includes articles on Euler's constant and Lagrange's multipliers.

The next four chapters are devoted to integration. The first of these treats the definite integral of a function of one variable, starting with a geometric approach and then giving an existence proof, covers the usual topics here, including the mean value theorems, differentiation under the integral sign, and approximate integration by Simpson's rule with the determination of the error by this rule. Then follows a chapter on indefinite integrals, dealing with integration of rational fractions in general terms and a brief introduction to elliptic integrals. The chapter on improper and infinite integrals includes tests of convergence, the Gamma and Beta functions and Stirling's formula and series. While Chapter VIII is entitled "Double and Triple Integrals," it also contains a discussion of line and surface integrals, with Green's and Stokes's theorems.

The following group of three chapters is devoted to series. The first of these, on "Infinite Series," gives a fairly extensive treatment for such a book as this. It covers the usual topics on tests of convergence, series of variable terms and uniform convergence, and double series, and includes an article on Weierstrass's example of a continuous function without a derivative. The reviewer was pleased to see here an adequate definition of the term "infinite series." Then follows a chapter on power series, covering the usual topics. Chapter XI is on trigonometric series and series of orthogonal functions; the treatment here is more extensive than in other advanced calculus texts.

The next two chapters are concerned with implicit functions and functional determinants, and applications to geometry, including applications to plane curves, skew curves, and surfaces.

In the last two chapters a brief introduction is given to the calculus of varia-

tions and to the theory of functions of a complex variable. In the former there is an interesting application of the calculus of variations to a problem in economics in addition to the usual applications to geometry and physics.

There is sufficient material in the book for a year's work. The style is clear, and the treatment is perhaps as rigorous as is possible for the average student at this stage of his development. L. L. SMAIL

Introduction to Mathematical Probability. By J. V. Uspensky. New York and London, McGraw-Hill Book Company, 1937. 9+411 pages. $5.00.

As stated by the author at the outset, the book begins in a truly elementary fashion, and expands in content to include much of recent work by such writers as Liapounoff, Markoff, Tshebysheff, Khintchine, Kolmogoroff, and "Student." By thus making available such a collection in English for the first time, Uspensky's book furnishes a fitting sequel to the final remark of Struik in his review of Fréchet's, *Généralités sur les Probabilités* (*Bulletin of The American Mathematical Society*, vol. 43, p. 603).

A partial list of topics treated will be of assistance in indicating the large scope of the text. They are: definitions, repeated trials, probabilities of hypotheses, difference equations and simple chains, Bernoulli's theorem, approximate evaluation of probabilities, mathematical expectation, law of large numbers, probabilities in continuum, general concept of distribution, fundamental limit theorems, normal distribution in two dimensions, method of moments. All through the text and particularly at the end of each chapter the theory is illustrated and many times introduced by means of an abundance of examples. From a teacher's standpoint the reviewer feels that this book is an exceptionally good one for giving self-reliance to students, through the interesting theory problems which have just enough hints to encourage them to actually attempt solution. The serious student of probability will find this a fine coördinating study for numerous topics of algebra, real and complex variable, *e.g.*, Lagrange's solution of finding the probability of exactly x successes in t independent trials with constant probability p, page 87.

Through the study of the composition formula for two distribution functions there is given in Chapter XIII the proof that the sum of normally distributed independent variables is again normally distributed with variance equal to sum of variances. In the following chapter one finds the rather long proof of Laplace's general theorem that the sum of n independent variables (with mild restrictions) divided by standard deviation of the sum is nearly normal if n is large. Of course one cannot include a proper statement of all the comparably relevant facts treated in this volume.

Only a few typographical errors were observed such as on pages 42, 94, and they would occasion no difficulty, being obvious. The book is evidently the work of an inspiring teacher and will undoubtedly take an important place among reference books and textbooks of probability.

J. A. GREENWOOD

Actualités Scientifiques et Industrielles. Paris, Hermann et Cie, 1936.

325. *Cinématique du solide et théorie des vecteurs*. By Ch. Platrier. 54 pages. 12 fr.
326. *La masse en cinématique et théorie des tenseurs du second ordre*. By Ch. Platrier. 81 pages. 18 fr.
327. *Cinématique des milieux continus*. By Ch. Platrier. 34 pages. 8 fr.
427. *Les axiomes de la mécanique newtonienne*. By Ch. Platrier. 59 pages. 14 fr.

The total of 169 pages in the first three of these little books constitute a good account of kinematics, suitable for an advanced course in mechanics. For the most part the material treated is of a classic nature and the exposition resembles what is found in the larger treatises. Nevertheless, the author has given a freshness and a liveliness to his account which make his work readable and interesting. He has also organized the material in such a way that theorems and propositions stand out with proper emphasis. The attentive reader should bring from the work clearly defined and usable ideas, though the absence of exercises prevents the attainment of the mastery which comes through intimate practice. The discussion of vectors and tensors which is contained in the work is quite limited, being restricted to the actual needs of the applications made.

The printing is of the high order that characterizes the useful series of monographs of which these are members.

In the fourth pamphlet one finds a discriminating and clear discussion of the axioms of classical mechanics. Every teacher of mechanics is aware of the perplexities and difficulties that arise when one seeks to present a foundation for the subject. Many of the terms used constantly throughout a course are inadequately understood by students. There has been for some time an extensive and critical literature dealing with the foundations of mechanics, but much of it requires considerable maturity and extensive study for its appreciation. The work under review meets a real need and instructors should be glad to have it available.

This monograph begins with a consideration of the ideas of solids, of periodic phenomena, and of quantity of matter; gives a discussion of the common basic units and of derived units; discusses homogeneity and similitude; sets forth fundamental axioms (giving with proper dates 28 names of men associated with the fundamental principles of mechanics, from Aristotle and Archimedes to Hertz and Painlevé); discusses such questions as the identity of gravitational mass and inertial mass; and ends with a consideration and classification of forces.

Although the work deals entirely with classical mechanics, references are made to relativity theory, and it is pointed out that relativity rejects certain postulates of classical mechanics when velocities comparable to those of light are being considered.

K. P. WILLIAMS

Bibliographia Kepleriana. A guide through the printed writings of John Kepler, with 80 facsimiles. Published by Max Caspar in collaboration with Ludwig Rothenfelder for the Bavarian Academy of Sciences. Munich, 1936, 158 pages, 79 plates.

This scholarly bibliography of the works of Kepler is based in the first instance upon an examination of the publications themselves in central European libraries. Kepler himself prepared in 1622 a list of his publications which is reproduced in facsimile. Fourteen other bibliographical lists published between 1711 and 1931 have been utilized by the author, and doubtless also that given by Poggendorff which is not cited despite its importance. To these bibliographies must be added as a source of information the *Joannis Kepleri Astronomi Opera Omnia*, Frankfort and Erlangen, 1858–1871, edited by Christian Frisch.

The importance of the present publication rests upon the complete bibliographical apparatus, including location of copies and reproduction of the title pages of all the known original editions and even the editions published during the author's lifetime. The bibliography prepared by Mr. Frederick Brasch for the History of Science Society Publication, *Johann Kepler, 1571–1630, A Tercentenary Commemoration of His Life and Work* (Baltimore, 1931), is properly praised as giving a much more complete list than the earlier ones. However, this list does not give collation or location of copies in America.

The usefulness of the bibliography would have been increased by the inclusion of the holdings of the Bibliothéque Nationale in Paris and the British Museum, both of which have notable collections of Kepleriana. However, with the bibliography it is comparatively easy to place the copies, given in the respective catalogues of the two libraries.

America has, so far as I know, no outstanding collection of Kepler's early works. Probably the John Crerar Library of Chicago has more of the original editions than any other American library. An edition there of the *Gründtlicher Bericht von einem ungewönlichen newen Stern Welcher im Oktober disz 1604 Jahrs erstmahlen erschienen.* (Prag. . . . Anno M.DC.V.) is a variant of 1605, not given by Caspar among the six variants of this title issued in the period 1604 to 1605.*

For the history of mathematics as well as astronomy and even physics, Kepler has superlative importance. This has been well indicated by D. J. Struik in his article, *Kepler as a Mathematician*, in the above-mentioned American Tercentenary publication on Kepler. To the evolution of the differential and integral calculus Kepler contributed fundamental material touching the most diverse phases of applications of the infinitesimal reckoning which connect closely with the most outstanding developments of the new universe of mathematics and of astronomy.

L. C. KARPINSKI

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* I am indebted to J. C. Bay, Librarian of the John Crerar Library, for confirmation of this title as given in the Union Catalogue.

Miscelanea Matematica. By J. Barinaga. Madrid, Nuevas Graficas, 1937. 4+127 pages.

The remarkable vitality of the Spanish people is indicated by a publication of this stimulating type created during such tragic times. The author is Professor of Mathematics in the University of Madrid and Director of the Mathematical Laboratory.

To many readers the biographical notices and notes which form a large part of the work will prove of particular interest. The dozen portraits—Ernst Steinitz, Otto Schreier, Jacques Herbrand, Cesáro, Riemann, Georg Cantor, Jacobi, Kelvin, Kronecker, Peano, Hilbert, and Abel—indicate the variety of fields touched and often illuminated in this volume. Largely the theoretical notes concern algebraic theory.

The Spanish students of mathematics today are fortunately not deprived of current material presented in an interesting way by scholars with broad, mathematical training and wide historical interests. The international character of the publication is worthy of comment; the portraits and biographical notes include Italian and German but no Spanish mathematicians; the author does not find it necessary to take a narrow, nationalistic attitude on scholarly matters.

More than 400 names appear in the Index, giving of living authors the date of birth, and of others the date of birth and of death.

Teachers will find in this work many practical suggestions for inspiring students to examine and to read the great classics of mathematical literature.

L. C. KARPINSKI

A CORRECTION

In the review of *Introduction to mathematics* by Cooley (*et al*) in the March (1938) issue of this MONTHLY the price of the book was incorrectly given. The price is $3.25 and not $3.75 as stated. E.J.M.

MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

All reports of club activities, suggestions, topics, with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

The new editor wishes to take this opportunity to thank Professor and Mrs. Owens in behalf of all of our Mathematics Clubs for the real help and inspiration they have been to us in conducting our work.

CLUB TOPICS

In January 1918, Professor R. C. Archibald of Brown University inaugurated this department of Mathematics Clubs. To him we are indebted for the beginning of suggested topics for club programs and accompanying bibliographies. These should be as helpful to our friends who prepare papers for mathematics clubs today as they were at that time. And so, for your convenience, we are making a reference list of many of these topics to which we hope, in the course of time, to make additions. Surely there are MONTHLY readers who have in their files fairly complete bibliographies on topics suitable for mathematics clubs which they would be willing to contribute to this department.

The following is a list of papers which have appeared in this MONTHLY.

The oldest mathematical work extant, vol. 25, p. 36.

Geometrography and other methods of measurement of geometrical constructions, vol. 25, p. 37.

Arithmetical prodigies, vol. 25, p. 91.

Ptolemy's theorem and formulae of trigonometry, vol. 25, p. 94.

Paper folding, vol. 25, p. 95.

Women as mathematicians and astronomers, vol. 25, p. 136.

The binary scale of notation, a Russian peasant method of multiplication, the Game of Nim, and Cardan's Rings, vol. 25, p. 139.

The logarithmic spiral, vol. 25, p. 189.

Golden section, vol. 25, p. 232.

A Fibonacci series, vol. 25, p. 235.

Euler Integrals and Euler's Spirals, sometimes called Fresnel Integrals and the Clotholde or Cornu's Spiral, vol. 25, p. 276.

Geometry of four dimensions, vol. 25, p. 316.

Constructions with a double edged ruler, vol. 25, p. 357.

The cattle problem of Archimedes, vol. 25, p. 411.

The number π, vol. 26, p. 209.

Codes and ciphers, vol. 26, p. 409.

The Chinese suan pan, vol. 27, p. 180.

Finite geometries, vol. 28, p. 85.

Functional equations, vol. 32, p. 428.

Fiedler's cyclography, vol. 32, p. 517.

The pasturage problem of Sir Isaac Newton, vol. 33, p. 155.

Theorem of Bang, isosceles tetrahedra, vol. 33, p. 224.

La courbe du diable, vol. 33, p. 273.
Additional references to club topics, vol. 43, p. 40; vol. 44, p. 537.
Calculating machines, vol. 43, p. 99.
Development of present day numerals, vol. 43, p. 99.
Four color problem, vol. 43, p. 181.

INTERCOLLEGIATE MATHEMATICS ASSOCIATIONS

The reports of club activities which have come to this department show an increasing interest in the formation of intercollegiate mathematics clubs in metropolitan areas. It would seem that such organizations might well justify their existence in promoting a friendly feeling among college students with the same major interests, encouraging a more ambitious mathematical program, stimulating research and developing forums which make possible the discussion of problems arising in separate clubs.

Perhaps, as time goes on, some of these organizations will find it possible to extend their contacts to include all of the mathematics clubs of their particular state or section of the country for at least one meeting of the academic year. It might even be possible to schedule this meeting at the same time as the state meeting of the Mathematical Association, so that professors and students might well be in conference at the same time.

This department would be interested in hearing of other organizations of this type. If you are promoting a particularly ambitious program, which you think would be suggestive to other clubs, we should very much appreciate a full account.

CLUB REPORTS

1937–38

Intercollegiate Mathematics Association, Milwaukee

This association held four meetings in the course of the academic year. The program included: at Milwaukee State Teachers College, "Life probabilities" by Mr. Fassel of the Northwestern Mutual Life Insurance Company; at Marquette University, an informal social meeting featuring a play entitled "Chums"; at Mount Mary College, five talks, one from a representative of each of the participating clubs, on the topics: Ideas of infinity and Lucretius; Practical mathematics in Roman Times; Mathematical fallacies; The geometric proof and Work of the C.C.C.; at University of Wisconsin Extension Division, the annual banquet with an address by Mrs. William Beckwith of Milwaukee Downer College.

President, Frederick Adler, M. U.; Vice-President, Margaret Kreiziger, M. M. C.; Secretary, Gordon Voigt, U. W. E. D.; Treasurer, Elmer Poppendieck, M. S. T. C.; Corresponding Secretary, Norma Fedders, M. D. C.

Pythagorean Club, Milwaukee State Teachers College

Topics discussed at the bi-monthly meetings were: Fundamental concepts of mathematics; Relation of mathematics to art; Place of mathematics in the curriculum; Geometric fallacies; How to teach the addition and subtraction of negative numbers; Origin of numerical forms. Social meetings of the year featured a play "A Mathematical Nightmare," spelldown using mathematical terms, a chili supper and a picnic.

President, Laura Gilbert; Vice-President, Elmer Poppendieck; Secretary, Jennie Brodi; Treasurer, Anna Goepfert; Faculty Advisor, Elizabeth Knight.

Junior Mathematics Club, University of Wisconsin, Extension Division

Each year this club entertains junior and senior high school students from Milwaukee and suburbs. At this year's meeting, motion pictures on geometry and astronomy were shown; two students spoke on "Mayan Number System" and "The Moon" respectively. An exhibit included calculating machines, astronomical models and instruments, slide rules, string and plaster and wooden models, charts, drawings and problems.

The Euler Prize awarded annually for the best paper presented on a mathematical subject was won by Yvonne Town with a paper entitled "Arithmetic based on Number Systems Other Than Ten."

President, Gordon Voigt; Secretary-Treasurer, Aimee Schultz; Faculty Advisor, Louise A. Wolf.

Mathematics Club, Milwaukee Downer College

Subjects for programs: Theory of relativity, Linkages and string figures, Mazes and labyrinths, Duo-decimal number system, The solar system and Meteorology. Social meetings: introductory meeting, a mathematical treasure hunt; Christmas meeting, mathematical bibliography; close of the year, a banquet.

President, Elizabeth Little; Secretary-Treasurer, Mary Frances McKee.

Mathematics Club, Mount Mary College, Milwaukee

This club sometimes met jointly with the Science Club of the same school. Talks at meetings included one by Dr. J. D. Ball on Dr. Steinmetz, another on "The mathematical education of women during the middle ages." At the annual party former members who had graduated were guests.

President, Margaret Kreiziger; Vice-President, Marion Clark; Secretary-Treasurer, Mary C. Neugent.

Joint Meetings in the Greater Boston Area

Three joint meetings have been held in Boston. The first was attended by members of the Boston University and Wellesley College Clubs. Tufts College club members met with the group at their second and third meeting. Each college group was the host for one meeting. Most of the talks at these meetings were made by students. At the last meeting, Professor Philip Franklin of Massachusetts Institute of Technology spoke on "Four color maps." Matching games were used to help members to get acquainted.

Plans are now being made to include the clubs at Northeastern College, Boston College and Massachusetts Institute of Technology as well as the above mentioned clubs. Two meetings are scheduled for next year, the first to be held in October at Boston University. The presidents of the clubs form the executive committee of the new society.

The Mathematics Club, Wellesley College

Subjects discussed at formal meetings were: Workers in the theory of numbers; Number systems; Probability and calculus.

President, Grace Mandeville; Vice-President, Evelyn Wicoff; Senior Executive, Doris Gasteiger; Junior Executive, Ruth Hawkes; Secretary, Gloria H. Sharp.

Mathematics Society, Northeastern University, Boston

Guest speakers at the monthly meetings of this organization were: Professor Elmer E. Haskins on "Use of series" and Professor Raymond K. Morley of Worcester Polytechnic Institute on "Polar coördinates," (using gear machines to demonstrate certain curves). The other meetings were devoted to the three famous topics of antiquity; Dimensional analysis; Mathematical games and puzzles; and the synthetic and analytic methods of solving the problem:—To find the point in the plane of any given triangle such that the sum of the distances from the point to the vertices of the triangle shall be a minimum.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 339. *Proposed by V. Thébault, Le Mans, France.*

Consider all the triangles inscribed in a given circle on a fixed chord as base, and determine the locus of the feet of the interior and exterior base-angle bisectors. Thus show that if two interior angle bisectors of a triangle are equal, the triangle is isosceles.

E 340. *Proposed by E. C. Kennedy, Texas College of Arts and Industries.*

If B is a positive integer, what rational values may be assumed by

$$R = \sqrt{\left(\frac{B}{7}\right)^2 + 5038}$$

E 341. *Proposed by J. H. Edmonston, Washington, D. C.*

A triangular octahedron (one with all faces triangular) may be regarded as a space analogue of a plane quadrilateral. On this basis, state and prove a space-analogue of the theorem that the midpoints of the sides of any plane quadrilateral are the vertices of a parallelogram.

E 342. *Proposed by W. E. Buker, Pittsburgh, Pennsylvania.*

Show how to draw three circles with radii a, b, and c, and common tangents d, e, and f.

E 343. *Proposed by H. E. Stelson, Kent State University.*

Prove that $ax^2+2hxy+by^2+2gx+2fy+c$ is the product of two factors, linear and rational in x and y, provided $[h+(h^2-ab)^{1/2}][f+(f^2-bc)^{1/2}] -b[g+(g^2-ac)^{1/2}]=0$, and that in this case, $abc+2fgh-af^2-bg^2-ch^2=0$.

E 344. *Proposed by V. W. Graham, High School, Dublin, Ireland.*

$ABCD$ is a parallelogram with vertices named in order around the perimeter. DA and CB are produced to P and Q respectively, so that $AP=BQ$. Any point X is taken on AB. PX meets the diagonal BD at Z. QZ meets DC at Y. Prove that $AX=DY$. Conversely, if $AX=DY$, prove that Q, Y, and Z are collinear.

CORRECTIONS. There was an error made in the statement of E 310 [1938, 47], which should read:

E 310. *Proposed by V. Thébault, Le Mans, France.*

In a certain system of notation there exists a four-place number of the form *aabb* which is the square of *bb*. Show that the numbers, b and $a^2+4(a-1)^2$ are perfect squares. Determine the base of such a system and the values of a and b, knowing that a is also a perfect square.

In line six on page fifty-two of the January issue (1938) the word "perimeter" should read "semi-perimeter."

SOLUTIONS

E 208. *Proposed by G. A. Whittemore, New York City.*

How may eight married couples play a seven-round bridge tournament, if each man plays one round with each lady except his wife, and everybody plays against everybody else except his or her spouse? How many solutions exist? Can this be generalized for $4n$ couples playing $4n-1$ rounds? Does a solution exist for $4n+2$ couples playing $4n+1$ rounds?

Partial Solution by W. E. Buker, Pittsburgh, Pennsylvania.

H. E. Dudeney, on page 203 of his *Amusements in Mathematics*, gives a solution of this problem for four couples. He notes that a solution for eight couples exist, and calls it a hard puzzle, but gives no solution.

Here is a solution:

Round 1	Round 2	Round 3	Round 4	Round 5	Round 6	Round 7
$Ab-De$	$Ac-Ef$	$Ad-Fg$	$Ae-Gh$	$Af-Hb$	$Ag-Bc$	$Ah-Cd$
$Cg-Hf$	$Dh-Bg$	$Eb-Ch$	$Fc-Db$	$Gd-Ec$	$He-Fd$	$Bf-Ge$
$Fh-Gc$	$Gb-Hd$	$Hc-Be$	$Bd-Cf$	$Ce-Dg$	$Df-Eh$	$Eg-Fb$
$Ed-Ba$	$Fe-Ca$	$Gf-Da$	$Hg-Ea$	$Bh-Fa$	$Cb-Ga$	$Dc-Ha$

Cyclic permutation of each of the letters yields seven new solutions.

E 304 [1937, 659]. *Proposed by Michael Goldberg, Washington, D. C.*

Superimpose a given circle upon a given polygon so that their common area is a maximum. Show that the segments which the circle intercepts on the sides of the polygon can form a closed polygon in which the directions of the segments are also preserved. Locate the center of the circle. When the polygon is a triangle, determine the locus of the center of the circle as its size is increased from that of the inscribed to that of the circumscribed circle.

Solution by the proposer.

Let the lengths of the sides of the polygon be a_i and the intercepts on them be q_i. If the circle is moved a distance dx in any direction, then the change in the area common to circle and polygon is $\sum q_i \sin A_i dx$ (except for infinitesimals of higher order) where the A_i are the angles which the sides of the polygon make with the direction of dx. For maximum area, $\sum q_i \sin A_i dx = 0$, or $\sum q_i \sin A_i = 0$.

But since this equation must hold for every direction of dx, the q_i must be equivalent to a set of balanced vectors. That is, they can form a closed polygon whose sides are parallel to the sides of the given polygon.

In particular, if the given polygon is a triangle, the polygon of the q_i is a similar triangle, and the q_i are proportional to the sides of the original triangle, a_i.

E 305 [1937, 659]. *Proposed by D. L. MacKay, Evander Childs High School, N. Y.*

If the external angle bisectors at A and B are equal, must the triangle ABC be isosceles?

Solution by W. E. Buker, Pittsburgh, Pennsylvania.

Let angle A be a right angle, let the bisector of external angle B meet CA produced at P, and the bisector of external angle A meet BC produced at Q.

In triangle BAP, $BP = c/(\sin B/2)$. In triangle BAQ, $AQ = c \sin B/\sin(45° - B)$. Now if $BP = AQ$, then $c/(\sin B/2) = c \sin B/\sin(45° - B)$, or $\sin^2 B + \cos B = 2 \sin A$. This equation is satisfied by a value of B slightly greater than 35°, so that the external angle bisectors of a triangle may be equal without that triangle being isosceles.

Also solved by E. P. Starke and the proposer.

E 306 [1937, 659]. *Proposed by W. B. Campbell, Ithaca, New York.*

Each license plate in a certain state bears from one to five characters, of which not more than two are chosen from the 26 letters of the English alphabet, the remainder being chosen from the nine digits, 1, 2, 3, 4, 5, 6, 7, 8, and 9. Any letter used can be in any position. Find the number of different plates possible.

Solution by E. P. Starke, Rutgers University.

If the license plate bears n characters, of which i are letters, the places in which the letters are to appear may be selected in ${}_nC_i$ ways, the letters for those places may be selected in 26^i ways, and the digits for the remaining places may be selected in 9^{n-i} ways. For such plates there are then ${}_nC_i 26^i 9^{n-i}$ possible choices. If this formula is summed for $i = 0, 1, 2$, and for $n = 1, 2, 3, 4, 5$, we shall have the desired result. It is thus found that there are 35 plates of one character, 1225 of two, 25,299 of three, 410,913 of four, and 5,840,019 of five. Hence in all, there are 6,277,491 possible different plates.

Several other answers were received to this problem, no two of which were alike.

E 307 [1937, 659]. *Proposed by V. Thébault, Le Mans, France.*

Locate the point P in the plane of the given triangle ABC such that the triangles PAB, PBC, and PCA may have equal perimeters.

Solution by the proposer.

It is easily shown that the desired point P is the center of a circle which is

externally tangent to three circles centered at the vertices A, B, and C of the original triangle, and with respective radii of $s-a$, $s-b$, and $s-c$, where s is the semi-perimeter. Furthermore, this point P is also the center of a circle tangent to three circles centered at the vertices A, B, and C of the original triangle, with respective radii of a, b, and c.

It may be further observed that there exists a point Q which is the center of the circle externally tangent to the first set of three circles named above, possessing the property that if it is joined to the vertices of the original triangle, the three new triangles thus formed have semi-perimeters s_1, s_2, s_3 such that $s_1-a=s_2-b=s_3-c$.

Also solved by W. B. Clarke and D. L. MacKay.

E 308 [1937, 659]. *Proposed by E. H. Clarke, Hiram College, Ohio.*

Find the triangle which contains an angle most nearly equal to one radian, from among all possible triangles whose sides are integers of one or two digits.

Solution by R. F. Schnepp, St. Mary's University of San Antonio.

Denote the sides by a, b, and c, and let A, opposite a, be the angle to approximate one radian. Now from the cosine law we have

$$a^2 = b^2 + c^2 - 2bc \cos A. \tag{1}$$

Hence $\cos A$ is rational. By means of continued fractions, the convergents to the close approximation, $\cos 1 = .54030230$ are found to be $1/1$, $1/2$, $6/11$, $7/13$, $20/37$, $47/87$, $67/124$, $181/335$, $248/459$, $429/794$, $20411/37777$, $\cdots$. To obtain the most suitable approximation, we choose the convergent farthest to the right that satisfies the conditions of the problem. Since $2bc \cos A$ is an integer, we need not examine convergents with denominators greater than 20,000. This rules out $20411/37777$ and all subsequent convergents. The three convergents before that are unable to make the right member of (1) a square.

If we set $\cos A = 67/124$, we have $a^2=b^2+c^2-67bc/62$, so that bc is a multiple of 62 and either b or c is 31 or a multiple of 31. While 31 itself will not work, if we let $b=62$ we have $a^2=62^2+c^2-67c$, or $(a+62)(a-62)=c(c-67)$. One obvious solution is $a=62$, $c=67$, and an examination of the c-discriminant shows that it is the only one which meets the given conditions. Hence the sides of the desired triangle are 62, 62, and 67. With these values, the formula $\cos A/2=\sqrt{s(s-a)/bc}$, gives $A=57°17'40''$, whereas 1 radian $=57°17'44''.8$.

Also solved by the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3882. *Proposed by J. R. Musselman, Western Reserve University.*

The pedal circle of the centroid G of a triangle $A_1A_2A_3$ passes through the centers of the hyperbolas of Kiepert and Jerabek. These hyperbolas are considered in Casey's *Analytic Geometry of the Point, Line, and Circle*, pp. 442–448.

3883. *Proposed by J. R. Musselman, Western Reserve University.*

The orthopole of the Euler line of a triangle $A_1A_2A_3$ as to the same triangle is the center of the hyperbola of Jerabek; the orthopole of the Brocard line of the triangle as to the same triangle is the center of the hyperbola of Kiepert. Orthopole is defined in Johnson's *Modern Geometry*, p. 247.

3884. *Proposed by H. S. M. Coxeter, University of Toronto.*

Prove that the points of contact of the real bitangents to a plane quartic (of genus 3) are its points of intersection with 1, 2, 4 or 7 conics. See Baker, *Principles of Geometry*, vol. 6, p. 14 for a theorem similar but not so complete.

3885. *Proposed by V. Thébault, La Mans, France.*

The product of n consecutive positive integers, n being odd, is divisible by their sum, except in the case where, n being prime, the arithmetic mean of the n integers is divisible by n. Examine the case where n is even.

3886. *Proposed by V. Thébault, La Mans, France.*

A parallelogram is inscribed in an ellipse and a point P is chosen arbitrarily on the ellipse. Two straight lines are drawn from P parallel to the sides of the parallelogram cutting them in four points. A third straight line is drawn from P parallel to one of the diagonals of the parallelogram cutting the tangents to the ellipse at the ends of this diagonal in two points. Show that the six points thus obtained are the vertices of a hexagon whose consecutive sides are parallel to two conjugate diameters of the ellipse, and that the area of the hexagon is the same as that of the parallelogram. See 3861 [1938, 122].

3887. *Proposed by V. Thébault, La Mans, France.*

Through the vertex A of a triangle ABC a straight line AM is drawn cutting the side BC in M. Let 2θ be the angle AMC; O and I the centers of the

circumscribed circle (O) and the inscribed circle (I) of ABC. The circles (ω_1) and (ω_2) with centers ω_1 and ω_2 and radii ρ_1 and ρ_2 are each tangent to (O) and the first is tangent also to the two sides of angle AMC while the second is tangent to the two sides of angle AMB. Prove that: (1) The straight line joining ω_1 and ω_2 passes through I. (2) The point I divides the segment $\omega_1\omega_2$ in the ratio $\tan^2\theta:1$; and $\rho_1+\rho_2=r^2\sec^2\theta$, where r is the radius of (I).

SOLUTIONS

3796 [1936, 500]. *Proposed by Don Wallace, Charlottesville, Va.*

Solve the following matric-differential equation using only ordinary methods, i.e., without the use of the matric product integral,

$$\frac{dZ}{dt} + ZA + BZ + ZPZ = 0,$$

where A and B are constant matrices and P is a function of t.

Solution by J. H. M. Wedderburn, Princeton University.

Let $Y=Z^{-1}$, then $dY/dt=AY+YB+P$. Put $C=e^{-At}$, $D=e^{-Bt}$, $X=CYD$; a short calculation gives $dX/dt=CPD$ and therefore $X=\int CPD\,dt$.

Solved also in a similar manner by the proposer.

3797 [1936, 500]. *Proposed by Don Wallace, Charlottesville, Va.*

Show that the locus of points collinear with their isogonal and isotomic conjugates is a conic passing through the median and exmedian points and the incenter and excenters. Show that its center is Steiner's point and that it is a rectangular hyperbola. Show also that the tangents at the median and exmedian points pass respectively through the symmedian and exsymmedian points, and that any point, its isogonal and isotomic conjugates are conjugate points with respect to the conic.

See in this MONTHLY, *On isogonal points*, by J. H. Weaver, 1935, pp. 496–499, and *Isogonal and isotomic conjugates and their projective generalization*, by P. H. Daus, 1936, pp. 160–164.

I. *Solution by Otto J. Ramler, Catholic University of America.*

If we take the lengths of the sides of the reference triangle to be a_1, a_2, a_3, and use trilinear normal coördinates, the isogonal conjugate of (x_1, x_2, x_3) is $(x_1^{-1}, x_2^{-1}, x_3^{-1})$ and the isotomic conjugate is $(a_1^{-2}x_1^{-1}, a_2^{-2}x_2^{-1}, a_3^{-2}x_3^{-1})$. Writing the condition that these points shall be collinear, we obtain in determinant form the equation of the locus of points (x_1, x_2, x_3) which together with their isogonal and isotomic conjugates are collinear,

$$(1) \qquad \begin{vmatrix} x_1^2 & x_2^2 & x_3^2 \\ 1 & 1 & 1 \\ a_1^{-2} & a_2^{-2} & a_3^{-2} \end{vmatrix} = \sum_{i,k} x_i^2 (a_k^{-2} - a_i^{-2}) = 0.$$

The equation shows that the fundamental triangle is self conjugate with respect to the conic locus. Another reason for this property will appear later.

The coördinates of the median and exmedian points are $(\pm a_1^{-1}, \pm a_2^{-1}, \pm a_3^{-1})$ and are readily seen to satisfy the equation of the conic, showing that these points lie on the conic. The same may be said of the coördinates of the incenter and excenters $(\pm 1, \pm 1, \pm 1)$. Since the incenter and excenters form an orthocentric set of points the conic must be a rectangular hyperbola. The fact that the sum of the coefficients in the equation (1) is identically zero also shows that the conic is a rectangular hyperbola.

The equation of the polar of (y_1, y_2, y_3) with respect to (1) is

$$\sum a_i^2 x_i y_i (a_j^2 - a_k^2) = 0, \qquad i \neq j \neq k \neq 1, 2, 3. \tag{2}$$

If (y_1, y_2, y_3) lies on the conic, (2) is the equation of the tangent at the point. Simple substitutions show that the symmedian point (a_1, a_2, a_3) lies on the tangent at the median point $(a_1^{-1}, a_2^{-1}, a_3^{-1})$, and that the exsymmedian points lie on the tangents at the exmedian points respectively; for instance $(-a_1, a_2, a_3)$ lies on the tangent at $(-a_1^{-1}, a_2^{-1}, a_3^{-1})$. The center may be obtained as the intersection of the polar of two points at infinity. It is readily shown that the polars of $(0, a_2^{-1}, -a_3^{-1})$ and $(a_1^{-1}, 0, -a_3^{-1})$ intersect at $[a_1^{-1}(a_2^2 - a_3^2)^{-1}, a_2^{-1}(a_3^2 - a_1^2)^{-1}, a_3^{-1}(a_1^2 - a_2^2)^{-1}]$ which is Steiner's point.

Equation (2) is satisfied if $x_i y_i = 1$ and if $a_i^2 x_i y_i = 1$, $i = 1, 2, 3$. But these conditions identify (x_1, x_2, x_3) and (y_1, y_2, y_3) as isogonal and isotomic conjugates respectively. Hence any point, its isogonal conjugate and isotomic conjugate are conjugate points with respect to the conic. Since in the isogonal and isotomic transformation the reference triangle is exceptional in that any point on a side corresponds to the opposite vertex, the reference triangle is self-conjugate as to the conic (1). Equation (1) is an analytical confirmation of this fact.

II. *Solution by P. H. Daus, University of California at Los Angeles.*

The transformation by isogonal or by isotomic conjugates is a special quadratic Cremona transformation. Either establishes a 1-1 point correspondence in the plane, except that the vertices and sides are exceptional. To any general point on a side corresponds the opposite vertex, while to a vertex corresponds the opposite side. However, if we associate with each vertex the directions through it, then to each direction corresponds a unique point on the opposite side and conversely. We assume that the 1-1 correspondence has been restored in this fashion and we discuss the locus by considering lines through a vertex of the triangle $A_1A_2A_3$. Let P, P', P'' be the point, its isogonal conjugate, and its isotomic conjugate. Suppose that P moves along a line through the vertex A_2, cutting A_1A_3 in Q, while A_2P' and A_2P'' cut A_1A_3 in A_2' and A_2'', the points which correspond to A_2. Then the point row on A_2Q is projective to those on A_2A_2' and A_2A_2'', with Q corresponding to A_2 on each of the others. Hence the point rows $u'(P')$ and $u''(P'')$ are perspective and the lines $p' \equiv P'P''$ pass through a point, say T, on A_1A_3. If we project P to T by p, we have two super-

posed projective pencils which have two self corresponding lines and for the corresponding positions of P, the points P, P', P'' are collinear. Hence the required locus is such that an arbitrary line through A_2 cuts it in two points T_1, T_2, and hence is a conic c. That it passes through the eight points I_i and G_i, $(i=0, 1, 2, 3)$, is evident since I_i coincides with its isogonal conjugate and G_i with its isotomic conjugate. That the conic is an equilateral hyperbola follows since I_i form an orthocentric system. (See Holgate, *Projective Pure Geometry*, p. 207.)

When the point P takes the position Q, p' cuts A_3Q in A_2, and when P takes the position A_2, p' cuts A_2Q in Q. Hence the projectivity at T considered above is an involution, so that A_2P cuts $P'P''$ in the harmonic conjugate of P with respect to T_1, T_2. The points A_2, Q form a special pair of such points, indicating that the triangle $A_1A_2A_3$ is self-polar with respect to c, since we may take any line through A_2, and also consider the other vertices. In a manner analogous to the above we see that A_3P cuts $P'P''$ in a point which is the harmonic conjugate of P with respect to the points where A_3P cuts the conic c. Hence $P'P''$ is the polar of P with respect to the conic c, that is, any point is the polar conjugate of both its isogonal and isotomic conjugates. In particular, the tangent at any point on c contains the isogonal and isotomic conjugates of the point of contact. This shows that the tangents at G_i pass through the corresponding symmedian or exsymmedian points K_i, $(i=0, 1, 2, 3)$, their isogonal conjugates, and the tangents at I_i pass through their isotomic conjugates.

The property of the center will be determined by the definition of Steiner's point S. For our immediate purpose we define it as the fourth intersection of the circumcircle and the so-called Steiner ellipse (see Sommerville, *Analytical Conics*, p. 59), the isogonal conjugate and isotomic conjugate, respectively, of the line at infinity. Since S' and S'' are both at infinity, it follows that S, the pole of $S'S''$, is the center of the conic c.

Solved also by J. A. Clawson, S. Kaplan, and the proposer.

Editorial Note. The remaining solutions were analytic with the given triangle as the basis of homogeneous coördinates. Daus gave also a similar analytic solution. Kaplan used a coördinate system as in I and observed that the conic passes through the four points $a_i^{-1}(a_j+a_k)^{-1/2}$ and also through the four points $[a_i(a_j+a_k)]^{-1/2}$. He would like to know if these points are of interesting geometric significance. He also stated that Ferrer's *Trilinear Coördinates* gives laborious methods for finding the foci and asymptotes; and he wishes to know if any reader can give simpler methods for this case.

The center and asymptotes of the rectangular hyperbola passing through four given points may be constructed in a way which proves some of the theorems in the above solutions and gives Steiner's generalized theorem. This construction is based on the fact that a given hyperbola of any type and its asymptotes cut from a straight line two corresponding segments with a common midpoint. It then follows that, if the hyperbola is rectangular, a right triangle

having for its hypotenuse a chord of the hyperbola and its two sides parallel to the asymptotes, is such that the straight line through the vertex of the right angle and the midpoint of the chord is a diameter of the hyperbola. Let A, B, C be the midpoints of the sides MN, NL, ML of a triangle whose vertices are on a rectangular hyperbola. On NM and ML as hypotenuses construct right triangles with sides parallel to the asymptotes, supposed known. Then the two diameters constructed as above meet in the center U of the hyperbola; and it is easily seen that angle CUA is equal to angle B of the triangle ABC, or is its supplement. Hence U is on the circumcircle of ABC, the nine-point circle of LMN. If a side is parallel to an asymptote, the hyperbola degenerates into this side and the altitude of it; but there is always at least one side not parallel to an asymptote and the construction does not fail. Conversely, if U is any point on the nine-point circle, the circle with a midpoint distinct from U, say A, as center and radius AU cuts MN in two points such that, if these are joined by straight lines with U, the two perpendiculars thus obtained cut from MN a segment with A for its midpoint. It will then be seen that the two perpendiculars cut from ML a segment with C as its midpoint. Hence the rectangular hyperbola with these lines as asymptotes passing through M and N will also pass through L. Moreover, if H is the orthocenter of LMN, the nine-point circles of LMN and HMN coincide; and, with U as center and the asymptotes determined as above, the rectangular hyperbola through M and N passes through both L and H. It now easily follows that any conic through the vertices of a triangle LMN and its orthocenter H must be a rectangular hyperbola. For the conic must be some kind of hyperbola as we see from the relative positions of the points. As shown above there exists a rectangular hyperbola through the same four points and having one asymptotic direction the same as one for the conic. Then the two conics, having five points in common, must coincide; and the given conic must be a rectangular hyperbola.

Now let X be any point chosen in the plane of L, M, N, but distinct from these points. Then the nine-point circles for LMN and XMN, if not coincident, meet in A and also in a point S. The rectangular hyperbola (S) with center S and the corresponding asymptotes, which passes through L, M, N will pass also through X. There are four such nine-point circles and they have S in common. If the two nine-point circles mentioned above coincide, then X must be at H. For, if two triangles have two vertices in common and also the same circle as nine-point circle, the two triangles either coincide or the third vertex of one is the orthocenter of the other triangle. If any two of the four nine-point circles coincide, it follows that all four coincide. If then X is not at H, the center S and the corresponding rectangular hyperbola (S) through L, M, N, X, are uniquely determined. Denote the midpoints of XL, XM, XN by L', M', N'. Then it is obvious that the points A, N', B, L', C, M' form a hexagon with its opposite sides equal and parallel. Hence a conic (K) passes through these six points with diameters AL', BN', CN' through its center K. Moreover, the circumcircles of $ABC, AM'N', BN'L', CL'M'$ meet in the point S. This theorem

says that, if three straight line segments AL', BM', CN', have a common midpoint K, the circumcircles of ABC, $AM'N'$, $BN'L'$, $CL'M'$ meet in a point S: a proof of this without reference to the hyperbola is simple. We now show that S is also on (K). The circle (ABC) cuts (K) in A, B, C, S'; hence BC and AS' are equally inclined in opposite senses to a principal diameter of (K). Also circle $(AM'N')$ cuts (K) in A, M', N', S'', and $M'N'$ and AS'' are equally inclined in opposite senses to the same diameter. But BC is parallel to $N'M'$, therefore AS' and AS'' are parallel; hence $S' \equiv S'' \equiv S$.

Let the feet of the altitudes of LMN be H_1, H_2, H_3, where LH_1 is one of the altitudes; then it is clear that $H_1H_2H_3$ is a self polar triangle for (S). There are three other such self polar triangles, for example, if α_1, α_2, α_3 are the feet of the altitudes of XMN, where $X\alpha\alpha_1$ is an altitude, the triangle $\alpha_1\alpha_2\alpha_3$ is self polar. Hence the intersection $\bar{L}$ of H_2H_3 and $\alpha_2\alpha_3$ is the pole of MN, and $\bar{L}M$, $\bar{L}N$ are the tangents to (S) at M and N. Also $\bar{L}A$ is a diameter; and, if $\bar{M}B$ is a diameter obtained in the same way, the intersection of $\bar{L}A$ and $\bar{M}B$ is S.

For an arbitrary point X, BC is not the polar of A with respect to (S), and we now consider the important special case where X is so situated that BC is the polar of A. Let AL cut (S) again in G and BC in $\overline{A}$, then $AG/G\overline{A} = AL/\overline{A}L = 2$; and, since $A\overline{A}L$ is a median for ABC and LMN, G must be the centroid for each triangle. It follows then that ABC is self polar with respect to (S), and if we take X at G, the centroid of ABC, this result is realized. In this case $K \equiv G$; (K) becomes (G), the Steiner ellipse with center G; and this ellipse cuts the nine-point circle of LMN in the Steiner point S, where S is the center of the rectangular hyperbola through L, M, N and the centroid G to LMN. The above constructions are now simplified, since the intersections of corresponding sides of ABC and $H_1H_2H_3$ are the poles of the corresponding sides of LMN. The Steiner ellipse is easily seen to be tangent to the sides of triangle LMN at their midpoints. This ellipse appears in the solution of 3565 [1933, 372] as the ellipse of least area circumscribing the triangle ABC, also in the solution of 3718 [1936, 442] in its relation to the roots of a cubic. From the theorem of the problem results the interesting corollary:

If a rectangular hyperbola passes through the vertices of a triangle and its centroid, it passes also through the incenter and the three excenters of the triangle formed by the midpoints of the given triangle.

Solution II gives an interesting synthetic proof of this corollary.

3798 [1936, 500]. *Proposed by N. A. Court, University of Oklahoma.*

Construct a sphere belonging to a given coaxal pencil and passing through the inaccessible point of intersection of a given line with a given plane.

Note. A similar problem in plane geometry was discussed in the *Educational Times*, Reprints, vol. 5, 1904, p. 83, Q, 15401.

Solution by the proposer.

First solution. Let I be the inaccessible point common to the given line t and the given plane (P), and let A be a point on the basic circle, assumed to be

real, of the given coaxal pencil Σ. The foot P of the perpendicular AP from A upon the plane (P) lies on the sphere (M) having AI for diameter, and the same is true about the feet Q, R of the perpendiculars from A upon any two planes (Q), (R) passing through t. The sphere $(M) \equiv APQR$ is therefore determined, hence also its center M.

Thus the midpoint M of the segment AI is known. The plane perpendicular to AM at M meets the line of centers of Σ in the center of the required sphere.

Second solution. The above solution breaks down, if Σ is a non-intersecting pencil. The solution which follows is applicable whatever the nature of the pencil Σ may be.

The spheres of Σ determine on the line t pairs of points in involution. This involution is projected from a line u taken in the plane (P) by an involution of planes (u). If I' is the trace on t of the plane (P') which corresponds to (P) in the involution (u), the sphere passing through I' and belonging to the coaxial pencil Σ solves the problem (Nathan Altshiller-Court, *Modern Pure Solid Geometry*, p. 178, art. 559. Macmillan, 1935).

Editorial Note. It is not clear as to the definition of the accessible region. O.D.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

A testimonial dinner was recently given by alumni of the University of Michigan to Dr. J. W. Glover, James Olney professor of mathematics and former chairman of the department, who retired from active teaching at the end of the academic year. In his honor the James W. Glover scholarship fund, made possible by contributions from his former students, was announced, and a check for $800 as the nucleus of this fund was presented to the University. Dr. Glover was presented with a volume containing many letters expressing admiration and affection.

On May 20, 1938, at the dedication of the Franklin Institute, the University of Pennsylvania conferred an honorary degree upon Dean G. D. Birkhoff of Harvard University. At the annual meeting of the American Academy of Arts and Sciences held on May 11 in Boston, Dean Birkhoff spoke on *Mathematical advances since 1900 and their influence on scientific thought.*

Professor M. H. Stone of Harvard University has been elected a member of the National Academy of Sciences.

Professor Solomon Lefschetz of Princeton University addressed the British Association for the Advancement of Science at its August 1938 meeting. His topic was *Fixed points of transformations.*

A Guggenheim Fellowship has been awarded to Assistant Professor D. H.

Lehmer of Lehigh University. Professor Lehmer will work at various English universities on the analytic theory of numbers.

The William Lowell Putnam Prize Scholarship for 1938 has been awarded to Mr. I. Kaplansky of the University of Toronto. This scholarship is awarded annually by the Division of Mathematics at Harvard University for study at that university to one of the first five contestants in the William Lowell Putnam Mathematical Competition. Mr. Kaplansky plans to use this award during the academic year 1939–40.

Professor J. W. Lasley, Jr. of the University of North Carolina has been serving as president of the North Carolina Academy of Science for the year 1938.

Assistant Professor R. D. Agnew of Cornell University has been promoted to a professorship.

Assistant Professor L. A. Aroian is on leave of absence from Colorado State College, Fort Collins, for 1938–39 and holds a fellowship for graduate study at the University of Michigan.

Professor Emil Artin, who came to the University of Notre Dame from the University of Hamburg in the fall of 1937, has been appointed professor of mathematics at Indiana University.

Dr. Theodore Bennett of Marietta College has been promoted to an assistant professorship.

Dr. M. T. Bird of Utah State Agricultural College has been promoted to an ssistant professorship.

Professor J. W. Bradshaw of the University of Michigan has leave of absence for the first semester of the year 1938–39.

At the North Carolina State College of Agriculture and Engineering, Assistant Professor J. W. Cell has been promoted to an associate professorship.

Assistant Professor R. V. Churchill of the University of Michigan has been promoted to an associate professorship.

Associate Professor H. B. Curry of Pennsylvania State College is on leave of absence for 1938–39 and will be at the Institute for Advanced Study.

At Indiana University Professor S. C. Davisson has retired after forty-eight years of service in the department of mathematics.

Associate Professor R. D. Douglass of Massachusetts Institute of Technology has been promoted to a professorship.

Dr. W. H. Durfee, professor of mathematics at Hobart College, has been made dean of the college.

Dr. A. L. Foster of the University of California has been promoted to an assistant professorship.

Dr. Gordon Fuller of New Mexico State College has been appointed an associate professor at Alabama Polytechnic Institute.

Dr. D. W. Hall has been appointed a National Research Fellow for 1938–39, and will study at the University of Pennsylvania.

Associate Professor Frank Irwin of the University of California has been given the title emeritus.

Dr. B. F. Kimball has been appointed engineering mathematician, Depreciation Unit of the Division of Research and Valuation of the State Public Service Commission, New York, N.Y.

Assistant Professor J. H. Kusner of the University of Florida has been promoted to an associate professorship.

Dr. Saunders MacLane of the University of Chicago has been appointed an assistant professor at Harvard University.

Dr. H. M. MacNeille of Harvard University has been appointed associate professor at Kenyon College.

Dr. W. T. Martin of Massachusetts Institute of Technology has been promoted to an assistant professorship.

Professor Karl Menger of the University of Notre Dame was visiting professor during the summer quarter at the University of California.

Assistant Professor A. B. Mewborn of the University of Arizona is on leave of absence for 1938–39.

Dr. F. J. Murray of Columbia University has been promoted to an assistant professorship.

Dr. S. B. Myers of the University of Michigan has been promoted to an assistant professorship.

E. A. Nordhaus is on leave of absence from the University of Wisconsin Extension Division and holds a fellowship for graduate study at the University of Chicago.

Dr. B. J. Pettis of the University of Virginia has been appointed a Sterling Fellow at Yale University.

Associate Professor C. H. Rawlins, Jr. of the Postgraduate School, U. S. Naval Academy, has been promoted to a professorship.

Dr. Evelyn Carroll Rusk, professor of mathematics at Wells College, has been appointed dean of the college.

Professor Hazel E. Schoonmaker of Hartwick College was married on July 14, 1938 to Professor L. T. Wilson of the United States Naval Academy.

Dr. Ruth G. Simond of the University of Michigan has been appointed to an assistant professorship at Hampton Institute.

Professor E. R. Sleight of Albion College is on leave of absence for the first semester 1938–39. He has been traveling in Norway and Sweden and expects to study in English and Scottish universities.

Assistant Professor C. H. Smiley of Brown University has been made chairman of the department of astronomy and director of Ladd Observatory, with the rank of associate professor.

Professor Virgil Snyder of Cornell University has retired with the title professor emeritus.

Assistant Professor A. W. Tucker of Princeton University has been promoted to an associate professorship.

C. B. Tucker of Kansas State Teachers College, Emporia, has been promoted to an assistant professorship.

Dr. R. J. Walker of Cornell University has been promoted to an assistant professorship.

Professor Agnes E. Wells, who has been dean of women in addition to teaching mathematics at Indiana University, has given up the work of dean to give full time to teaching.

Associate Professor W. M. Whyburn of the University of California at Los Angeles has been promoted to a professorship.

Professor K. P. Williams of Indiana University has been appointed chairman of the department of mathematics.

Associate Professor W. L. Williams of the University of South Carolina has been promoted to a professorship.

The following appointments to instructorships have been announced:
Agricultural and Mechanical College of Texas: R. R. Lyle
Bryn Mawr College: M. P. Fobes
Case School of Applied Science: Dr. J. M. Dobbie
University of California at Los Angeles: Dr. A. E. Taylor
Georgia School of Technology: Dr. G. B. Lang
Harvard University: Dr. L. A. Pipes
Haverford College: Dr. C. B. Allendoerfer
University of Illinois: Dr. P. T. Maker, Dr. E. L. Welker
Lehigh University: Dr. A. E. Pitcher, Dr. M. F. Smiley

Los Angeles City College: Dr. D. C. Duncan
Michigan College of Mining and Technology: V. O. York
University of Michigan: Dr. R. F. C. Bartels, Dr. P. C. Hammer, Dr. C. J. Nesbit
Middlebury College: L. B. Hedge
University of Nevada: Ingo Maddaus
North Park Junior College, Chicago: C. G. Erickson
University of Notre Dame: Dr. P. M. Pepper
Princeton University: W. C. Strodt
Purdue University, Dr. D. R. Shreve, Dr. M. S. Webster
Rose Polytechnic Institute: T. P. Palmer
University of Texas: Dr. H. S. Kaltenborn
United States Naval Academy: Dr. H. C. Ayres
Virginia Military Institute: I. G. Foster
Washburn College: Paul Eberhart
Wellesley College: Melita A. Holly
Wesleyan University: Dr. J. W. Wrench, Jr.
Yale University: Dr. D. C. Murdoch

Emeritus Professor E. W. Brown of Yale University died on July 22, 1938, at the age of seventy-one years. He was a charter member of the Mathematical Association.

Professor Carl Gundersen of Oklahoma Agricultural and Mechanical College died April 11, 1938. He was a charter member of the Mathematical Association.

Doctor D. N. Lehmer, emeritus professor of mathematics at the University of California, died September 8, 1938. He was a charter member of the Mathematical Association and had served both as vice-president and as trustee.

PROPOSED AMENDMENT TO THE BY-LAWS OF THE ASSOCIATION

The experience of the Mathematical Association is that the life membership fee has not in recent years been in accordance with actuarial practice. The Trustees have therefore authorized the Secretary-Treasurer to propose the following amendment to Article VII of the By-Laws, the amendment to be presented for adoption at the Annual Meeting in December 1938. In the meantime life membership fees will not be accepted on the old basis.

Section 6, Article VII shall read:

> The life membership fee shall be the present value, according to the American Annuitants' Table (Male) based upon three and one-half (3½) per cent interest, of an annuity due of Four Dollars ($4) a year at the attained age of the member; an annual valuation of the life membership fund shall be made under the American Annuitants' Table (Male), three and one-half (3½) per cent; and the reserve thus computed shall be held as a liability.

W. D. CAIRNS, *Secretary-Treasurer*

THE OCTOBER MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The ninth regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at the University of Pittsburgh, Pittsburgh, Pennsylvania, on Saturday, October 23, 1937. Professor L. L. Dines, chairman of the Section, presided at both the morning and afternoon sessions.

Forty-six representatives from twelve colleges, three research laboratories, and twelve high schools attended the meeting, including the following twenty-four members of the Association: O. F. H. Bert, H. L. Black, A. M. Bryson, Helen Calkins, W. E. Cleland, Elizabeth B. Cowley, L. L. Dines, H. L. Dorwart, F. A. Foraker, H. C. Hicks, B. P. Hoover, R. P. Johnson, V. V. Johnston, H. R. Leifer, M. L. Manning, L. T. Moston, J. H. Neelley, E. G. Olds, J. B. Rosenbach, E. M. Starr, J. S. Taylor, R. W. Thomas, W. J. Wagner, E. D. Wells.

At the annual business meeting the following officers of the Section were elected: Chairman, H. L. Black, Westminster College; Secretary-Treasurer, J. S. Taylor, University of Pittsburgh; Member of the Executive Committee, H. C. Hicks, Carnegie Institute of Technology. Professor F. W. Owens, Pennsylvania State College, continues in office for the second year of his term as the additional member of the executive committee.

The following five papers were read:

1. "A problem in heat conduction with application to the ignition of gases" by H. G. Landau, Coal Research Laboratory, Carnegie Institute of Technology, introduced by the Secretary.

2. "Mathematics for the millions" by Professor W. P. Cunningham, California State Teachers College, introduced by C. S. Atchison.

3. "The Tarry-Escott problem" by Professor H. L. Dorwart, Washington and Jefferson College.

4. "An application of the derivative for classes in elementary calculus" by A. M. Bryson, University of Pittsburgh.

5. "Cryptography" by Professor H. C. Hicks, Carnegie Institute of Technology.

Abstracts of these papers follow, with the numbers corresponding to the numbers in the list of titles:

1. Mr. Landau's problem arose from a consideration of the processes which occur when a mixture of combustible gases is ignited by a local source, such as an electric spark. A heat conducting medium of infinite extent is initially at temperature T_0, except within a sphere of radius R, where the initial temperature is T_1. At the start this sphere is filled with active particles which diffuse through the medium; each active particle generates Q units of heat in unit time and they increase in number at a rate proportional to their concentration. It is desired to determine the condition for the temperature at the center of the sphere never to decrease. The partial differential equations for concentration of active particles and temperature are solved and from this the required condition is obtained.

2. A recent study made by the Committee for Curriculum Revision in the State Teachers Colleges of Pennsylvania revealed that the current practice of colleges is to minister chiefly to the needs of mathematics majors, with little or no provision for others. To meet this situation Professor Cunningham suggested that college teachers enrich their courses by introducing more of the applications of mathematics to science, industry, and the study of social phenomena.

3. This is a report of a portion of a joint paper of the same title by H. L. Dorwart and O. E. Brown which appeared in this MONTHLY, December 1937, pages 613–626.

4. Mr. Bryson developed a method suitable for classes in elementary calculus whereby approximate values of the elementary transcendental functions may be obtained to any desired accuracy from a known initial value of such a function and a knowledge of its derivatives. Not only does this treatment throw early illumination on the rôle and significance of the derivative, but by passing to the limit an opportunity is offered to introduce Taylor's series in a somewhat novel but straightforward manner.

5. In discussing code making and code deciphering Professor Hicks explained the nature of the different types of codes employed and methods of deciphering and presented many illustrations of historical interest. In particular he emphasized the practical feature that a code should be simple enough to permit its easy use without being decipherable quickly enough to destroy the purpose for which it was designed. In conclusion, Professor Hicks challenged the ingenuity of the audience in deciphering a code message appropriate to the meeting, which was based upon the key "The Mathematical Association of America."

J. S. TAYLOR, *Secretary*

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The tenth regular meeting of the Allegheny Mountain Section of the Mathematical Association of America was held at the Sharon Works of the Westinghouse Electric and Manufacturing Company, Sharon, Pennsylvania, on Saturday, May 14, 1938. Professor H. L. Black, chairman of the Section, presided at both the morning and afternoon sessions, which were held in the auditorium of the new research building of the Sharon Works. The last paper on the program included a demonstration in the special laboratory for the production of artificial lightning.

The meeting was attended by sixty-eight representatives from eleven colleges, ten research laboratories and industrial concerns, and two high schools, including the following seventeen members of the Association: C. S. Atchison, H. L. Black, Helen Calkins, W. E. Cleland, L. L. Dines, H. L. Dorwart, C. W. Foard, B. P. Hoover, V. V. Johnston, A. V. Karpov, M. L. Manning, L. T. Moston, C. T. Oergel, E. G. Olds, J. S. Taylor, W. J. Wagner, E. D. Wells.

Following a welcoming address by W. M. Dann, assistant manager, Transformer Engineering Department, Westinghouse Electric and Manufacturing Company, the following five papers were read:

1. "Scientific approach to engineering problems" by A. V. Karpov, chairman of Committee on Fundamentals Controlling Structural Design, American Society of Civil Engineers; Consulting Engineer, Pittsburgh.

2. "The content of a course in higher algebra for prospective secondary school teachers of mathematics" by Professor W. H. Erskine, Bethany College, introduced by the Secretary. Leader of discussion: Professor E. G. Olds, Carnegie Institute of Technology.

3. "Use of mathematics in engineering" by H. V. Putman, manager, Transformer Engineering Department, Westinghouse Electric and Manufacturing Company, introduced by Mr. Manning.

4. "Analysis of transient voltages in transformer networks" by P. L. Bellaschi and A. J. Palermo, Westinghouse Electric and Manufacturing Company, introduced by Mr. Manning.

5. "Demonstration of laboratory lightning" by M. L. Manning, Westinghouse Electric and Manufacturing Company.

Abstracts of these papers follow, with the numbers corresponding to the list of titles:

1. Mr. Karpov outlined in a comprehensive fashion the similarity in the present day developments of the fine arts, science, and engineering. The method of carrying human knowledge beyond the perceptual world by the introduction of a conceptual world was discussed and illustrated by significant examples in these fields. The mathematical bases of the exact sciences were outlined, as well as their application to engineering problems. Particular attention was given to the concept of stress, its limitations, and the necessity of extending this concept.

2. Professor Erskine gave a description of the aims, content, and methods of a course in higher algebra given at Bethany College during the current year for

prospective secondary school teachers of mathematics. The aims were to widen the high school teacher's interest in and appreciation of the subject matter of algebra and arithmetic. The content included topics from the theory of integers and the fundamental concepts of algebra and arithmetic; the method involved lectures, assigned readings, and exercises showing the dependence of algebra and arithmetic upon fundamental concepts, and included the experimental contact with algebras differing from that taught in high schools. The discussion, led by Professor Olds, brought out many other features of general interest.

3. Mr. Putman thought it might be possible in mathematics classes for engineering students to place greater emphasis on the application of mathematics than on manipulative technique. Comparatively few engineers attain great proficiency in the real use of mathematics, and much of what is learned is forgotten because of inability to apply it. It was even suggested that for students who do not go to college it might be possible to develop a high school course in elementary calculus and linear differential equations which would be less difficult than some parts of intermediate algebra and much more useful. A summary of mathematical applications in transformer design was included to show the importance of a thorough training in mathematics in this branch of engineering. In response to the suggestion that an endeavor be made to evaluate the usefulness of different branches of mathematics in engineering fields in order that students may select those courses likely to be of greatest value to them, a motion was passed that the Society for the Promotion of Engineering Education be informed of the interest that would attach to such a study.

4. This investigation by Mr. Bellaschi and Mr. Palermo was a study, both from an engineering and a mathematical analysis viewpoint, of transformer networks subjected to lightning. The solution for the response of the network showed completely the amplitudes and periods of natural oscillations. A geometric method of approach aids materially the analytic work. The mathematical analysis formulated a complete system of solutions for obtaining the network responses. A pattern of "alpha" operators was developed which reduced the work to relatively simple algebra. To form this pattern, only the operational solutions for one and two mesh networks were required. From this pattern solutions can be directly written for networks having any specified number of meshes. The analysis was applied to transformer windings and is applicable to many fields of electrical engineering.

5. Following a description of the laboratory equipment by Mr. Manning, the most important of which are the high voltage generator (3,000,000 volts) and a heavy current generator (150,000 amperes), a very spectacular demonstration was given. A wood pole used for distribution service was badly splintered when subjected to a discharge from the generator. A similar lightning stroke was then applied to a distribution transformer protected by deion gaps; the transformer was unharmed. The paper concluded with a discussion of the many developments which lightning stroke generators may aid in producing.

J. S. TAYLOR, *Secretary*

FIFTEENTH ANNUAL MEETING OF THE INDIANA SECTION

The fifteenth annual meeting of the Indiana Section of the Mathematical Association of America was held Friday and Saturday, May 6 and 7, 1938, at Indiana State Teachers College, Terre Haute, Indiana.

Eighty-one registered at the meetings including the following twenty-three members of the Association: W. C. Arnold, Juna Lutz Beal, C. S. Doan, J. E. Dotterer, Olive M. Draper, W. E. Edington, P. D. Edwards, Louis Green, H. E. H. Greenleaf, W. R. Hardman, L. C. Karpinski, M. W. Keller, Cornelius Lanczos, Florence Long, T. E. Mason, Karl Menger, H. A. Meyer, H. R. Pyle, C. K. Robbins, L. S. Shively, W. O. Shriner, L. H. Whitcraft, K. P. Williams.

At the business session on Saturday the following officers were elected for next year: Chairman, C. K. Robbins, Purdue University; Vice-Chairman, L. S. Shively, Ball State Teachers College; Secretary, P. D. Edwards, Ball State Teachers College. The sixteenth annual meeting will be held at Ball State Teachers College on April 28 and 29, 1939.

Professor K. P. Williams of Indiana University made the report for the committee appointed to encourage and recognize superior preparation for the teaching of secondary mathematics. On the basis of examinations conducted April 23 and April 30 a Certificate of Merit in Mathematical Preparation was awarded to Margaret Stump of Butler University.

Following the annual dinner on Friday night the following program was presented:

1. "Report on the work of the Joint Commission of the Association and the National Council on the place of mathematics in the secondary schools" by Professor K. P. Williams, Indiana University.

2. "Report on the study of a committee of the American Association of Teachers Colleges on desirable attainments for teachers of secondary mathematics" by Professor L. H. Whitcraft, Ball State Teachers College, introduced by the Secretary.

3. "The need for greater emphasis on the mathematics of finance" by Professor W. O. Shriner, Indiana State Teachers College.

At the meetings on Saturday the following program was presented:

4. "The three classical problems of geometry" by Professor Karl Menger, University of Notre Dame.

5. "Sources of material which we present in freshman mathematics" by Professor L. C. Karpinski, University of Michigan.

6. "Interpolating and extrapolating power series" by Professor Cornelius Lanczos, Purdue University.

7. "A practical general formula for annuity problems" by Professor H. E. H. Greenleaf, DePauw University.

8. "Concerning a certain functional equation" by Professor K. P. Williams, Indiana University.

9. "Abstract group definition for groups of finite order" by Professor W. E. Edington, DePauw University.

10. "Fractional radices" by Professor H. A. Meyer, Hanover College.

11. "A solution of the quartic equation" by M. E. Gamble, Purdue University, introduced by Professor Robbins.

Abstracts of the papers follow, the numbers corresponding to the numbers in the list of titles:

1. Professor Williams gave a discussion of the work of the Joint Commission of the Mathematical Association of America and the National Council of Teachers of Mathematics, which is studying the place of mathematics in the secondary schools. He commented upon the first four chapters of the Preliminary Report which had already appeared, and gave an outline of the contents of the remaining part, which was to appear about July 1, 1938.

2. Professor Whitcraft gave an account of the method employed to secure a list of desirable attainments for teachers of secondary mathematics. The attainments were classified under the headings: (a) Subject matter, (b) Professional understanding and skills relating to mathematics, and (c) Personal qualifications. The study originated in the Teachers College Conference Group at the University of Chicago, 1937, with fourteen mid-western institutions cooperating.

3. With the mathematics of finance as the central theme, Professor Shriner stated that high school algebra could be made the most vital part of secondary education and an essential part of the general education of every adult. The algebraic principles involved in analyzing problems in home finance, savings, installment buying, amortization, depreciation, insurance, *etc.*, reveal the need for all the topics of elementary algebra.

4. Professor Menger pointed out that the answer to the questions whether one can double the cube, trisect the angle, and square the circle depends, of course, on the means of construction allowed. A conchoidograph allows us to construct exact solutions for the first two. An involutograph, of which simple forms have recently been devised, enables us to construct, from a circle, with radius r, a segment of length πr, and consequently πr^2. Approximate solutions of great accuracy are possible for all three problems by means of ruler and compasses. Noteworthy among them is a simple trisection of the angle with an inaccuracy of less than 20″ due to Kopff.

5. The mathematical material presented to freshmen in American colleges is directly connected by Professor Karpinski with the most ancient mathematical documents of the Babylonians and Egyptians. Recent discoveries touch vitally the story of the progress of algebraic and trigonometric ideas. The intensive development of algebra and the applications of its methods to geometry have been extended backwards in time to some two thousand years before the Christian Era. The new problems are concerned primarily with the perimeter and with linear, quadratic, and even cubic functions of the sides of triangles and rectangles. Arithmetical series are involved in the division into trapezoidal areas of triangles and trapezoids. Trigonometric development, notably computation of chords in a circle by the Pythagorean theorem, are now definitely connected

with the highly scientific Babylonian astronomy. The progress of geometrical algebra and trigonometry in Greece, and similar work among the Arabs, is made a natural continuation of the Babylonian ideas. European and modern developments are revealed as vitally bound to the past by the new material. The analytic geometry of Descartes and Fermat is indicated as another phase of the fruitful association of geometry and algebra as begun in Babylon and Egypt and as continued in Greece, India, and the Mohammedan World.

6. The different nature of extrapolating and interpolating series was discussed by Professor Lanczos. The Taylor series and the so-called "asymptotic expansions" illustrate the former, the Fourier series the latter. The Tshebysheff polynomials permit the Fourier series to be transformed into an ordinary power series and thus increase the convergence of both Taylor and asymptotic expansions, by transforming then into interpolating series. For the Taylor series a mere re-arrangement of the series in Tshebysheff's polynomials yields a series which approximates closer in a given range with the same number of terms, for the asymptotic expansions trigonometric interpolation is necessary to obtain an "economized" series. Illustrations were given.

7. Professor Greenleaf presented a simple symbol which has proved useful in teaching the solution of practical problems in annuities. This symbol, used with three simple rules for reducing it to tabulated symbols, helps eliminate many of the errors frequently made.

8. It is well known that if H_n is a spherical harmonic of order n then H_n/r^{2n+1}, where $r^2=x^2+y^2+z^2$, is also a solution of the Laplace equation. Professor Williams considered the question of determining m from a functional equation in such a way that H_n/r^m will satisfy the equation of Laplace. The equation arrived at is $f(n)+f(n-f(n))=0$. Some of the properties of this equation were noted and different solutions were given. If $g(n)=n-f(n)$, then $g(g(n))=n$, or $g(n)=g^{-1}(n)$, so that $g(n)$ is its own inverse, and $y=g(x)$ is symmetric with regard to $y=x$.

9. Professor Edington surveyed some of the work that has been done in the definition of finite abstract groups defined by two operators whose product is of the second or third order in which one additional condition is imposed. He stated a number of theorems and relations that are useful in determining the orders of operators. The existence of groups and systems of groups was made to depend on substitutions and the use of mathematical induction.

10. The use of number systems with base other than ten is familiar to mathematicians. Professor Meyer demonstrated the peculiarities of such systems in case the base is a proper fraction.

11. Mr. Gamble gave a general solution of the quartic equation in which he derived the resolvent cubic by a method somewhat more straight-forward than in the solution of Ferrari.

P. D. EDWARDS, *Secretary*

THE p-ADIC NUMBERS OF HENSEL*

C. C. MacDUFFEE, University of Wisconsin

1. Introduction. One cannot blame a respectable mathematician for looking twice at the equation

$$-1 = 4 + 4\cdot 5 + 4\cdot 5^2 + 4\cdot 5^3 + \cdots .$$

However, if we add 1 to both sides of this equation, we have

$$\begin{aligned} 0 &= 5 + 4\cdot 5 + 4\cdot 5^2 + 4\cdot 5^3 + \cdots \\ &= 0 + 5\cdot 5 + 4\cdot 5^2 + 4\cdot 5^3 + \cdots \\ &= 0 + \quad 0 + 5\cdot 5^2 + 4\cdot 5^3 + \cdots \\ &= 0 + \quad 0 + \quad 0 + 5\cdot 5^3 + \cdots \\ &= 0 + \quad 0 + \quad 0 + \quad 0 + \cdots \end{aligned}$$

with 0's as far out as we care to carry it.

It may also seem a trifle strange to write

$$2/3 = 4 + 1\cdot 5 + 3\cdot 5^2 + 1\cdot 5^3 + 3\cdot 5^4 + \cdots ,$$

where the coefficients beyond the first are alternately 1 and 3. Yet multiplying by 3 gives

$$\begin{aligned} 2 &= 12 + 3\cdot 5 + 9\cdot 5^2 + 3\cdot 5^3 + 9\cdot 5^4 + \cdots \\ &= 2 + \quad 0 + \quad 0 + \quad 0 + \quad 0 + \cdots . \end{aligned}$$

Furthermore

$$\sqrt{7} = 1 + 1\cdot 3 + 1\cdot 3^2 + 0\cdot 3^3 + 2\cdot 3^4 + \cdots .$$

For if we square this series, retaining only terms whose exponents are $\leqq 4$, we have

$$\begin{aligned} 7 &= 1 + 2\cdot 3 + 3\cdot 3^2 + 2\cdot 3^3 + 5\cdot 3^4 + \cdots \\ &= 1 + 2\cdot 3 + \quad 0 + \quad 0 + \quad 0 + \cdots . \end{aligned}$$

2. Justification of the p-adic numbers. No one will deny that the above examples put a heavy strain on our earlier conceptions of the terms *equality* and *convergence*. It is obvious that the statement

$$-1 = 4 + 4\cdot 5 + 4\cdot 5^2 + 4\cdot 5^3 + \cdots$$

is absurd if ordinary convergence is intended. The whole point to Hensel's theory is that this is not ordinary convergence, but a new type of convergence which, from the point of view of abstract algebra, is equally worthy of the name.

A relation of equality for a mathematical system Σ is defined as follows. Let a, b, and c be elements of Σ. Then

* Presented for the Slaught Memorial Volume of the MONTHLY.

1. Either $a=b$ or $a\neq b$ (Determinative property).
2. $a=a$ (Reflexive).
3. If $a=b$, then $b=a$ (Symmetric).
4. If $a=b$ and $b=c$, then $a=c$ (Transitive).

These four properties of equality are all that are needed in mathematics, and consequently constitute an abstract formulation of the concept.

When Hensel* introduced the p-adic numbers, his treatment was somewhat informal, but he had a perfectly sound feeling for what he was doing. The present vogue is to introduce the p-adic numbers by a method due to Kürschák,† similar to the well known development of the real numbers by Cauchy sequences.‡

Let $a, b, \cdots$ be rational numbers. A function ϕ is called a *valuation* if

1. $\phi(a)$ is a positive number or 0,
2. $\phi(a)>0$ for $a\neq 0$, $\phi(0)=0$,
3. $\phi(ab)=\phi(a)\cdot\phi(b)$,
4. $\phi(a+b)\leqq\phi(a)+\phi(b)$.

From (3) with $b=1$, we have $\phi(1)=1$. From (3) with $a=b=-1$, we have $\phi(-1)=1$. Then, with $a=-1$, we have $\phi(-b)=\phi(b)$.

Clearly ordinary absolute value, $\phi(a)=|a|$, is a valuation. Furthermore, the four properties listed above constitute an abstract formulation of the concept of absolute value in the sense that only these properties are needed for the development of the real numbers from the rational numbers by the method of regular sequences.

We recall that the ordinary integers or whole numbers $0, \pm 1, \pm 2, \pm 3, \cdots$ are called the *rational integers*, to distinguish them from algebraic integers such as $-\frac{1}{2}-\frac{1}{2}\sqrt{-3}$ which are not rational. A *rational prime* such as $\pm 2, \pm 3, \pm 5, \pm 7 \cdots$ is a rational integer neither 0 nor ± 1 such that, if it is resolved into a product of two rational integral factors, one of the factors must be 1 or -1. Two integers are *relatively prime*, or *prime to each other*, if their only common divisors are ± 1.

Let p be a fixed rational prime. Every rational number $a\neq 0$ is uniquely expressible in the form

$$a = (r/s)p^n, \qquad s > 0,$$

where r and s are rational integers prime to each other and to p, and n is a rational integer. We define

$$\phi(a) = p^{-n}, \qquad a \neq 0, \qquad \phi(0) = 0.$$

The function $\phi(a)$ is a valuation for the rational field.

Properties (1) and (2) are evident. If

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* K. Hensel, Theorie der algebraischen Zahlen, Teubner, 1908.

† J. Kürschák, Journal fur die reine und angewandte Mathematik, vol. 142, 1913, pp. 211–253.

‡ See B. L. van der Waerden, Moderne Algebra, 2nd ed. I, Springer 1937, p. 221.

$$a = (r_1/s_1)p^m, \qquad b = (r_2/s_2)p^n,$$

where r_1, s_1, r_2, and s_2 are prime to p, then

$$ab = (r_1r_2/s_1s_2)p^{m+n},$$

where r_1r_2 and s_1s_2 are prime to p. Hence

$$\phi(ab) = p^{-m-n} = \phi(a)\cdot\phi(b).$$

Without loss of generality assume that $m \leqq n$. Then

$$a + b = \frac{r_1s_2 + r_2s_1p^{n-m}}{s_1s_2}\,p^m,$$

where s_1s_2 is prime to p, so that

$$\phi(a + b) \leqq p^{-m} = \phi(a),$$
$$\phi(a + b) \leqq \phi(a) + \phi(b).$$

Let p be a fixed prime, and let ϕ be defined relative to p. A sequence

$$\{a_i\} = (a_1, a_2, a_3, \cdots, a_i, \cdots)$$

of rational numbers is called *regular* if for every positive rational number ϵ there is a positive integer n_ϵ such that

$$\phi(a_i - a_j) < \epsilon \qquad i, j > n_\epsilon.$$

Denote by Ω_p the set of all regular sequences $\{a_i\}$. Two such sequences are defined to be *equal* if, for every ϵ, there is an n_ϵ such that

$$\phi(a_i - b_i) < \epsilon \qquad i > n_\epsilon.$$

Equality as defined above is determinative, reflexive, symmetric, and transitive.

The determinative property is evident. The reflexive property follows from the definition of regularity. Symmetry follows from the fact that $\phi(-a) = \phi(a)$. Transitivity follows from the "triangle property" $\phi(a+b) \leqq \phi(a)+\phi(b)$.

The theory now proceeds as in the usual treatment of real numbers as regular sequences. We define

$$\{a_i\} + \{b_i\} = \{a_i + b_i\}, \qquad \{a_i\}\cdot\{b_i\} = \{a_ib_i\}.$$

The sum and product of regular sequences are regular. The set Ω_p of all regular sequences, with equality, addition, and multiplication as we have defined them, is a field of characteristic zero—that is, it is a field which contains a subfield isomorphic with the rational field. Indeed, two fields Ω_p and Ω_q where p and q are distinct primes are non-isomorphic so that we obtain infinitely many essentially different fields, each, of course, different from the real field. But like the real field every Ω_p is perfect—that is, incapable of further extension by means of regular sequences based on a valuation which extends the valuation by which Ω_p was defined.

It is one of the standard procedures in analysis to show that every real number can be represented as an infinite decimal—that is, a series of the type

$$a_{-\nu}\left(\frac{1}{10}\right)^{-\nu}+\cdots+a_0+a_1\left(\frac{1}{10}\right)+a_2\left(\frac{1}{10}\right)^2+\cdots$$

or the negative of such a series, with $0\leqq a_i<10$. This process can be carried over intact to the p-adic fields. It may be proved that every regular p-adic sequence is equal in the p-adic sense to a sequence

$$p^{-\nu}\{d_i\}, \qquad \{d_i\}=(a_0,\ a_0+a_1p,\ a_0+a_1p+a_2p^2,\cdots), \qquad 0\leqq a_i<p.$$

That is, every p-adic number may be represented by a power series

$$a_{-\nu}p^{-\nu}+\cdots+a_0+a_1p+a_2p^2+\cdots, \qquad 0\leqq a_i<p.$$

Just as the infinite decimal is automatically convergent in the Cauchy sense, so is the above series automatically convergent in the p-adic sense.

It is to be emphasized that for every rational prime p there is a field Ω_p quite comparable with the field of real numbers, but not isomorphic with it, nor with any other Ω_p. The field Ω_2 is the field of all *diadic* numbers, Ω_3 of all *triadic* numbers, Ω_5 of all *pentadic* numbers, etc. Once the field has been selected, all calculations remain in this field. We cannot add or multiply a triadic number and a pentadic number, for instance.

3. Solution of equations. Now that we understand the meaning of a statement such as

$$\sqrt{7}=1+1\cdot 3+1\cdot 3^2+0\cdot 3^3+2\cdot 3^4+\cdots,$$

it remains to show how any desired number of terms of the expansion can be derived.

First we note that every p-adic number

$$a_{-\nu}p^{-\nu}+\cdots+a_0+a_1p+a_2p^2+\cdots, \qquad 0\leqq a_i<p,$$

is the sum of a rational number

$$a_{-\nu}p^{-\nu}+\cdots+a_{-1}p^{-1}$$

and a number

$$a_0+a_1p+a_2p^2+\cdots$$

having no negative exponents, which is called an *integral p-adic number*, or a *p-adic integer*.

Two p-adic integers

$$\alpha=a_0+a_1p+a_2p^2+\cdots, \qquad 0\leqq a_i<p,$$
$$\beta=b_0+b_1p+b_2p^2+\cdots, \qquad 0\leqq b_i<p,$$

are equal in the p-adic sense, according to the definition of §2, if for every $\epsilon>0$

there is an n_ϵ such that for $i>n_\epsilon$,

$$\phi((a_0-b_0)+(a_1-b_1)p+(a_2-b_2)p^2+\cdots+(a_i-b_i)p^i)<\epsilon.$$

Suppose that $a_0=b_0$, $a_1=b_1$, $\cdots$, and that k is the first integer for which $a_k\neq b_k$, so that a_k-b_k is prime to p. Then

$$\phi((a_k-b_k)p^k+(a_{k+1}-b_{k+1})p^{k+1}+\cdots+(a_i-b_i)p^i)=1/p^k,$$

so that if we take $\epsilon<1/p^k$, the condition that $\alpha=\beta$ is not met. Thus *if* $\alpha=\beta$, *corresponding coefficients are equal.* The converse is evident.

Now if

$$\alpha=a_0+a_1p+a_2p^2+\cdots, \qquad 0\leqq a_i<p,$$

then clearly

$$\begin{aligned}
&\alpha\equiv a_0 \pmod{p},\\
&\alpha\equiv a_0+a_1p \pmod{p^2},\\
&\alpha\equiv a_0+a_1p+a_2p^2 \pmod{p^3},\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\alpha\equiv a_0+a_1p+\cdots+a_{i-1}p^{i-1} \pmod{p^i}.
\end{aligned}$$

Thus the coefficients a_0, a_1, a_2, $\cdots$, in the expansion of α can be successively determined from the residues of α modulo p^i, $i=1, 2, \cdots$. In other words, $\alpha=\beta$ *if and only if*

$$\alpha\equiv\beta \pmod{p^i}$$

for every positive integer i.

Let $f(x)$ be a polynomial with rational integral coefficients, and let p be a fixed rational prime. We wish to find out if $f(x)=0$ has a solution α in Ω_p, and to determine an arbitrary number of its coefficients.

First, suppose that $f(x)=0$ has an integral p-adic solution, *e.g.*,

$$\alpha=a_0+a_1p+a_2p^2+a_3p^3+\cdots, \qquad 0\leqq a_i<p.$$

Denote

$$\alpha_{n-1}=a_0+a_1p+a_2p^2+\cdots+a_{n-1}p^{n-1}.$$

Thus α is a solution of $f(x)=0$ in Ω_p if and only if

$$f(\alpha)\equiv 0 \pmod{p^i} \qquad (i=1, 2, 3, \cdots);$$

that is to say, if and only if each of the infinitely many congruences

$$\begin{aligned}
&f(\alpha_0)\equiv 0 \pmod{p},\\
&f(\alpha_1)\equiv 0 \pmod{p^2},\\
&f(\alpha_2)\equiv 0 \pmod{p^3},\\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

$$f(\alpha_{n-1}) \equiv 0 \pmod{p^n},$$

. . . . . .

holds.

The problem is now reduced to a familiar one in congruences. Whether $f(x) \equiv 0 \pmod{p}$ has a solution or not is usually best determined by trial. If there is a solution α_0, $0 \leqq \alpha_0 < p$.

Now there is a well known step-by-step process for finding a solution (mod p^{n+1}) when a solution (mod p^n) is known.* Suppose that

$$\alpha_{n-1} = a_0 + a_1 p + a_2 p^2 + \cdots + a_{n-1} p^{n-1}, \qquad f(\alpha_{n-1}) \equiv 0 \pmod{p^n}.$$

We wish to find a_n so that

$$\alpha_n = \alpha_{n-1} + a_n p^n, \qquad f(\alpha_n) \equiv 0 \pmod{p^{n+1}}.$$

By the binomial theorem,

$$\begin{aligned} f(\alpha_n) &= f(\alpha_{n-1}) + a_n f'(\alpha_{n-1}) p^n + \cdots \\ &\equiv f(\alpha_{n-1}) + a_n f'(\alpha_{n-1}) p^n \pmod{p^{n+1}}, \end{aligned}$$

where f' denotes the derivative. Since $f(\alpha_{n-1}) \equiv 0 \pmod{p^n}$, there exists an integer h_{n-1} such that

$$f(\alpha_{n-1}) \equiv h_{n-1} p^n \pmod{p^{n+1}}, \qquad 0 \leqq h_{n-1} < p.$$

Hence a_n can be determined from the congruence

$$a_n f'(\alpha_{n-1}) p^n + h_{n-1} p^n \equiv 0 \pmod{p^{n+1}},$$

which is equivalent to the congruence

$$a_n f'(\alpha_{n-1}) + h_{n-1} \equiv 0 \pmod{p}. \tag{1}$$

Clearly a_n will exist unless $f'(\alpha_{n-1}) \equiv 0$, $h_{n-1} \not\equiv 0 \pmod{p}$. If it exists, it can be chosen in the interval $0 \leqq a_n < p$.

In order that

$$x^2 = 7$$

be solvable in triadic numbers, it is first necessary that 7 be a quadratic residue† modulo 3. This condition is met, for both 1 and 2 are solutions of $x^2 \equiv 7 \pmod{3}$. Let us take the first solution. Then $\alpha_0 = a_0 = 1$.

$$f(x) = x^2 - 7, \qquad f(\alpha_0) = -6 \equiv 3 \pmod{9}, \qquad h_0 \equiv 1 \pmod{3},$$
$$f'(x) = 2x, \qquad f'(\alpha_0) \equiv 2 \pmod{3}.$$

Then (1) becomes

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* L. E. Dickson, Introduction to the theory of numbers, University of Chicago Press, 1929, p. 16, ex. 4.

† Dickson, *l.c.*, p. 30.

$$2a_1 + 1 \equiv 0 \pmod 3,$$

which has the solution $a_1 = 1$. Thus $\alpha_1 = 1 + 1 \cdot 3 = 4$ is a solution of $x^2 \equiv 7 \pmod 9$.

The second step gives $\alpha_1 = 4$,

$$f(\alpha_1) = 9, \qquad h_1 = 1, \qquad f'(\alpha_1) = 8 \equiv 2 \pmod 3,$$
$$2a_2 + 1 \equiv 0 \pmod 3, \qquad a_2 = 1, \qquad \alpha_2 = 1 + 1 \cdot 3 + 1 \cdot 3^2.$$

Two more steps give

$$\alpha_4 = 1 + 1 \cdot 3 + 1 \cdot 3^2 + 0 \cdot 3^3 + 2 \cdot 3^4,$$

which was checked in §1.

The other value of a_0, namely 2, is the first term of another triadic solution,

$$\beta = 2 + 1 \cdot 3 + 1 \cdot 3^2 + 2 \cdot 3^3 + 0 \cdot 3^4 + \cdots.$$

There are no other triadic solutions of $x^2 = 7$.

The equation

$$x^2 + x + 1 = 0,$$

whose roots in the complex field are not real has no solution in pentadic numbers, since

$$x^2 + x + 1 = 0 \pmod 5$$

has no solution. However, it has two heptadic solutions,

$$\alpha = 2 + 4 \cdot 7 + 6 \cdot 7^2 + \cdots,$$
$$\beta = 4 + 2 \cdot 7 + 0 \cdot 7^2 + \cdots$$

with the usual relations $\alpha^2 = \beta$ and $\beta^2 = \alpha$. It is incorrect to think of these solutions as being complex numbers—they belong to the field Ω_7.

So far we have looked only for integral p-adic solutions of $f(x) = 0$. But if the leading coefficient of $f(x)$ is divisible by p while not all of the other coefficients are divisible by p, $f(x) = 0$ may have a p-adic solution which is not integral. But this situation involves no difficulty, for a simple transformation reduces this case to the preceding.

Consider the equation

$$9x^2 = 7, \qquad \Omega_3.$$

This has no integral triadic solution, since

$$9x^2 - 7 \equiv 0 \pmod 3$$

has no solution. But a transformation $3x = y$ yields an equation $y^2 = 7$ which we have solved in Ω_3. Then the given equation has as solutions the fractional triadic numbers

$$\alpha = 3^{-1} + 1 + 1 \cdot 3 + 0 \cdot 3^2 + 2 \cdot 3^3 + \cdots,$$
$$\beta = 2 \cdot 3^{-1} + 1 + 1 \cdot 3 + 2 \cdot 3^2 + 0 \cdot 3^3 + \cdots.$$

4. Rational numbers in Ω_p. The close analogy of the p-adic series with infinite decimals is well exemplified in the behavior of the rational numbers. A rational number r/s with r prime to s can be expressed as a finite decimal if and only if every prime factor of s is 2 or 5. A positive rational number r/s with r prime to s can be expressed by a finite p-adic series if and only if s is a power of p.

A decimal is finite or periodic if and only if it is equal to a rational number. Analogously

A p-adic series is finite or periodic if and only if it is equal to a rational number.

It will be sufficiently general to consider the series

$$\alpha = A + p^k B + p^{k+l} B + p^{k+2l} B + \cdots,$$

where

$$A = a_0 + a_1 p + \cdots + a_{k-1} p^{k-1}, \qquad 0 \leqq a_i < p,$$
$$B = b_0 + b_1 p + \cdots + b_{l-1} p^{l-1}, \qquad 0 \leqq b_i < p.$$

We shall call B the *period* of α. Then

$$\alpha - A = p^k B + p^l [p^k B + p^{k+l} B + \cdots] = p^k B + p^l [\alpha - A];$$

that is,

$$\alpha = A + \frac{p^k B}{1 - p^l},$$

which is clearly rational.

To prove the converse, first suppose that $\alpha = r/s$ is a negative proper rational fraction, r prime to s, s prime to p and positive. There exist positive integers l such that

$$p^l \equiv 1 \pmod{s}$$

by Euler's theorem. Let l be the smallest such integer—that is, l is the exponent to which p belongs* modulo s. Let

$$1 - p^l = ms, \qquad m < 0, \qquad mr > 0.$$

Then

$$\alpha = r/s = mr/(1 - p^l).$$

Since α is proper, mr is expressible in the form

$$mr = B = b_0 + b_1 p + \cdots + b_{l-1} p^{l-1}, \qquad 0 \leqq b_i < p.$$

Then

$$\alpha = B + p^l B + p^{2l} B + \cdots$$

is periodic.

* Dickson, *l.c.*, p. 16.

If α is positive, it can be written as the sum of a polynomial in p and a negative proper fraction. The development of $-\alpha$ can be obtained by subtracting the development of α from

$$0 = p \cdot 1 + (p-1) \cdot p + (p-1) \cdot p^2 + (p-1) \cdot p^3 + \cdots .$$

Neither of these operations will destroy the eventual periodicity of the series.

The methods of this paragraph are quicker and more effective for rational numbers than the more general method of §3.

5. Generalizations. The ideas which were disclosed in the development of the p-adic numbers have inspired much modern research. Hensel* himself extended the theory far beyond the simple p-adic fields: to g-adic rings where g is not a prime, to p-adic extensions of algebraic fields, and to functions over such fields and rings. The concepts of valuation and p-adic extension are of great importance in the modern theory of linear algebras,† and their ramifications are still being explored.

A NOTE ON THE USE OF THE LAPLACE TRANSFORMATION

H. P. THIELMAN, College of St. Thomas

In recent years a great deal of work has centered around the Laplace transform of a function $F(t)$.‡ In this note we shall prove an elementary theorem concerning such a transform, and show how it may be used in evaluating some definite integrals.

By definition the Laplace transform of $F(t)$ is the function $f(\alpha)$ given by the formula

$$f(\alpha) = \int_0^\infty e^{-\alpha t} F(t)\,dt.$$

We shall first prove the following:

THEOREM: *If $f(\alpha)$ is the Laplace transform of $F(t)$, then*

$$\int_0^\infty f(\alpha)\,d\alpha = \int_0^\infty \frac{F(t)}{t}\,dt,$$

provided these integrals exist.

The proof of this theorem goes as follows: From the hypothesis that the second one of the last two integrals exists, it follows that the integral

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* K. Hensel, Zahlentheorie, Berlin 1913. Mathematische Zeitschrift, vol. 2, 1918, pp. 433–452.

† Deuring, Algebren, Ergebnisse der Mathematik, vol. 4, Springer, 1935, p. 99.

‡ D. V. Widder, The inversion of the Laplace integral and the related moment problem, Transactions of the American Mathematical Society, vol. 36, 1934, p. 107; Necessary and sufficient conditions for the representation of a function by a doubly infinite Laplace integral, Bulletin of the American Mathematical Society, vol. 40, 1934, p. 321. H. T. Davis, The Theory of Linear Operators, Principia Press, Bloomington, Indiana, 1936, pp. 24, 25, 28 ff.

$$\phi(\alpha) = \int_0^\infty e^{-\alpha t} \frac{F(t)}{t}\, dt \tag{1}$$

exists, is uniformly convergent for all $\alpha \geqq 0$, and is a continuous function of α for $\alpha \geqq 0$.* The integral

$$f(\alpha) = \int_0^\infty e^{-\alpha t} F(t) dt$$

converges uniformly for all α greater than any positive number k. This is also a consequence of the same hypothesis, and a well known theorem.† It follows that

$$\phi'(\alpha) = -\int_0^\infty e^{-\alpha t} F(t) dt = -f(\alpha), \qquad \alpha > 0,$$

where $f(\alpha)$ is the Laplace transform of $F(t)$. Integrating the last equation, and noting that $\phi(\alpha)$ as defined by (1) approaches zero as α goes to infinity, we obtain

$$\phi(\alpha) = \int_\alpha^\infty f(t) dt.$$

Since $\phi(\alpha)$ is a continuous function of α, as was noted above, we can take the limit as α goes to zero of both sides of the last equation, and obtain the required result.

To illustrate the use of our theorem, we shall evaluate some integrals. Consider the well known integral

$$\int_0^\infty \frac{\sin t}{t}\, dt.$$

Here $F(t) = \sin t$. Its Laplace transform is

$$f(\alpha) = \int_0^\infty e^{-\alpha t} \sin t\, dt = \frac{1}{1+\alpha^2}, \qquad \alpha > 0.$$

But the integral of $f(\alpha)$ from zero to infinity is obviously $\pi/2$, hence this is the value of the original integral.

Next, consider

$$\int_0^\infty \frac{\sin^2 t}{t^2}\, dt,$$

which is easily seen to exist. Here

* James Pierpont, The Theory of Functions of Real Variables, Boston, 1905, vol. I, sections 643, 661, and 669.

† Pierpont, *loc. cit.*, p. 468.

$$f(\alpha) = \int_0^\infty e^{-\alpha t} \frac{\sin^2 t}{t}\, dt.$$

The integral

$$-\int_0^\infty e^{-\alpha t} \sin^2 t\, dt$$

is uniformly convergent for α greater than any positive number since

$$\int_p^\infty -e^{-\alpha t} \sin^2 t\, dt < \int_p^\infty e^{-\alpha t} dt = \alpha^{-1} e^{-\alpha p}, \qquad \alpha > 0,\ p > 0,$$

and taking p large enough the last expression can be made arbitrarily small. Therefore

$$f'(\alpha) = -\int_0^\infty e^{-\alpha t} \sin^2 t\, dt, \qquad \alpha > 0.$$

This last integral can be evaluated by elementary methods, and is found to be $2\alpha^{-1}(\alpha^2+4)^{-1}$. Integrating this expression one finds, after determining the constant of integration, that

$$f(\alpha) = -\frac{1}{4} \log \frac{2}{\alpha^2 + 4}, \qquad \alpha > 0.$$

Thus by our theorem the given integral is equal to the integral from zero to infinity of the last displayed expression, which can be evaluated by elementary means, and is found to be $\pi/2$. Hence this is the value of the given integral, which is also a known result.*

By the same method the following results were obtained

$$\int_0^\infty \frac{\sin^3 t}{t^2}\, dt = \frac{3}{4} \log 3,$$

$$\int_0^\infty \frac{\sin^m t \cos t\, dt}{t} = \frac{(m-1)!\pi}{2^m \left[\left(\frac{m-1}{2}\right)!\right]^2 (m+1)} \quad \text{if } m > 0 \text{ and odd},$$

$$= \frac{m!}{2^m} \sum_{p=1}^{m/2} \frac{(-1)^{p+1}(2p+1)\log(2p+1)}{\left(\frac{m}{2}-p\right)!\left(\frac{m}{2}+p+1\right)!} \quad \text{if } m > 0 \text{ and even}.$$

* Pierpont, *loc. cit.*, p. 485.

POSSIBLE BOUNDARIES IN THE EARLY HISTORY OF MATHEMATICS*

DAVID EUGENE SMITH, Columbia University

Having been invited by the editors of this Memorial Volume to contribute an article upon some appropriate topic in which Professor Slaught would be interested if he were with us at this time, a brief historical topic has been selected.

Having known Dr. Slaught for many years, often discussing with him the teaching of mathematics and the types of literature which were best adapted to our work, we were naturally led to considering at various times the story of the origin and development of the subject of our major interest. I well recall that in one of our visits we dwelt upon the possibility of the opening of new regions in the early history of the subject, necessarily in the fields of numbers, and then of geometry, and finally of some crude form of algebra. We were both familiar with the current literature of the subject, but we visualized a much older era to be revealed by earlier material which scholars in the field of archeology might discover. He was greatly interested in the work being done at that time in his own university circle, particularly in the excavations then being carried on in Iraq. It was natural, therefore, that we should be led to discuss the possibility of discoveries in other fields of archeology and even of medieval documents.

Recalling that meeting of only a few years ago, I seem to feel again his enthusiasm in discussing the possibility that we were then upon new thresholds of the habitations of ancient mathematics. It seems therefore appropriate to mention a few of the regions which have opened up since that time and a few which have found place in our recent histories. Briefly stated, since my talks with Dr. Slaught these regions have been widely extended and—what is proving to be of even more importance—the cuneiform records have been deposited in many large museums in most of the leading centers of culture. This spread of original material has allowed scholars, conversant with cuneiform, hieroglyphic, Sanskrit, and other alphabets and related languages, to have access to original documents.

We pass over the well known list of excavations of some decades along the Euphrates and Tigris; there are others in progress at this time, with which students are not so familiar. For example, excavations are now being carried on in the Tepe Gawra, near Mosul in Iraq, where more than twenty strata (levels) have been found, each revealing a period in the life of a great city. There is also the Sumerian Uruk, the Biblical Erech, in which Level XIII dates from the fifth millennium B.C., and has brought one of the richest "finds" of recent years, a rare field for the discoveries of mathematical-commercial material. Level XV dates from *c.* 4500 B.C., and the work upon it is progressing in this very year. The evidence of number values is also being revealed at the present time in the discoveries at Chagar Bazar in northeast Syria, of *c.* 1900 B.C., and

* Presented for the Slaught Memorial Volume of the MONTHLY.

others near by at Tall Brak where contract tablets are being found, some seventy recording the receipt and sale of grain, and there are others in the same region which go back to 1500 and even to 2500–2300 B.C.

Another region in Syria is also revealing a large number of specimens relating to numerals and values. This is at Megiddo, the ancient Armageddon of the Bible, where excavations are being made under the auspices of The University of Chicago. Here is found some fine work in ivory of about 3100 years ago. One of the plaques shows the offerings to the prince, showing that numeral values must have been considered, a supposition which is confirmed by the discovery of a gaming board, requiring a knowledge of numbers. At Megiddo there has also been found a treasure house in one of the palaces dating from *c.* 1650 B.C., and in an earlier stratum there are houses of the late Hyksos kings of *c.* 2000 B.C., those whom the kings of Ahmes's time and name drove out of Egypt and into Palestine.

Iraq has contributed more definitely to our knowledge of ancient mathematics as already mentioned. To it we owe the most of our knowledge of the cuneiform treatment of ancient algebra and its applications. There are, however, many possibilities for finding more material, as in Tell Agrab, about forty miles northeast of Baghdad. This runs back to *c.* 3000 B.C. In Iran too there are still greater possibilities, especially for later material, in a large area near Persepolis and in other regions which have been only superficially explored, particularly for its painted pottery with numbers and their use.

Historians are concerned not merely with the ancient numerals or with computations, but are equally interested with the interchange of numeral systems between countries and cities. Such evidence was found, for example, in last year's excavations in Lachish on the border line between Egypt and Palestine, the scene of the war of about 580 B.C., referred to in the Bible where the name of Nebuchadnezzar (more precisely Nabu-kudurri-usur) stands out prominently. The present excavations near Gaza have shown that the numerals then used for 2, 4, 26, 420, 900, and 1000 reveal a closer relation between the civilizations of Egypt and Palestine than was generally thought. Such interchanges between countries often relate to the calendar, land measure, their products, and especially to important measures of wheat.

The measures required not only a knowledge of commercial values but also of the zodiac and the prediction of solstices and eclipses. They led to an interchange of learning, and one of the most interesting features of the early history of mathematics is concerned with the values and measurements used in different countries. This field might well attract the attention of students of languages and the sciences. In the last two decades excavations have been made in Mohenjo-daro, Harappa, and other regions in India, giving evidence of Aryan migration to this territory in the third millennium B.C., which means a wide spread of numerals and their use.

As to the Mediterranean regions, there have been only a few important discoveries of late relating to ancient mathematics with which we are not al-

ready familiar. Italy contributed little before the founding of Rome, and the contributions of Greece are well known. Egypt, however, has still to add to our store of knowledge. Here as yet undiscovered papyrus rolls and fragments may still have much to be revealed. It is only within two or three years that the tomb of Pharaoh Aha, first king of Egypt, at Sakkara has brought to light material of great value. Although attention has not been called to mathematical items of interest, the mere finding of such a mass of material shows the possible discovery of numerals and measures not yet revealed.

The earliest document relating to the mathematics of the Mediterranean region is, as is well known, the Ahmes (A'h-mosè) papyrus of about 1700 B.C. There is also the so-called Moscow papyrus, but these are not related to the later discoveries with which this article is concerned. As to this recent material there have been found at Kuhan, Egypt, a few fragments dating from about 1900 B.C.

It is not here, however, that we must look for the earliest documents. The oldest mathematical ones of great importance are two Babylonian tablets, one in the Louvre and the other in the British Museum. The latter of these dates from the first Babylonian Dynasty, about 2186–1961 B.C. These reveal a surprising knowledge of quadratic equations for that date. Other ancient mathematical material has of late been found in Sumer, Babylon, Iran, India, and adjacent regions.*

As to the source of the cuneiform tablets, it now seems established that the Sumerians, even before 3500 B.C., knew the cuneiform characters on clay, developing them in or carrying them to the Iraquian regions. These were later developed by the Babylonians and Assyrians into the cuneiform numerals which were in use at least in the first Dynasty of Babylon about 2200 B.C. It seems evident that the temple schools of Nineveh and Babylon developed a science of mathematics, and especially the branch which we now call algebra. This has long been familiar to readers of the current histories of mathematics, but the details of their algebra have only recently become well known. This knowledge has come to us largely through the efforts of such distinguished scholars as Thureau-Dangin, editor of and contributor to the *Revue d'Assyriologie*, with important articles like those of Deimal, Stenzel, Vogel, Bortolotti, and Professor O. Neugebauer, formerly of Göttingen but now of Copenhagen, whose *Mathematische Keilschrift-Texte* has appeared within the last three or four years. Still more recently there has appeared an extended essay by Dr. Solomon Gandz (*Osiris*, vol. 3, pp. 1–146) on *The origin and development of the quadratic equations in Babylonian, Greek, and early Arabic algebra.*

So far as we can now see, the solution of the quadratic equation is due to the Sumerians, the Assyrians, and the Babylonians; but the relation of the Sumerians to the early Hindu civilization as well as to the Babylonian, would lead

* For an account of these discoveries see Professor L. C. Karpinski's list in the American Journal of Semitic Languages and Literatures, vol. 52, p. 73, and this author's article in Scripta Mathematica, vol. 4, pp. 111–125.

us to feel that the first knowledge of this equation is far more remote than the Hindu literature reveals. We are here confronted, however, by the fact that we have no written and contemporary documents to verify any such conjectures as to Indic algebra.

We know by tradition that mathematics was known thousands of years ago, but not through authentic documents. It is only by traditions that we learn through astronomical lore that the zodiac was studied 3000 years B.C., or, as some historians claim, 17,000 B.C. Written documents of China give us interesting traditions of life and science three thousand years ago, just as we have Greek manuscripts containing traditional evidence of the Odyssey, the Iliad, the Bible, and especially the Hindu traditions relating to religious history. We know that calendars existed in Egypt and Babylonia several thousand years ago, but this knowledge comes to us chiefly by documents which themselves are records of traditions.

When we consider that the observation of the equinoxes and solstices as early as 3100 B.C. are recorded and are vouched for by such scholars as Sri Ramakrishna in his *Cultural Heritage of India* (vol. 3, p. 344), we must remember that these statements are themselves based upon a series of unrecorded accounts. We have similar evidence that the Indo-Sumerian peoples contributed to the Assyrian civilization as early as *c.* 2500 B.C., and that these Sumerians had developed a numeral system some three centuries earlier, using clay tablets with cuneiform characters. If this is correct, as is probably the case, we may expect that some documentary evidence of algebra among the Sumerians may soon be found. We also know that King Sagon, who flourished *c.* 2750 B.C. in Akkad, just north of Sumer, may have had commercial intercourse with its people and might have known of their numerals, their calendar, and their astrological-astronomy. We have good reason to say that Sagon's people had a workable system of weights and measures, and that they used common as well as sexagesimal fractions in their work in astronomy. With the Babylonians they also knew how to solve certain types of quadratic and cubic equations.

As to documentary evidence on paper, this medium was known in China as early as *c.* 300 B.C., and became known in Europe through the Arabs in the eighth century, and in Greece about 1300 of our era.

Some historical material is being found in wall paintings and reliefs. The tax collectors, for example, were pictured on various temple walls in Egypt about 2000 B.C., but this merely tells that simple computation and the keeping of accounts were not unknown.

From these sources we may conclude that the possible boundaries in the early history of mathematics, before our era, seem at present to be limited as follows:

1. *Range of knowledge.* Calculation by means of some type of abacus, including counters, tally-sticks, and the like. Mensuration related to areas and volumes. Astronomy, often applied to astrology. Solution of single or simultaneous equations as high as the third degree, used as puzzles. Geometry, including

shadow reckoning by proportion, and attention given to demonstrations as an application of logic. Applications to building—pyramids, arches, areas, volumes.

2. *Sources of information.* We cannot depend upon documents written on wood, paper (including papyrus), leather (including parchment and vellum), cloth, none of which is durable enough to answer alone the purpose of the historian. It is necessary, however, to continue the search for such written or painted material in the cuneiform tablets and in papyrus fragments which have pushed the boundaries back so much in the last few years.

We may therefore hope that some of the younger scholars of our generation will follow their predecessors and continue the study of the clay tablets which have revealed so much ancient history in the field of mathematics. This is the most fruitful field at the present moment. There is next the study of papyri. In spite of the fact that these sources are not as permanent as the clay tablets, they are usable as long as they last and may be valuable. The wall paintings and reliefs have proved of great value, especially when including numerals and their applications. There is also the possibility of the traditions of the Orient in the manipulation of numbers which may reveal something worth considering in number theory.

The possible boundaries therefore seem to be sought in the solution of special equations of higher degrees, a further approach to the euclidean method of proof of geometric theorems and the construction of figures, the European methods of computation, the relation of the equation, which has made such progress of late, to navigation, astronomy, and measurements, and the theory of numbers. When we look at the boundaries of a decade ago and the way in which they have been extended, we may be justified in anticipating a further extension in the future.

A DEVICE FOR DEMONSTRATING THE PROPERTIES OF A SIMPLE PERMUTATION GROUP

VLADIMIR KARAPETOFF, Professor of Electrical Engineering, Cornell University

As the Theory of Finite Groups begins to find its place in physics and chemistry, with eventual possibilities in such applied fields as engineering and biology, it becomes advisable to illustrate the fundamental properties of at least a few simplest groups by means of mechanical or electrical devices, drawings, and other visual means which are customary ways of representation for applied scientists. The device shown in Figure 1 is intended to show that the following permutations form a group: 1, $(xyzwu)$, $(xzuyw)$, $(xwyuz)$, $(xuwzy)$, $(uy)(zw)$, $(xu)(yw)$, $(xw)(yz)$, $(xz)(uw)$, $(xy)(zu)$.*

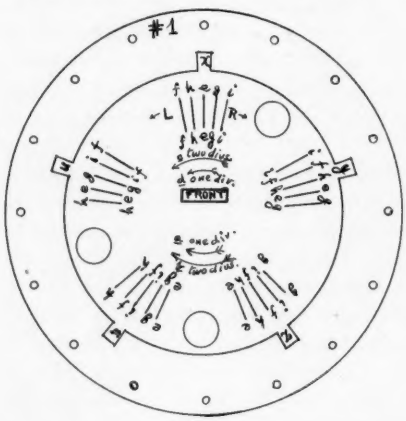


FIG. 1

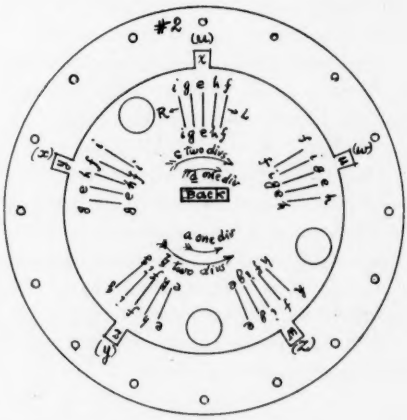


FIG. 2

The device consists of two identical circular cardboard or celluloid slide-rules, No. 1 and No. 2. On No. 1 a desired permutation is set in two steps and on No. 2 in one step. This, of course, is one of the fundamental properties of a group, which is thus directly checked on the device. Each slide-rule consists of an outer rim and an inner disk. The two sides of a slide-rule will be called the front and the back. The letter markings on the front and on the back are identical, in the sense that if the slide-rules were transparent, only one side would have to be marked. The material used being opaque, both sides are marked, and consequently the letters L and R (left and right) seem reversed when looking at the back of the device. The disk can be turned by any desired angle with respect to the rim; the three holes are provided for the operator's fingers, to facilitate turning.

In Figure 3, the outer portion of the disk, concealed by the rim, is shown

* See, for example, H. Hilton, An Introduction to the Theory of Groups of Finite Order. Oxford, 1908; page 53, Example 21 (ii).

separately. It contains the letters *xyzwu* in various permutations; in any position of the rim only one letter is visible for each set of letters, through the slots in the rim. The radial lines marked *e*, *g*, *h*, *i*, and *f* (Fig. 1) are axes of rotation of the slide-rule, to change from the front to the back, or vice versa. There are five axes marked *e*, five marked *f*, and so on, but in any position of the device only that axis marked with a particular letter is to be used which is opposite one of the slots of the rim. Thus, in a specific case, a rotation with respect to a desired letter axis becomes perfectly definite, even though there are four other axes denoted by the same letter.

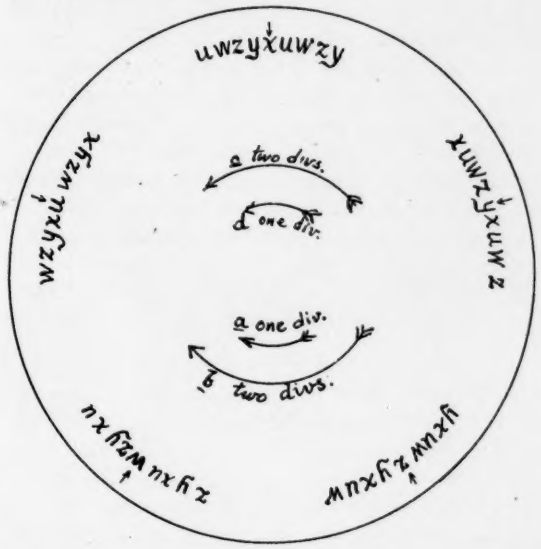
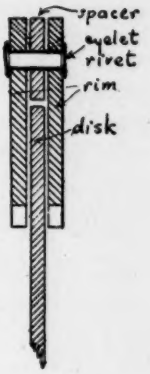


FIG. 3



F.G. 4

A cross-section of one of the slide-rules is shown in Figure 4. In assembling one, a rim is first laid on a table and is covered with a spacer and the disk, both in the same plane. A second rim is then laid over them, and the two rims and the spacer are riveted together at several places, near the circumference.

For the sake of brevity, all the given permutations of the group will be denoted by single letters, as follows:

$$(xyzwu) = a \qquad (uy)(zw) = e$$
$$(xzuyw) = b \qquad (xu)(yw) = f$$
$$(xwyuz) = c \qquad (xw)(yz) = g$$
$$(xuwzy) = d \qquad (xz)(uw) = h$$
$$(xy)(zu) = i$$

To perform the permutation a means to replace each letter by the one immediately to its right, the first letter becoming the last; that is, *xyzwu* is con-

verted into *yzwux*. In the position of the slide-rule No. 1 shown in Figure 1 *xyzwu* is read clockwise, *x* being on top. Turning the disk clockwise by one division will bring *y* on top, the new order of letters being *yzwux*. It is for this reason that one of the arrowheads near the center of the disk is marked "*a* one division." Moving the disk farther by one division clockwise will bring *z* on top, but the order of the letters will always be *xyzwu*. Thus, turning the disk clockwise will give the operations, a, a^2, a^3, and so on, whereas turning it counter-clockwise will produce a^{-1}, a^{-2}, and so on. The letters are so placed on the disk (Fig. 3) as to satisfy these conditions.

It will be seen from the foregoing table that the permutation d is the inverse of a, the letters being written in the inverse cyclic order. Thus, $ad = I$, I being the identical element. To perform the operation d, the disk is moved by one division counter-clockwise, as is indicated by the arrowhead denoted by d. Similarly, it will be found that to perform the permutation b, it is necessary to turn the disk by two divisions clockwise ($b = a^2$). The permutation c is the inverse of b and requires turning the disk by two divisions counter-clockwise; thus, $bc = I$. To demonstrate on the device that to perform the permutation b the disk must be moved by two spaces clockwise, proceed as follows: Set the slide-rule No. 2 in its normal position, that is, with the front towards the observer and the letter x on top. Set the slide-rule No. 1 in the identical position and then turn its disk by two divisions clockwise. Comparing now the letters on the two slide-rules it will be found that

z has taken the place of x,
w " " " " " y,
u " " " " " z,
x " " " " " w,
y " " " " " u.

Writing this result in the form z to x, x to w, w to y, y to u, u to z, it is abridged to (*zuywx*) which is the permutation b. Thus, a result is to be read and interpreted by comparing the two dials, even though the order of the letters on a dial may not be the same as in the permutation itself. Instead of being originally set on x, the two slide-rules could have been set on y or any other letter.

To perform the permutation e means to interchange u and y, and simultaneously to interchange z and w. In the position of device No. 1 shown in Figure 1 this means a rotation about the vertical axis e by 180 degrees, thus changing from the front to the back of the device. The letter y will then be to the reader's left and u to his right. The letters z and w will also be interchanged (Fig. 2). Now let the operation a have been performed previous to e. Letter x is now to the left and so is the e line opposite that slot. Turning by 180 degrees about this new axis gives the arrangement of letters shown in parentheses on the slide-rule No. 2 (Fig. 2). A little consideration (or an actual trial if such a device is available) will show that the same result could be achieved in one operation, by starting with the position of No. 1 shown in Figure 1, by turning the device by

180 degrees about the f axis, opposite letter z. Thus, we have the relationship $ae=f$.

These examples show that the device enables one to demonstrate that the same permutation may be achieved in two steps, say on No. 1, and in one step on No. 2. All the possible combinations of the elements of the group are shown in its multiplication table below.

	I	a	b	c	d	e	f	g	h	i
I	I	a	b	c	d	e	f	g	h	i
d	d	I	a	b	c	i	e	f	g	h
c	c	d	I	a	b	h	i	e	f	g
b	b	c	d	I	a	g	h	i	e	f
a	a	b	c	d	I	f	g	h	i	e
e	e	i	h	g	f	I	d	c	b	a
f	f	e	i	h	g	a	I	d	c	b
g	g	f	e	i	h	b	a	I	d	c
h	h	g	f	e	i	c	b	a	I	d
i	i	h	g	f	e	d	c	b	a	I

The elements of the group are not commutative. It will be seen from the multiplication table that whereas $ae=f$, the product $ea=i$. The order of the operations must be strictly adhered to.

The device shown in Figure 1 does not give any more information about the group than the multiplication table. It only illustrates the group and its properties to the eye, and permits the group to be represented in the form of a piece of apparatus, thereby making it more tangible and "practical." This is of importance if groups are to be used in applied sciences.

The device brings out in particular the two fundamental properties of a group, namely

A. The product of any two (or the square of any one) of the elements is itself an element of the group.

B. The set contains the inverse of each element of the set. Thus, d and a are the inverse elements, and so are b and c, because they simply mean rotations of the disk clockwise and counter-clockwise respectively. The remaining elements, e, f, g, h, i are their own inverse elements. Each of these means a rotation by 180 degrees, so that the square of each element is a rotation by 360 degrees, that is, an identity.

While a fairly simple group has been chosen in this case, a person interested in practical applications of groups could build similar devices for much more complex groups, to illustrate some of their properties and theorems, including sub-groups. Groups represented by colored electric bulbs, with switches to permute the colors or the positions of the lamps, seem to offer a particularly attractive possibility, enabling a demonstrator before a large audience to visualize the fundamental properties of a group.

SOME PROPERTIES OF LINES ASSOCIATED WITH A HYPERBOLA

J. H. WEAVER, Ohio State University

Introduction. In his article "On four lines and their associated parabola" published in the October, 1937 issue of this MONTHLY, J. R. Musselman has defined a parabola by means of a complex variable and has determined some of the properties of its tangent lines. It is the purpose of the present note to define a hyperbola by means of a complex variable, to determine certain lines associated with each point on the hyperbola, and to show that these lines are tangent to a parabola, and hence have all the properties of the lines described in Musselman's paper.

1. Lines associated with a point on a hyperbola. Let the hyperbola be defined by the equation

$$z = \frac{2}{(t-1)(t-t_1)}, \tag{1}$$

where t and t_1 are complex variables with absolute value 1, and t_1 is fixed and $t_1 \neq 1$. The equation of the hyperbola (1) may also be written in the form

$$(\bar{z} + zt_1^2 - 2t_1)^2 = z\bar{z}t_1(1+t_1)^2. \tag{2}$$

From equation (2) the following facts are easily verified:

(a) One focus of the hyperbola is the point $z=0$.

(b) The center of the hyperbola is the point $z=-4/(t_1-1)^2$.

(c) The equation of the axis of the hyperbola is

$$\bar{z} - zt_1^2 = 0. \tag{3}$$

(d) The equations of the asymptotes of the hyperbola are

$$\bar{z} - zt_1 = \frac{-4t_1}{t_1-1}, \qquad \bar{z} - zt_1^3 = \frac{4t_1^2}{t_1-1}. \tag{4}$$

Equation (3) shows that the axis of the hyperbola passes through the origin and lies along the conjugate of t_1.

Equations (4) show that the angle between the asymptotes is the amplitude of t_1.

Consider the equation

$$z = \frac{2}{(t-1)(T-t_1)}, \tag{5}$$

where T is fixed, and $1 \neq T \neq t_1$. This represents a straight line which cuts the hyperbola at the point where $t=T$. Consider also the equation

$$z = \frac{2}{(T-1)(t-t_1)}. \tag{6}$$

This also represents a line which cuts the hyperbola at the point where $t=T$. Hence equations (5) and (6) determine two lines which are associated with a point on the hyperbola. These equations may be written in the following forms:

$$\bar{z} - zTt_1 = 2Tt_1/(T - t_1), \tag{5'}$$

$$\bar{z} - zTt_1^2 = 2Tt_1/(T - t_1). \tag{6'}$$

These equations show that the two lines associated with any point on the hyperbola (2) intersect at an angle which is one half the amplitude of t_1.

The line (6') intersects the hyperbola (2) in the points

$$z = \frac{2}{(T-1)(T-t_1)}, \qquad z = \frac{2t_1^2}{(T-t_1)(T-t_1^2)}. \tag{7}$$

At the latter point we have again two lines determined and their equations are

$$z = \frac{2t_1^2}{(T-t_1)(t-t_1^2)}, \tag{8}$$

$$z = \frac{2t_1^2}{(t-t_1)(T-t_1^2)}. \tag{9}$$

The line (8) is precisely the line (5') and the line (9) may be written in the form:

$$\bar{z} - zT = \frac{2Tt_1}{T-t_1^2}. \tag{9'}$$

Hence the choice of a point A_1 on the hyperbola (2) determines two lines m and m_1. If m_1 intersects the hyperbola again in A_2, then the point A_2 determines the lines m_1 and m_2. If the line m_2 intersects the hyperbola again in the point A_3, then A_3 determines m_2 and m_3, *etc.* If we continue this process we find the equation of the point A_n to be

$$z = \frac{2t_1^{2(n-1)}}{(T-t_1^{n-1})(T-t_1^n)}. \tag{10}$$

The point A_n determines the line m_n whose equation is

$$\bar{z} - zTt_1^{2-n} = \frac{2Tt_1}{T-t_1^n}. \tag{11}$$

If we assign to n positive or negative integral values or zero in equations (10) and (11) we may obtain the equations for as many points or lines as we may desire. If $t_1^n=1$ we have a series of just n points and a series of just n lines. Otherwise the number of points or of lines is infinite.

2. Properties of the associated lines. If in equation (11) we consider T as a parameter we may obtain the equation of the envelope of the family of lines

(11). It is easily shown to be

$$(\bar{z} - zt_1^2)^2 - 4t_1(\bar{z} + zt_1^2) + 4t_1^2 = 0. \tag{12}$$

This equation represents a parabola with the focus at $z=0$, and whose axis coincides with the axis of the hyperbola (2). We are now in a position to state the following theorem.

THEOREM: *The set of lines* (11) *associated with the hyperbola* (2) *are tangent to the parabola* (12) *and hence possess all the properties of the lines defined in Musselman's paper referred to above.*

In case $t_1^4=1$ we have a system of four points on an equilateral hyperbola. These four points determine an orthocentric system, that is, the fourth point of the system is the orthocenter of the triangle determined by the other three points.

If $t_1^2=1$ the hyperbola degenerates into a straight line counted twice, the lines determined at each point of the line are perpendicular to each other. These lines envelope a parabola and the double line is its directrix.

AN EASY METHOD FOR CONSTRUCTING POLYHEDRAL GROUP-PICTURES

H. S. M. COXETER, University of Toronto

Consider a finite number of planes, passing through a point O, and so arranged that the whole figure is symmetrical by reflection in each plane. The following six cases arise:

(a) one plane,

(b) n planes symmetrically disposed about a common line,

(c) the same, together with one plane perpendicular to the line,

(d) the six planes of symmetry of the regular tetrahedron,

(e) the nine planes of symmetry of the octahedron or of the cube,

(f) the fifteen planes of symmetry of the icosahedron or of the dodecahedron.

These planes divide a sphere with center O into a number of congruent (or symmetric) regions, each of which is (*a*) a hemisphere, or (*b*) a lune, or (*c*) a triangle of angles $\pi/2$, $\pi/2$, π/n, or (*d*)–(*f*) a triangle of angles $\pi/2$, $\pi/3$, π/n ($n=3, 4,$ or 5). The networks of triangles in cases (*d*)–(*f*) are conveniently represented on a plane by stereographic projection.* The great circles naturally remain circles and cut one another at the proper angles. In fact, such a figure may be described in purely two-dimensional terms, as consisting of a finite number of circles (and possibly straight lines) so arranged that the whole figure is symmetrical by inversion (or reflection) in each circle (or line). The purpose of the

* See, for instance, Burnside, Theory of Groups, frontispiece, and Klein, Lectures on the Icosahedron, frontispiece.

present note is to point out how easily the centers and radii of these circles may be found.

Let $PQRS$ (Fig. 1) be a square, center O, and let A, B, C be the mid-points of the sides of an equilateral triangle $A'B'C'$ (Fig. 2). For case (d), draw four equal circles, centers P, Q, R, S, radius PQ, and the two straight lines through O parallel to the sides of the square. Alternatively, draw three equal circles, centers A, B, C, radius AB, and the three medians AA', BB', CC'; let the first of these circles meet AA' at D.

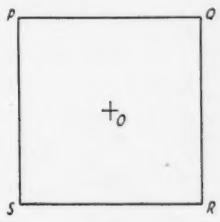


FIG. 1

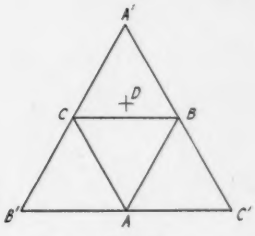


FIG. 2

For case (e), add either the circumcircle and diagonals of the square, or three equal circles, centers A', B', C', radius $B'D$.

Let the sides of the regular pentagon $VWXYZ$ (Fig. 3) be produced so as to form the star-pentagon $V'X'Z'W'Y'$. Let the sides of the equilateral triangle FGH (Fig. 4) be divided according to the "golden section" (or in "extreme and mean ratio"*), GH at I and J, HF at K and L, FG at M and N; and let NK,

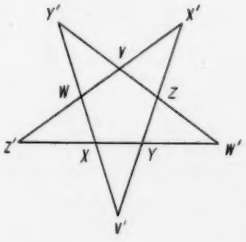


FIG. 3

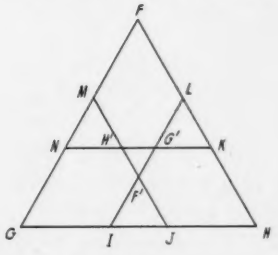


FIG. 4

JM, LI intersect in pairs to form the triangle $F'G'H'$. For case (f), draw five equal circles, centers V, W, X, Y, Z, radius VX' ($=VX$); another five equal circles, centers V', W', X', Y', Z', radius $V'W$ ($=V'W'$); and the five straight lines VV', WW', XX', YY', ZZ'. Alternatively, draw three equal circles, centers F', G', H', radius $F'I$; six equal circles, centers I, J, K, L, M, N, radius IG; three equal circles, centers F, G, H, radius FK; and the three straight lines FF', GG', HH'.

* So that FI is parallel to LJ.

When the circles (and lines) have been drawn, a very pleasing effect is obtained by coloring alternate curvilinear triangles. Two adjacent triangles (one white and one colored) then represent a fundamental region for the tetrahedral, octahedral, or icosahedral group. So far as the construction of group-pictures is concerned, the lines of Figures 1–4 are mere scaffolding; they should be drawn in pencil, and erased or ignored after serving their purpose. On the other hand, the same systems of lines are significant in a quite different connection.

Figure 5 is the "complete face" of the icosahedron, *i.e.*, it consists of the eighteen lines in which the plane of one face ($F'G'H'$) of the icosahedron is cut

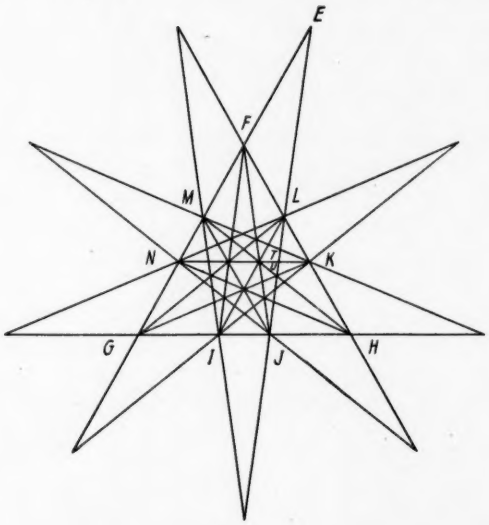


FIG. 5

by the planes of the other faces. (FGH is one of the twenty faces of the *great icosahedron*.*) This figure is evidently derivable from Figure 4 by adding six lines like $FH'I$ and six lines like IK. It is interesting to observe that the circles of the icosahedral group-picture pass through six points like E (where FG meets LJ), six points like T (where NK meets LJ), and six points like U (where GK meets HM).

Analogously, Figures 1 and 2 are "complete faces" of the cube and the octahedron, and Figure 3 becomes the complete face of the dodecahedron if we add the sides of the star-pentagon $VXZWY$. Perhaps some reader can throw further light on this strange fact, that the "complete face" of any regular polyhedron determines the centers of the circles which are stereographic projections, from

* It seems that Figure 5 was first drawn by E. Hess, "Über die zugleich gleicheckigen und gleichflächigen Polyeder," 1876. *Cf.* Coxeter, DuVal, Flather and Petrie, "The 59 icosahedra," University of Toronto Studies, Mathematical Series No. 6, 1938, p. 11.

the center of a face, of the great circles in which the inscribed sphere is cut by the planes of symmetry.

Perhaps, also, the same reader (or another) can explain the fact that the number of planes of symmetry (namely 6, 9, or 15) is three times the order of the commutator of the generators of the polyhedral group.

$$S^3 = T^2 = (ST)^n = 1, \qquad (n = 3, 4, \text{ or } 5).$$

TABLE OF DEFINITE INTEGRALS

GEORGE RUTLEDGE and R. D. DOUGLASS, Massachusetts Institute of Technology

The authors present herewith a systematic tabulation of definite integrals which they have found useful in studying the function*

$$(1) \qquad Q(x) = \frac{\sin \pi x}{\pi} \int_0^1 u^{x-1} \left(\frac{1+u}{1-u}\right)^x du.$$

If we write

$$(2) \qquad \int_0^1 u^{x-1} \left(\frac{1+u}{1-u}\right)^x du = \frac{1}{x} + \int_0^1 \frac{2\binom{x}{1} u^x + 2\binom{x}{3} u^{x+2} + \cdots}{(1-u)^x} du,$$

where $\binom{x}{k}$ is a binomial coefficient, there results an expansion in a series of Beta functions, whence

$$(3) \qquad \begin{aligned} Q(x) &= 1 + \binom{x}{1}\binom{x}{1} + \binom{x+1}{2}\binom{x}{2} + \binom{x+2}{3}\binom{x}{3} + \cdots \\ &= 1 + \frac{1}{1^2} x^2 - \frac{1}{2^2} x^2 \left(1 - \frac{x^2}{1^2}\right) + \frac{1}{3^2} x^2 \left(1 - \frac{x^2}{1^2}\right)\left(1 - \frac{x^2}{2^2}\right) - \cdots \\ &= 1 + \frac{\pi^2}{12} x^2 + A_4 x^4 + \cdots . \end{aligned}$$

The constant A_4 is defined by the series

$$(4) \qquad A_4 = \left(\frac{1}{1^2}\right)\frac{1}{2^2} - \left(\frac{1}{1^2} + \frac{1}{2^2}\right)\frac{1}{3^2} + \left(\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2}\right)\frac{1}{4^2} - \cdots ,$$

and has been computed from this definition. The value to thirteen decimals is

$$(5) \qquad A_4 = .16265\ 46673\ 974^+ = \frac{\pi^4}{598.87055\ 559^+}.$$

We include at the end of this paper a statement of the independent controls which ensure the accuracy of this computation.

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* This MONTHLY, vol. 43, 1936, pp. 27–32.

We first bring together in one group integrals which are evaluated in terms of π^4 and A_4, and refer to them for convenience as integrals of *order* 4. Integrals followed by the Roman numeral (I) have been evaluated in a previous paper.* N. Nielsen† has developed a general theory of the integrals

$$s_{n,p} = \frac{(-1)^{n+p-1}}{(n-1)!p!}\int_0^1 \frac{\log^{n-1} u}{u}\log^p(1-u)du, \tag{6}$$

$$\sigma_{n,p} = \frac{(-1)^{n-1}}{(n-1)!p!}\int_0^1 \frac{\log^{n-1} u}{u}\log^p(1+u)du, \tag{7}$$

and obtains $s_{2,2}$, $s_{3,1}$, $s_{1,3}$, $\sigma_{3,1}$ in terms of π^4, and $s_{2,1}$, $s_{1,2}$, $\sigma_{2,1}$, $\sigma_{1,2}$ in terms of s_3.‡ Concerning $\sigma_{1,3}$ and $\sigma_{2,2}$, which we have evaluated in terms of A_4, we have from §20, (1), p. 202 of Nielsen's monograph,

$$L_{3,1}\left(\frac{1}{2}\right) - L_{2,2}\left(\frac{1}{2}\right) = \frac{\pi^4}{120} + \sigma_{1,3} + \sigma_{2,2}, \tag{8}$$

and from §20, (5), p. 202,

$$L_{3,1}\left(\frac{1}{2}\right) - L_{2,2}\left(\frac{1}{2}\right) = \frac{\pi^4}{144} + 3\sigma_{1,3} + 2\sigma_{2,2}, \tag{9}$$

the L's being definite integrals the definition of which is immaterial to our present purpose. From (8) and (9) we have the important linear relation

$$2\sigma_{1,3} + \sigma_{2,2} = \frac{\pi^4}{720}. \tag{10}$$

Corresponding to our integrals 7. and 4., respectively, we have

$$\sigma_{1,3} = \frac{1}{4}A_4 - \frac{\pi^4}{5760}, \qquad \sigma_{2,2} = -\frac{1}{2}A_4 + \frac{\pi^4}{576}, \tag{11}$$

with (10) serving as an independent check.

Integrals followed by the Roman numeral (II) are evaluated in the present paper. The remaining integrals listed are evaluated by use of the simple power

* This MONTHLY, vol. 41, 1934, pp. 29–36.

† N. Nielsen, Der Eulersche Dilogarithmus, Monograph, 1909, Abhandlung der Kaiserlischen Leop.-Carol. Deutschen Akademie der Naturforscher, vol. 90, pp. 121–211. Referred to hereafter as Nielsen.

‡ Here s_3 denotes the sum of the reciprocals of the cubes of the integers. T. J. Stieltjes (Tables des valeurs des sommes $s_k = \sum_{n=1}^{\infty}(1/n^k)$, Acta Mathematica, vol. 10, 1887, p. 299) evaluated the sums of the series defined in the title of his paper, up to exponent 70, to 32 places of decimals. The values given by Stieltjes for $s_2(=\pi^2/6)$, s_3, and $s_4(=\pi^4/90)$ are as follows:

$s_2 = 1.64493\ 40668\ 48226\ 43647\ 24151\ 66646\ 03,$
$s_3 = 1.20205\ 69031\ 59594\ 28539\ 97381\ 61511\ 46,$
$s_4 = 1.08232\ 32337\ 11138\ 19151\ 60036\ 96541\ 18.$

series development of $\log(1+u)$ or $\log(1-u)$, this being indicated by the notation "(directly)," or are related to such directly evaluated integrals by means of a simple integration by parts, or by a simple combination. A notation follows each integral indicating its source.

INTEGRALS OF ORDER 4

1. $\displaystyle\int_0^1 \frac{\log^2 u}{u} \log(1+u)du = \frac{7\pi^4}{360}.$ (directly)

2. $\displaystyle\int_0^1 \frac{\log^2 u}{u} \log(1-u)du = -\frac{\pi^4}{45}.$ (directly)

3. $\displaystyle\int_0^1 \frac{\log^2 u}{u} \log\frac{1+u}{1-u}du = \frac{\pi^4}{24}.$ (from 1., 2.)

4. $\displaystyle\int_0^1 \frac{\log u}{u} \log^2(1+u)du = A_4 - \frac{\pi^4}{288}.$ (I)

5. $\displaystyle\int_0^1 \frac{\log u}{u} \log^2(1-u)du = -\frac{\pi^4}{180}.$ (I)

6. $\displaystyle\int_0^1 \frac{\log u}{u} \log^2\frac{1+u}{1-u}du = 2A_4 - \frac{\pi^4}{60}.$ (I)

7. $\displaystyle\int_0^1 \frac{1}{u} \log^3(1+u)du = \frac{3}{2}A_4 - \frac{\pi^4}{960}.$ (II)

8. $\displaystyle\int_0^1 \frac{1}{u} \log^3(1-u)du = -\frac{\pi^4}{15}.$ (from 2.)

9. $\displaystyle\int_0^1 \frac{1}{u} \log^3\frac{1+u}{1-u}du = \frac{\pi^4}{8}.$ (I)

As a check on the integrals 3., 6., and 9., we have the fact that the series (3) is the product of the two series

$$\frac{\sin \pi x}{\pi} = x - \frac{\pi^2}{6}x^3 + \frac{\pi^4}{120}x^5 - \cdots, \tag{12}$$

$$\int_0^1 u^{x-1}\left(\frac{1+u}{1-u}\right)^x du = \frac{1}{x} + \frac{\pi^2}{4}x + \left(A_4 + \frac{\pi^4}{30}\right)x^3 + \cdots, \tag{13}$$

and that consequently

$$\frac{1}{3!}\int_0^1 \frac{1}{u}\left\{\left(\log u + \log\frac{1+u}{1-u}\right)^3 - (\log u)^3\right\}du = A_4 + \frac{\pi^4}{30}. \tag{14}$$

The integral in (14) involves the integrals 3., 6., and 9., thus yielding the check

$$\frac{1}{3!}\left[3\left(\frac{\pi^4}{24}\right)+3\left(2A_4-\frac{\pi^4}{60}\right)+\frac{\pi^4}{8}\right]=A_4+\frac{\pi^4}{30}. \tag{15}$$

We next present a group of integrals which are evaluated in terms of s_3, the sum of the reciprocals of the cubes of the integers, and refer to these as integrals of *order* 3.

INTEGRALS OF ORDER 3

10. $\int_0^1 \frac{\log u}{u}\log(1+u)du=-\frac{3}{4}s_3.$ (directly)

11. $\int_0^1 \frac{\log u}{u}\log(1-u)du=s_3.$ (directly)

12. $\int_0^1 \frac{\log u}{u}\log\frac{1+u}{1-u}du=-\frac{7}{4}s_3.$ (from 10., 11.)

13. $\int_0^1 \frac{1}{u}\log^2(1+u)du=\frac{1}{4}s_3.$ (II)

14. $\int_0^1 \frac{1}{u}\log^2(1-u)=2s_3.$ (from 11.)

15. $\int_0^1 \frac{1}{u}\log^2\frac{1+u}{1-u}du=\frac{7}{2}s_3.$ (I)

Finally we append for the sake of completeness the three integrals of this type which are commonly found in standard collections, referring to them as integrals of *order* 2.

INTEGRALS OF ORDER 2

16. $\int_0^1 \frac{1}{u}\log(1+u)du=\frac{\pi^2}{12}.$ (directly)

17. $\int_0^1 \frac{1}{u}\log(1-u)du=-\frac{\pi^2}{6}.$ (directly)

18. $\int_0^1 \frac{1}{u}\log\frac{1+u}{1-u}du=\frac{\pi^2}{4}.$ (from 16., 17.)

We now derive the formulas 7. and 13. by integrating over a path in the complex plane consisting of the negative real-axis from 0 to -1 and the half of the unit circle in the upper half plane from -1 to $+1$. The integral 7. then becomes

$$\int_0^1 \frac{1}{t}\log^3(1-t)dt + \int_0^\pi \left(\frac{3}{2}\phi\log^2 2\cos\frac{\phi}{2} - \frac{1}{8}\phi^3\right)d\phi + i\left\{-\int_0^\pi \log^3 2\cos\frac{\phi}{2}\,d\phi + \frac{3}{4}\int_0^\pi \phi^2\log 2\cos\frac{\phi}{2}\,d\phi\right\}. \tag{16}$$

The values of the real integrals in (16) are derived by the authors in the paper previously referred to as (I). The integral 7. is consequently found to be equal to

$$\left(-\frac{\pi^4}{15}\right) + \left(\frac{3}{2}A_4 + \frac{93}{960}\pi^4\right) + \left(-\frac{\pi^4}{32}\right), \tag{17}$$

which reduces to the value given in our table.

With the same contour as a path of integration the integral 13. becomes

$$\int_0^1 \frac{1}{t}\log^2(1-t)dt + \int_0^\pi \phi\log 2\cos\frac{\phi}{2}\,d\phi + i\left\{-\int_0^\pi \log^2 2\cos\frac{\phi}{2}\,d\phi + \int_0^\pi \frac{\phi^2}{4}\,d\phi\right\}. \tag{18}$$

The first of the two real integrals is equal to $2s_3$, and the second* is equal to $-7s_3/4$ and consequently the integral 13. is equal to $s_3/4$.

For the purpose of obtaining a controlling check on our computation we exhibit a relation between the number A_4 and the series

$$a_4 = \frac{1}{1^4 2} + \frac{1}{2^4 2^2} + \frac{1}{3^4 2^3} + \frac{1}{4^4 2^4} + \frac{1}{5^4 2^5} + \cdots. \tag{19}$$

In terms of the notation (7) Nielsen† obtains

$$a_4 = \frac{\pi^4}{90} - \frac{7}{8}s_3\log 2 + \frac{\pi^2}{24}(\log 2)^2 - \frac{1}{24}(\log 2)^4 - \sigma_{1,3}. \tag{20}$$

Combining this with our value for $\sigma_{1,3}$ given in (11) we have

$$A_4 = -4a_4 + \frac{13\pi^4}{288} - \frac{7}{2}s_3\log 2 + \frac{\pi^2}{6}(\log 2)^2 - \frac{1}{6}(\log 2)^4. \tag{21}$$

The constant a_4 has been computed from the definition (19) by Burrau‡ as follows:

$$a_4 = .51747\quad 90616\quad 73899\quad 38633. \tag{22}$$

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* Nielsen, §21, (13), p. 206.

† Nielsen, §20, (2), p. 202.

‡ Burrau, Abhandlungen der Kopenhagener Akademie, (6), vol. 8, 1898, p. 443. Burrau's results are reprinted in Nielsen's monograph.

Therefore we have the following controlling check on our computation of A_4.

$$
\begin{aligned}
-4a_4 &= -2.06991\ 62466\ 95597\ 54^+ \\
+\frac{13\pi^4}{288} &= +4.39693\ 81369\ 51498\ 90^+ \\
-\frac{7}{2}s_3 \log 2 &= -2.91620\ 82365\ 41921\ 97^+ \\
+\frac{\pi^2}{6}(\log 2)^2 &= +0.79031\ 35301\ 13954\ 60^+ \\
-\frac{1}{6}(\log 2)^4 &= -0.03847\ 25164\ 30513\ 90^+ \\
\hline
A_4 &= .16265\ 46673\ 97420^+
\end{aligned}
$$

Our value (5) to thirteen decimals agrees with this result. Not only A_4, but also the coefficients $A_6, A_8, \cdots, A_{16}$ of the series (3) have been computed* (to eleven decimals). A further controlling check on the computed value of A_4 is then the relation

$$(23) \qquad Q(1) = 1 + \frac{\pi^2}{12} + A_4 + A_6 + A_8 + \cdots = 2.$$

The known values of $Q(\frac{1}{2})$, $Q(2)$, $Q(3)$, furnish additional checks on the computed values of the coefficients of the Maclaurin expansion of the function $Q(x)$.

We remark finally that, by virtue of the relation (21), the integrals 4., 6., and 7. may also be expressed (though much less simply) in terms of a_4, but that although A_4+4a_4 is expressed rationally (as a homogeneous polynomial of "degree" four) in terms of π, s_3, and log 2, the evidence seems to indicate that probably A_4 and a_4 are not individually so expressible.

* This MONTHLY, vol. 43, 1936, p. 30.

MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

This Department of the MONTHLY *has been created as an experiment to afford a place for the discussion of the place of mathematics in education. With this topic will naturally be associated other matters emphasizing the educational interests of those who teach mathematics. It is not intended to take up minute details of teaching technique. The columns are open to those who have thoughtful critical comment to make, be it favorable or adverse to the cause of mathematics. The success of this department obviously will depend upon the cooperation of the readers of the* MONTHLY. *Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

A PROGRAM FOR THE ASSOCIATION*

G. B. PRICE, University of Kansas

It behooves every organization to examine its activities from time to time to guard against having only a horse-and-buggy program in a streamlined airplane age. I shall consider the program of the Association and make certain suggestions for increasing the effectiveness of its work.

My subject has been considered most often in retiring presidential addresses of the Association, and those of Slaught [1],† Young [2], and Dresden [3] are especially significant. The job of keeping up-to-date is never finished, however, and I need not apologize for my much-used subject.

I propose a program for the Association based on the following two aims:

AIM I. *To foster adequate, high quality instruction in mathematics in high schools and in colleges and universities.*

AIM II. *To assimilate the results of mathematical research and to interpret mathematics to scientists and the public.*

These aims are actually less inclusive than those suggested by Young. I do not disapprove of the other features of his program, but I feel that the Association might concentrate its efforts on a smaller one dealing with the more pressing problems at the present time.

In any consideration of Aim I, it is necessary to take notice of the following fact in the present situation: For the year 1919–20 the enrollment in public high schools (colleges and universities) in this country was 2,200,000 [4, p. 1] (356,694 [5, p. 381]); in 1933-34 it had grown to 5,656,000 (919,176).

This rising tide of enrollments has been accompanied by a certain amount of destruction. In the first place, many students have been swept into high schools and colleges who are not able to master the traditional mathematics courses. As a result a clamor has arisen to remove mathematics from the curriculum, and mathematics has been and is being removed from the curriculum in both high schools and colleges. Thus, *there is no longer adequate instruction in mathematics* [4, pp. 17–18], [6, pp. 98–99]. In the second place, the influx of poor students into high schools has led to a lowering of the standards for promotion

* Condensed from a paper read before the Kansas Section of the Association, April 2, 1938.

† Numbers in square brackets refer to the references at the end.

and graduation. Formerly the elementary and high schools insisted upon the possession by the pupil of certain definite, clearly defined skills, knowledges, and habits as the basis of promotion of the pupil from grade to grade in the elementary school, and from the elementary school to high school, and as the basis of graduation from high school. This is no longer the prevailing practice; in most cases pupils are promoted through the elementary and high school on the basis of age and time spent in a grade, and they are graduated from high school on the same basis ([4] and [6] are my authority for this statement). The final blow to the efforts of many state universities to maintain standards of instruction and achievement is a state law requiring them to accept any graduate of a recognized high school. Thus, *there is no longer high quality instruction in mathematics (and of course in other subjects too)*.

The acute problem of the mathematics departments in many of our colleges and universities arises from the presence in the freshman classes of students of widely varying ability and preparation. If the instructor adjusts his instruction to the average of the class, both extremes suffer. In many cases 30 per cent or more of a class fail to pass the course, but it is being recognized now that the greatest loss is suffered by those at the top of the class. Instruction designed for the bottom or middle section of a nonhomogeneous class fails to develop the students of real ability into those scholars and useful citizens whose production only can justify the sums being spent on education. In this connection Tildsley [4, pp. 17–18] says:

> "Have we any reason to believe that with the best 75 per cent of our pupils, a unitary-planned curriculum of English, foreign language, mathematics, science, social science, music, art, and health education would not have resulted in an even larger degree of educational well-being for the boys and girls than the free-for-all curriculum of the average American city high school of today?
>
> "In our floundering, unintelligent, spasmodic efforts to find something which this minority of 25 per cent would be willing—and in some cases be able—to do, we have sacrificed unnecessarily the well-being of the 75 per cent majority. We have not merely given them an education of greatly impaired value, but, of far graver import, we have permanently dwarfed the possibilities of many of these boys and girls through fostering in them habits and attitudes which are destructive of growth. At the same time we have failed to grow in them the knowledges, understandings, skills, qualities, habits, attitudes which should be the outcomes of the educational process—an education which they were capable of receiving in varying degrees and which was within our power to give them, even under the present unfavorable conditions."

Johnston [6, pp. 86–87] says:

> "The lives of scores of naturally competent boys and girls are wrecked every year under our eyes because of their association with incompetents in high school. Others manage to get on with great difficulty because what they

got out of their high school subjects gives them such poor foundation for further building in the same subjects. The association of competent students with incompetents is now being continued in the freshman year of college."

I have described briefly the situation. It is not my purpose to suggest a solution or remedy, but rather to insist that there is a problem in which we as mathematicians are interested, and toward whose solution the Association can make contributions. Remedies are being proposed already [4, pp. 29–30, 54, *etc.*], [6, pp. 93–113], [7], [17]. Probably within the next ten years the present difficulties will be resolved, but in the meantime something of an emergency exists. When I go to meetings of the Association, I find various ones discussing these problems as well as the methods being tried by several universities for solving them. I have not found them receiving the attention which, it seems to me, they deserve on the programs of the Association.

To accomplish Aim I, I suggest the following:

PROGRAM I. *Conferences and symposia at the national meetings designed to achieve Aim I; a department in the* MONTHLY *to treat these matters.**

I am not advocating that the Association consider problems of pedagogy and methods of instruction. But in any effort to accomplish Aim I, many problems of administration are met which the Association could study profitably. In a recent note Langer [18] discussed an administrative problem and its solution at Wisconsin. The remedies which are being proposed for our present troubles deal entirely with matters of administration [4], [6], [7], [17]. Our problems concern our administrators first of all, therefore, but the advice of the mathematicians will certainly be sought. Let them take the initiative in matters which concern them so vitally.

I do not want to leave the impression that the Association has done nothing. Although it has not taken up the problem which I have described, it recently studied another one through its Commission on the Training and Utilization of Advanced Students in Mathematics [20], and the Joint Commission on the Place of Mathematics in the Secondary Schools [8] is making an extensive investigation at the present time.

If the work of the Association is to be successful, however, the major emphasis must be placed on *mathematics*. And here I turn to the consideration of a program designed to accomplish Aim II as set forth above.

Before I outline my second suggestion, let me review certain facts in the history of the Association and in the present situation.

In the first place, the need for expository addresses and publications has been recognized throughout the history of the Association. It is probably true that no policy is more firmly established than that of fostering mathematical exposition.

* *Note by the Editor.* The Department of Mathematical Education was being planned by the editors without the knowledge of Dr. Price at the time this paper was written. It offers an opportunity for writers to follow Dr. Price's suggestions. E.J.M.

I proceed to point out some of the outstanding expressions and results of this policy. (a) Almost from its founding the Association has given a subvention to the *Annals of Mathematics* [9]. Originally, suitable expository papers were to be published up to 100 pages a year, but in recent years this restriction has been dropped by mutual agreement. At the present time the *Duke Mathematical Journal* likewise receives a subvention. (b) Beginning in 1917 and for many years thereafter the Association studied a proposal for the publication of a mathematical dictionary [10]. The prohibitive cost of the undertaking prevented its being carried out. (c) The five Carus Monographs are the result of one of the Association's most successful efforts to secure mathematical exposition [11]. (d) Many speakers and writers throughout the history of the Association have emphasized the need and desirability of expository addresses and publications, including large treatises and encyclopedias; as an example I mention an address by Murnaghan [12]. (e) The Chauvenet Prize for Mathematical Exposition [13], although an effort in the right direction, apparently has not been very effective in securing expository articles. (f) Several editors, however, have put forth special efforts to secure the publication of such articles in the MONTHLY (see especially [15], [16], [21] and current numbers of the MONTHLY).

In the first place, therefore, I say the need for exposition has been recognized from the beginning of the Association, and various steps have been taken to foster it. In the second place, however, the efforts to stimulate exposition, although effective to some extent in the beginning, are not effective and adequate at the present time. In the third place, the efforts put forth have been more and more infrequent.

As a result of these facts, I wish to suggest a second activity for the Association, designed to accomplish Aim II, at least partially:

PROGRAM II. *Herbert Ellsworth Slaught Lectures.*

I proceed to explain my idea of the nature and operation of these lectures.

A major portion of the program of one, or of each one, of the two annual meetings should be devoted to a study of one general topic. The Herbert Ellsworth Slaught Lectures, given by a person chosen and designated the Slaught lecturer for that meeting, should constitute the major portion of the program. These lectures should be expository in character and should be an introduction to, or survey of, some field of mathematics rather than a summary of latest results of research as in the case of the Colloquium Lectures of the Society. Provision should be made for the immediate publication of these lectures; it might even be desirable to have the printed lectures available when the lectures are delivered. These volumes, to be known as the Herbert Ellsworth Slaught Lectures of the Mathematical Association of America, should be complete with bibliographies, glossaries of terms, and lists of symbols in a form designed to secure the maximum usefulness to a non-specialist in the field.

It is suggested that these lectures be named after and in honor of one of those most responsible for the founding of the Association, and one who gave to it freely of his time and energy for many years. The Association would honor itself in so honoring him.

I would like to add the following comments:

(1) There has been at least one meeting of the Association at which the entire program was devoted to one general topic [14], and the records indicate that it was highly successful. On the typical program there are five or six papers, each on a different subject. In spite of the fact that they are excellent papers, the result is apt to be a feeling of discursiveness and haphazardness rather than completeness and thoroughness. At the present time the *conference*, devoted to an intensive study of one subject, is in vogue.

In this connection mention should be made of the symposia organized by the Section on Medical Sciences (N) of the American Association for the Advancement of Science. In a recent note in *Science* [19] the situation is described which led to the decision to organize the program of each meeting in the form of a symposium, and the success of the new type of program is emphasized. There is every reason to believe that the programs of the Association could be improved in the same way by devoting each one to a thorough study of one subject. In many cases it would be desirable to arrange the program as a symposium with two, or even three, Slaught Lecturers.

(2) A program built around a symposium or series of lectures should be effective also in improving the meetings of the sections of the Association.

(3) A program with the Slaught Lectures as the central feature should make a strong appeal to our best undergraduate majors in mathematics and to our beginning graduate students.

(4) There are many fields of mathematics at the present time that are well suited for study in the Slaught Lectures. Many would like an introduction to statistics, an active field of research. Just now there is a pressing need for exposition in algebra. On every hand we hear of fields, rings, lattices, Boolean algebras as well as many other varieties, groups, isomorphisms, *etc.* Many additional subjects for Slaught Lectures could be obtained from the fields of topology and abstract spaces. One or more programs could be devoted to a study of invariants. There has been much recent work on logic and the foundations of mathematics. A program could be given over to a study of the work of the intuitionists and the formalists. In fact, subjects for Slaught Lectures could be found in each of the numerous fields of research.

(5) Under ideal conditions the Slaught Lectures would be published as a monograph, a book of whatever size might be required to cover the subject adequately. The Lectures, as delivered, would vary between three and six in number and would not cover all the material in the published Lectures. The delivered Lectures probably would not be able to do more than outline the subject, *i.e.*, state the fundamental problems and the nature of the results obtained. The delivered Lectures should be designed to serve as the best possible introduction to a further study of the subject in the published Lectures and in the original sources.

(6) If funds for more adequate publication are not available, I suggest that a modest beginning might be made by publishing the Slaught Lectures as a supplement to a number of the MONTHLY. The Committee on Mathematical

Dictionary recommended that a dictionary should not be attempted unless a large and thoroughly complete one could be put out. I suggest that a beginning be made with the Slaught Lectures even if the accompanying publication is forced to begin on a small scale.

(7) The honor of being named a Slaught Lecturer should be sufficient to attract able speakers, and should be a more effective inducement to exposition than any means now being employed.

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QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the MONTHLY *is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.*

TRANSFORMATIONS OF EQUATIONS

E. J. FINAN, Catholic University of America

The purpose of this paper is to present a different method, which seemingly is superior to the ones now ordinarily employed, of applying certain transformations to polynomial equations. In 1911 A. Chatelet published an easy method of applying a rational transformation to the roots of an equation by means of integral matrices. This may be described briefly as follows: Let it be required to find the equation whose roots are the rational function ϕ of the roots of $f(x)=0$. Let B be the companion matrix of $f(x)=0$. Then $|yI-\phi(B)|=0$ is the required equation.*

Another problem, which occurs frequently in algebraic number theory, may be solved with integral matrices by a method somewhat similar to the above. Let x satisfy an equation $g(x)=0$ of degree m with coefficients in $F(\theta)$. It is known x satisfies an equation

$$h(x) \equiv g(x)g'(x) \cdots g^{(n-1)}(x) = 0$$

with coefficients in F. The calculation of $h(x)$ by means of symmetric functions is not easy. However the following theorem furnishes an entirely practicable rational method of obtaining the required equation. Also as the last illustration shows one may perform certain irrational transformations on the roots by this method.

THEOREM. *Let $F(\theta)$ be an algebraic extension of a field F, θ being a root of the irreducible equation $f(x)=0$, of degree n with coefficients in F. Let C be the companion matrix of $f(x)=0$. Let $A=(a_{rs}(\theta))$ be a square matrix of order m with elements in $F(\theta)$, $a_{rs}(\theta)=\sum_{i=1}^{n}\alpha_{irs}\theta^{i-1}$. Define $C_{rs}=\sum_{i=1}^{n}\alpha_{irs}C^{i-1}$. Let P be the mn rowed compound matrix (C_{rs}) with elements in F. Then the characteristic function of P is the product of the characteristic function of A by its conjugates.*†

In order to obtain the $h(x)$ mentioned in the introduction we take the companion matrix of $g(x)=0$ as the matrix A of the Theorem. Then $h(x)\equiv|P-xI|$. This method is quite superior computationally. For example let

$$g(x) \equiv x^2 - (3+\sqrt[3]{2})x + 2\sqrt[3]{2} - 2\sqrt[3]{4} = 0$$

* For a more detailed description of the process together with definitions of the terms see C. C. MacDuffee, The Theory of Matrices, pp. 20 and 24.

† This theorem was read before the American Mathematical Society on October 26, 1935. Since then my attention has been called to the fact that a similar result was published by Williamson in August 1931 in the Bulletin of the American Mathematical Society. Hence no proof will be offered here.

be an equation with coefficients in $R(\sqrt[3]{2})$ where R is the rational field. We desire an equation with rational coefficients which contains among its roots those of $g(x)=0$. Then the companion matrices of $g(x)=0$ and $x^3=2$ are respectively

$$A=\begin{Vmatrix} 0 & 1 \\ -2\sqrt[3]{2}+2\sqrt[3]{4} & 3+\sqrt[3]{2} \end{Vmatrix}, \qquad C=\begin{Vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \end{Vmatrix}$$

and according to the theorem

$$h(x)=\begin{vmatrix} -x & 0 & 0 & 1 & 0 & 0 \\ 0 & -x & 0 & 0 & 1 & 0 \\ 0 & 0 & -x & 0 & 0 & 1 \\ 0 & -2 & 2 & 3-x & 1 & 0 \\ 4 & 0 & -2 & 0 & 3-x & 1 \\ -4 & 4 & 0 & 2 & 0 & 3-x \end{vmatrix}=0$$

is satisfied by x.

Although it is often not the most direct this theorem furnishes a method for applying an algebraic irrational transformation to the roots of an equation. For example let it be required to find an equation whose roots are the cube roots of those of $f(x)=0$ which is satisfied by x. Then $Z-\sqrt[3]{x}=0$ from which we get $Z^3-x=0$ which is an equation of the same type as the one appearing in the problem just solved. Naturally more complicated algebraic transformations may be performed in the same way.

Some very interesting additional theorems on applications of the theory of matrices to equations are given in MacDuffee's *The Theory of Matrices* on pages 83 and 84.

NOTE ON THE DIVISION OF LABOR

FRANK MORLEY

[*Note:* Among some papers left behind by the late Professor Frank Morley, were certain ones marked for sending to me. One of these is the present little note, which appears to have been written in 1937 since with the MS. there is a letter of this year, from an old friend, referring to the manuscript's return. In this connection a problem which Morley proposed for solution forty years earlier may be recalled (*Educational Times Reprint*, vol. 67, 1897, p. 80): "*A* can dig a field of potatoes in one hour, and pick the roots from the plants in two hours. *B* can dig that field in three hours and pick it in four. *C* can dig in five hours and pick in six. What is the least time in which all working together can dig and pick the field? Is a general formula known or obtainable for the least time in which m men can do n things, where no two things can be profitably done simultaneously by the same man?" Solutions by various contributors but

not a general formula, and a discussion of the solutions, are given. The following note suggests that the discussion was by Morley.

Brown University, May 1938. R. C. ARCHIBALD]

Suppose that there are three men and two jobs, and that the time in which each man can do each job is given. What is the least time required to complete the jobs?

Let the times for the first man be a_1 and b_1 and the fractions which he does be x_1 and y_1. Similarly for the other men. Then

$$x_1 + x_2 + x_3 = 1$$
$$y_1 + y_2 + y_3 = 1$$

and the time t taken is

$$t = a_1x_1 + b_1y_1 = a_2x_2 + b_2y_2 = a_3x_3 + b_3y_3.$$

If we eliminate x_i then

$$t\sum 1/a_1 = 1 + \sum b_1y_1/a_1 \tag{1}$$

whence

$$(a_1x_1 + b_1y_1)\sum 1/a_1 = \sum (a_1 + b_1)y_1/a_1.$$

This gives a linear transformation between x_1, x_2, x_3 and y_1, y_2, y_3 and taking them as barycentric coördinates of two points x and y of a plane there is a collineation, say

$$y = xC.$$

A fixed point of this collineation is given by

$$x_1(a_1 + b_1) = x_2(a_2 + b_2) = x_3(a_3 + b_3).$$

The other two are at infinity (which here is a line).

Now in the problem the x_i and y_i are non-negative, so that the points x and y are on or in the triangle of reference T.

But if x be on or in T, then y is on or in TC, the transform of T.

Hence y is on or in the polygonal region common to T and TC. There must be such a region for the fixed point is in both.

Now (1) shows that the time depends on the distance from y to the line

$$\sum b_1y_1/a_1 = 0, \tag{2}$$

a line which is outside the region.

The minimum time is then when y is at the vertex of the polygon which is nearest to (2).

The problem of m men doing two jobs is solved in the same manner by a collineation in a higher space.

THE TRILINEAR POLAR OF THE THIRD BROCARD POINT

O. J. RAMLER, Catholic University of America

In his interesting article *A generalization of the circles of Apollonius* in the January 1938 issue of this MONTHLY, p. 17, J. H. Weaver remarks that he had not identified equation (20) with any of the known lines of the triangle. The following considerations may be of interest.

From the way in which the circles whose equations are (4) and (10) pp. 18, 19, are defined, it follows at once that the circles represent three pairs of equal circles, and the centers of each pair lie on a side of the triangle symmetrically situated with regard to the midpoint of that side. Hence lines of centers whose equations are (6) and (11) cut the sides of the triangle in points symmetrically situated with respect to the midpoints of the respective sides. In areal coördinates two lines related in this manner will have equations

$$a_1x_1 + a_2x_2 + a_3x_3 = 0 \qquad \text{and} \qquad a_1^{-1}x_1 + a_2^{-1}x_2 + a_3^{-1}x_3 = 0.$$

The trilinear poles of these lines are points whose coördinates are (a_1, a_2, a_3) and $(a_1^{-1}, a_2^{-1}, a_3^{-1})$ respectively. But these points are isotomic conjugates, and hence we have the

THEOREM: *If two lines are isotomically related, so also are their trilinear poles.*

Now the trilinear pole of the Lemoine axis is the symmedian point, and the isotomic conjugate of the symmedian point is the center of perspective of the fundamental triangle and its first Brocard triangle (Johnson, *Modern Geometry*, §462). This point is sometimes called the third Brocard point of the triangle. Hence we can identify equation (20) of Professor Weaver's article as the trilinear polar of the third Brocard point of the triangle.

The line is also the line of flexes of the point cubic which is the locus of cusps of three-cusped hypocycloids inscribed to the fundamental triangle (*Cf.*, H. A. Converse, *On a system of hypocycloids of class three inscribed to a given 3-line and some curves connected with it*, Annals of Mathematics, 2nd series, vol. 5, 1904, p. 124, equation (10)).

RECENT PUBLICATIONS

EDITED BY TOMLINSON FORT, Lehigh University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

Das Parallelenproblem und seine Lösung. Eine Einführung in die hyperbolische nichteuklidische Geometrie. By Max Zacharias. Leipzig and Berlin, B. G. Teubner, 1937. 44 pages. RM 3.60.

This little book appears in the mathematical-physical library under the editorship of W. Lietzmann and A. Witting. Since it is intended for the widest possible circle of readers, it does not aim at scientific completeness, suppressing the more difficult proofs and referring the interested reader to several more pretentious treatises.

After a brief historical introduction, there follow four sections: (1) parallels, parallel angle, and parallel distance; (2) angle sum and area of the triangle; (3) equidistant curves; and (4) circles, to which is appended a discussion of the question whether our universe is euclidean or non-euclidean and a few remarks on elliptic geometry. The treatment is elementary throughout and well suited to the public for which it is written. It is closely printed with narrow margins so that the author is able in the few pages to exhibit the salient facts against an appropriate historical background.

R. M. WINGER

Historical Development of the Graphical Representation of Statistical Data. By H. G. Funkhouser. Osiris, Studies on the History and Philosophy of Science, and on the History of Learning and Culture, vol. III, Part 1, pp. 269–404. Bruges, Belgium, The Saint Catherine Press Ltd., 1937.

The contribution of this study is the summarization and collection for reference of the scattered material on the graphical representation of statistical data published during the centuries of its origination and early development. Some very limited use of graphs occurred during the tenth or eleventh century, but the first systematic and extensive use of various types of graphical representation of statistical data was published by an Englishman, William Playfair, toward the end of the eighteenth and during the first years of the nineteenth century. Mr. Funkhouser calls Playfair the "apparent inventor of statistical graphs."

This work started by Playfair appears to have attracted little attention outside of France; hence, the development of graphs during the next fifty years is associated with such names as Quetelet, Guerry, and Minard.

Then followed a period, 1860–1900, which Funkhouser entitled, "The age of enthusiasm in graphics" and there is no doubt that this was an age of rapid development. Graphs received a large share of the discussion at international statistical conferences, and serious attempts were made to classify graphic forms.

E. J. Morey's classic work of more than 700 pages appeared and there was a rather extensive use of graphs in official publications.

Even with all this development, the reviewer has difficulty conceiving more enthusiasm for graphics than is exhibited at the present time, especially for the picture statistics for popular consumption.

The last chapter, entitled "Recent developments in graphic methods, a critical attitude in their use" probably will be of most interest to those readers who are not serious students of history. The following topics are discussed in order: Picture statistics, three dimensional graphs, graphs in the school room, the use of grids, curves characterized by special names, a glossary of graphic terms, and nomography.

The text proper is followed by a 20-page bibliography which adds materially to the value of the book for reference purposes.

It is apparent that Mr. Funkhouser has made a very careful study of the history of the development of the graphical method. The length of time this method has been in use may surprise many readers, particularly those who did not receive extensive training in mathematics. Mathematicians are already familiar, in a general way, with much of the early development of graphs because of its relation to the work of some of the famous mathematicians of the fifteenth, sixteenth, and seventeenth centuries.

The text is interspersed with illustrations of outstanding charts which appeared at different stages in the development of graphical presentation. Some of these strikingly compare with, and others sharply contrast with, present-day charts. In places, the discussion of the history does not make particularly interesting reading, but this probably cannot be avoided in a work of this nature. This book probably is the only one which gives a comprehensive view of the origin and development of certain graphic forms and the general growth of the graphic method of presentation, and no doubt the reading of it will enhance almost any interested reader's perspective in regard to present-day practices.

J. M. THOMPSON

An Introduction to Business Statistics. By J. R. Stockton. Boston, D. C. Heath and Company, 1938. 378 pages. $3.00.

Mr. Stockton is of the opinion that the first course in business statistics should emphasize the practical value of statistical analysis to business men. Hence, the whole work is written to appeal more to those people whose minds and inclinations turn in the direction of business affairs than to those who plan to go into the field of statistics as a professional career. It is limited to the more simple and common statistical methods, and the author places much emphasis upon the idea that statistical analysis is very useful in the management of business enterprise. The entire tone of the book reflects what must have been a close association on the part of the author at some time with the use of statistics in the solution of definite problems in business.

The chapter titles indicate how the subject is covered, the scope of which is

similar to that of other elementary books on business statistics: "Use of Statistical Data," "Collecting Original External Data," "Organizing and Summarizing Data," "Presenting Statistical Data," "Charts," "Averages," "Dispersion," "Time Series," "Business Barometers," "Methods of Constructing Business Indexes," "Correlation," and "Measurement of Correlation." The appendices include the problems to be worked by the students, a list of statistical formulas, a discussion of significant figures, a description of machines used in statistical work, and tables to be used in arithmetical calculations.

The author has devoted more than the usual amount of space to a discussion of time series and what he calls business barometers. He presents an interesting discussion of a few of the more outstanding indexes of various forms of business activity, and he points out a few important sources which should give the business executive access to much of the available data on general business activity. He gives several very good practical suggestions and illustrations for makers of questionnaires in his treatment of the collection of data which depends on the voluntary cooperation of many busy people. His discussion of charts and maps is exceptionally clear, concise, and instructive.

In a few places there seem to be accidental slips, but perhaps these indicate a lack of experience with mathematics. On page 118 the text reads "The geometric mean is zero if any value of the variable is zero, and it may become imaginary if negative values occur. Except for these cases the value is always determinate and is rigidly defined." The geometric mean is just as determinate and rigidly defined in these cases as in any other; however, it is not *useful* for ordinary statistical purposes if it is zero or imaginary. On page 228 it is stated that dividing a certain equation by 12 gives an entirely different equation. Dividing an equation by 12, of course, leaves the equation unaltered. The statement should have indicated that dividing the constant term and the coefficient of X by 12 would give the desired result. The discussion of least squares on page 306 says that this least squares line gives the line of best fit, but it doesn't explain the significance of the term "best fit" in this connection.

The book entirely omits all derivations and anything bordering on algebraic manipulations. It is easy to read, and at times seems a little long and wordy, but perhaps that is necessary for the group of students for which it is written.

J. M. THOMPSON

Nécessaire mathématique P.C.B., S.P.C.N., By Maurice Curie and Maurice Prost. Paris, Hermann et Cie., 1937. 112 pages. 20 fr.

This book, no. 502 of *Actualités scientifiques et industrielles*, is designed to meet the mathematical requirements for certain certificates of studies in France. The authors wish to be convincing rather than rigorous, and to appeal to the intuition of readers having limited mathematical ability. Throughout, the emphasis is on physical applications. In subject matter, the topics are such as one might expect to cover in two or three years of the required mathematics for an engineering course in America.

S. S. CAIRNS

Mathematical Analysis for Economists. By R. G. D. Allen. London, Macmillan and Co., 1938. 16+548 pages. $9.00.

Mr. Allen has performed a great service both to mathematics and to economics by writing this book. It is most competently written from the mathematical point of view. The influence of the school of G. H. Hardy (of Cambridge) is very noticeable. Primarily an introduction to mathematical analysis for economists it should also be of interest to the mathematician for several reasons.

It is probably not more than a historical accident that the development of mathematics is very closely connected with the evolution of physics and astronomy. But this close connection had a great deal of influence upon the history of mathematics. Whereas there is no necessary relationship, it is safe to say that we would have another kind of mathematics nowadays if its development had been closely linked with the history of, say, biology or the social sciences. A certain one-sidedness results from the connection with physics which, perhaps, was not always to the advantage of the development of mathematics since it led to the neglecting of certain problems which are interesting from a purely mathematical point of view. Any mathematician who feels that he should correct this bias and take into account some of the more modern applications of mathematics to economics will find rich material in Mr. Allen's book.

One of the important contributions of his book is found in the examples taken from economics. Would it not be interesting for any mathematician to supplement the well known and worn out examples from physics by some economic applications? To quote some, the problem of the location of industries provides an interesting instance of the application of analytical geometry (pp. 80 f.). Demand curves and demand functions (pp. 108 ff.), total-revenue functions and curves (pp. 116 ff.), cost functions and curves (pp. 117 ff.), and indifference curves (pp. 124 ff.) show the application of the concept of functions to economic problems. Marginal revenue and marginal cost (pp. 152 ff.) illustrate derivatives. Monopoly problems show the application of the theory of maxima (pp. 196 ff.). The same is true for problems of duopoly (pp. 200 ff.). Problems in capital and interest (pp. 248 ff.) and the concept of "elasticity" (in the economic sense) (pp. 252 ff.) illustrate the idea of logarithmic derivation. The production function (pp. 284 ff.) shows an interesting application of the theory of functions of several variables. The same is true of the utility function (pp. 289 ff.). Partial derivatives are illustrated by demand functions depending upon the prices of several commodities (pp. 310 ff.) and by the theory of marginal productivity (pp. 312 ff.). Euler's theorem is important for the theory of production (pp. 320 ff.). Substitution in production and utility theory (pp. 340 ff.) shows the idea of differentials. Problems of joint production (pp. 359 ff.) and of production, capital, and interest (pp. 362 ff.) illustrate the application of the theory of maxima and minima in several variables. The demand for factors of production (pp. 369 ff.) and for consumers' goods (pp. 374 ff.) are examples in the theory of relative maxima and minima. Capital values (pp. 401 ff.) and a problem of durable capital goods (pp. 404 ff.) are applications of

integration. "Dynamic" demand and supply functions (pp. 434 ff.) illustrate the theory of linear differential equations. The problem of the stability of demand for factors of production (pp. 502 ff.) shows the application of the theory of maxima and minima of several variables under restraining conditions. The same is true for the theory of demand for a consumers' good (pp. 509 ff.). Finally, a "dynamic" problem of monopoly (pp. 533 ff.) and the theory of saving (pp. 536 ff.) illustrate the calculus of variations.

This should give enough material to a teacher of mathematics who wants to supplement a course in calculus or advanced calculus by some economic examples. But Mr. Allen's book also provides an excellent text for a graduate course in mathematical economics. The applications given in his book cover almost the whole field and it would be only necessary to supplement them in a few points by another text, *e.g.*, G. C. Evans *Mathematical Introduction to Economics*. A course of this nature should be of equal interest to graduate students in mathematics with interest in social problems and to graduate students in economics with some mathematical knowledge.

Finally, Mr. Allen's book can be recommended for study to mathematicians who interest themselves in the social sciences. They can be assured that it is most competently written from the point of view of the economist. Its author is one of the most prominent of the younger English economists. Although some of the points are naturally controversial, this book will tend to give the mathematician much of the necessary background for the understanding of contemporary economic science and an idea of the treatment of economic problems at its best.

GERHARD TINTNER

A Text-Book of Convergence. By W. L. Ferrar. London and New York, Oxford University Press, 1938. 7+192 pages. $3.50.

In this book we have an excellent introduction to the theory of convergence of infinite processes. It is quite elementary and leisurely in treatment, and includes most of the fundamental topics for general use and is properly rigorous, but, as the author says, "excludes topics appropriate to post-graduate or to highly specialized courses of study." The chapters are short, with only a few fundamental related ideas in each chapter. One or more lists of exercises are included in each chapter; the early exercise lists are fairly easy and elementary, and the majority of the exercises are "reasonably straightforward." Numerous details throughout the book show skillful ways of handling the tricks of technique. For most of the book, real numbers only are used; complex numbers are used occasionally, particularly in connection with power series.

The book is divided into two parts: Part I, "A First Course in the Theory of Sequences and Series," containing seven chapters, and Part II, "The General Theory of Infinite Series," containing thirteen chapters. Part I opens with a chapter called "Preliminary Discussion," giving simple examples of convergent and divergent series. The next two chapters deal with infinite sequences: con-

vergence, properties of convergent sequences, divergence, bounds of sequences, monotonic sequences. To avoid the necessity of a detailed discussion of the rigorous theory of irrational numbers at the beginning of the course, the treatment is based on two fundamental assumptions. These assumptions refer to the upper bound of a bounded set and to the representation of an irrational number as the limit of a sequence of rational numbers; they are proved in the Appendix. Also the Dedekind theory of real numbers is treated in an Appendix.

Ferrar's use of the term "non-convergent" for sequences which are neither convergent nor properly divergent (page 12) does not seem quite appropriate. A notation or shorthand used systematically throughout the book for the definition of limit is the following: $x_n \to l$ as $n \to \infty$ if

$$\epsilon > 0;\ \exists N \cdot \left| x_n - l \right| < \epsilon \text{ when } n \geqq N.$$

The next four chapters are devoted to "Series of Positive Terms," "The Comparison Test; the Ratio Test," "Theorems on Limits," and "Alternating Series." There is no satisfactory explicit definition of the meaning of the term "infinite series" (such as is given by Knopp, for example). The author uses a convenient device with the ϵ-technique, by making one or more of the expressions early in a proof $< k\epsilon$ and then later making a choice of k so as to make the final result $< \epsilon$. Part I ends with a collection of "Miscellaneous Examples on Chapters I–VII."

Part II opens with a chapter on "The General Convergence Principle"; there follow chapters on "Absolute and Non-Absolute Convergence," "The Product of Two Series," "Uniform Convergence," "Binomial, Logarithmic, Exponential Expansions," "Power Series," "The Integral Test," "The Order Notation," "Tannery's Theorem," "Double Series," "Infinite Products," "Theorems on Limits: Cesàro Sums," "Fourier Series," and another collection of "Miscellaneous Examples."

The discussion of change of order of terms in a series is placed at the beginning of the chapter on multiplication of series. Cauchy's and Abel's theorems on multiplication of series are given but Mertens' theorem is not even mentioned. The chapter on order notation is motivated by giving Gauss's test of convergence in ordinary notation, then the order notation is introduced and illustrated, and applied to a restatement of Gauss's test. Tannery's theorem is given in a more general form than that of the original Tannery theorem.

A very brief treatment of double series is given. No explicit definition of convergence or divergence of double series is stated, only a definition of sum by rows, by columns, and by rectangles. The discussion of infinite products is based on the relation of the infinite product $\Pi(1+a_n)$ to the corresponding series $\sum \log(1+a_n)$. The treatment of summability of series is limited to the definition of $(C, 1)$ summability and the corresponding consistency theorem, which is used later in the discussion of Fourier series. The book ends with a good brief chapter on Fourier series.

L. L. SMAIL

Preview of Mathematical Analysis. By A. Freilich, H. H. Shanholt, and J. P. McCormack. New York, Silver Burdett Company, 1937. 4+137 pages. $0.60.

This short text has been written to prepare high school students for the Mathematics Attainment Test (Gamma) and to be used in conjunction with a second-course algebra textbook to meet the requirements of the New York State Syllabus in advanced algebra. It includes (1) elementary aspects of the theory of equations, 38 pages; (2) selected material on permutations, combinations, and probability, 44 pages; and (3) certain fundamentals of differential calculus, 46 pages. The topics from analytic geometry suggested by the College Entrance Board were included in another text by the same authors and omitted in this one.

As in previous books by the authors, the inductive approach is maintained and by easy stages the student is led through interesting examples to definitions and theorems.

The first chapter is not so carefully done as the second, and, in particular, certain definitions and theorems should be more clearly and accurately stated. On page 15 is this corollary: "Every root of a rational integral equation of the form $x^n+b_1x^{n-1}+b_2x^{n-2}+\cdots+b_n=0$ is a factor of the constant term." In the first place, there is nothing in the text to prevent each b_i from being any rational number; and in the second, the proof is invalid unless every number is considered to be a factor of every other number.

There is a decided improvement in the chapter on permutations, combinations, and probability, and it is strongly recommended to students, and to potential writers of college algebra texts. The numerous well chosen examples are carefully worked, and the exercises should arouse enthusiasm in many otherwise "untouchable" students. Definitions and theorems are well stated. This treatment has a minimum of the dry formalism of many texts, particularly in the exercises; if more chapters of algebras were correspondingly well written, the anxiety to hasten into analytic geometry and calculus might be checked.

The chapter on differential calculus contains brief discussions of velocity, acceleration, and applications to analytic geometry. The approach, without assumptions or theorems on limits, is fundamentally that of many texts on analytic geometry. It is more detailed than the latter treatises, has interesting examples and exercises, and should be of interest to calculus students. Unfortunately, different units are chosen on the two axes in each of their illustrated examples after the slope of a line has been defined as the tangent of the angle of inclination. This is confusing, at best.

The book should be of value and decided interest to college and advanced high school students.

E. R. Ott

MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

SLIDES AND FILMS RELATING TO MATHEMATICS AND ITS APPLICATIONS

It has been suggested that films, both sound and silent, might be used to advantage on some of our mathematics club programs. For some topics, silent films have supplemented and clarified prepared lectures. At the present time many colleges and universities are provided with projection and sound equipment as well as film libraries. Although the subject matter of mathematics and its history is full of material which could be used in pictures, very few films related to mathematics or its applications are at present available. Rental bureaus have expressed a willingness to list more films if good ones could be procured. Rental prices vary from 75 cents to $2.50 per reel per day (exclusive of transportation charges). The descriptive list below is not exhaustive. As new titles are called to our attention we shall be glad to mention them in these pages.*

We are listing the following films, all of which are 16 mm. (The numbers in parentheses refer to the renting bureaus listed in the footnote.†)

Frequency Curves. ½ reel silent. (5), (8).

This film is an explanation of the nature of a statistical frequency distribution. It shows the importance of the sampling process and the application of probability formulas.

Einstein's Theory of Relativity. 2 reels silent. (3), (8).

An introduction to the relativity of speed, time, size, motion and measurements.

Parabola. 1 reel sound.

Possible uses of the parabola in the field of design and architecture are presented by the author, Rutherford Boyd. In planning and making this film, he used a large collection of designs and parabolic forms of his own making.

The Development of Astronomical Knowledge. 1 reel silent. (8).

A historical treatment of the growth of astronomy from the time of the Chal-

* The attention of our readers is called to a discussion on mathematical films at the annual meeting of the British Mathematical Association, January 3, 1936. See "Exhibition of mathematical films," Mathematical Gazette, vol. 20, 1936, pp. 110–114.

† (1) University Extension Division, University of Arizona, Tucson.
(2) Bureau of Visual Instruction, University of Colorado, Boulder.
(3) Wm. H. Dudley Visual Education Service, Chicago.
(4) Division of General Extension, University System of Georgia, Atlanta.
(5) Indiana University Extension Division, Bloomington.
(6) Department of Visual Instruction, State University of Iowa, Iowa City.
(7) University Extension Division, University of Kansas, Lawrence.
(8) Bureau of Visual Instruction, University of Wisconsin, Madison.

deans, through the works of Copernicus, Kepler, and Galileo. Includes a diagrammatic explanation of Ptolemy's Theory of epicycles and detailed illustrations of various discoveries.

The Evolution of the Universe. 1 reel silent. (8).

The origin of the planets in terms of the nebular hypothesis.

Gravitation; the Moon; the Constellations. 1 reel silent. (8).

This picture shows Newton at work on his Law of Gravitation. There are many diagrams illustrating his theory and its applications to the orbits of the planets and the moon.

The Sun—Its Influence on the Earth. 1 reel silent. (8).

A study of the nature of the sun, its spots and their effects, the causes of an eclipse, and solar prominences.

Through the Kaleidoscope. ¼ reel silent. (2), (7).

Possibility of this instrument with respect to optical illusions.

The Earth in Motion. 1 reel sound. (1), (4), (5), (6), (7), (8).

Demonstrates the sphericity of the earth, its rotation as established by star trails and the Foucault pendulum, its revolution about the sun, characteristics of orbit, law of areas and the inclination of axis.

Exploring the Universe. 1 reel sound. (4), (5), (8).

Dynamic aspects of the stars within our galaxy and of the galaxies themselves.

The Moon. 1 reel sound. (1), (4), (5), (7), (8).

Explains concepts of the tides, phases of the moon, moon's orbit, lunar month, lunar and solar eclipses.

The Solar Family. 1 reel sound. (1), (4), (5), (6), (8).

Suggests possible origin of the solar system, movements of the solar system in space. Describes planets, planetoids and comets.

The Play of Imagination in Geometry. 1 reel sound. (5), (8).

A teaching aid to help students in plane and solid geometry to grasp spatial relationships. Includes a picturization of the ideas of infinity.

Sound Waves and their Sources. 1 reel sound. (1), (4), (5), (6), (7), (8).

Clearly demonstrates the characteristics of sound waves, such as frequency, amplitude, wave length, fundamentals and harmonics.

Light Waves and their Uses. 1 reel sound. (5).

Explains in an elementary fashion the principles of refraction and also of reflection with plane, concave and convex mirrors. Leads to an introduction of quantum theory.

Dynamic Symmetry. 41 plain slides. (5).

A presentation of the theory of dynamic symmetry beginning with historic art and architecture, and including more recent illustrations from the biological field.

Famous Mathematicians. 50 plain slides. (5).

A discussion of outstanding contributions of famous mathematicians from the early Greeks to the present accompanies this set of pictures. Also some applications of mathematics.

CLUB REPORTS

1937–38

Undergraduate Mathematics Club, State University of Iowa

During the past year, an effort has been made to bring the talks to this club to an undergraduate level and wherever possible to have material presented by undergraduate speakers. This has been an active interest in the club and the activities were not limited by lack of speakers but by lack of time. Subjects discussed at meetings were: Elementary topological surfaces; A topic in statistics; History of mathematics; Simple harmonic motion; The cone and its plane sections; Recreational mathematics; Projective geometry; Some geometric transformations; Talk on astronomy. Social features were a picnic at the City Park and a banquet at the Memorial Union.

President, Leo Nordquist; Secretary, Margaret Krueger.

Naperian Club, DePauw University

It is the policy of this honorary organization to include professors, outside speakers and student members on their programs. Once a year an open meeting is held. A calculus contest providing an additional opportunity for student participation was won by the president, Henry Heintzberger. Programs: Antiquated textbooks in mathematics; The nine-point circle; Modern trends in high school mathematics; History of magic squares; Methods of forming magic squares; Mathematics of music; The cubic and quadratic slide-rule; The Rhind papyrus; Mathematics in chemistry; Line integrals.

President, Henry Heintzberger; Vice-President, Ralph Mann; Secretary, Robert Springer; Treasurer, George Dickson; Faculty Advisor, Dr. H. E. H. Greenleaf.

Phi Chi Mu, Washington and Jefferson College

An organization of honor students who have completed five semesters of college work. Elected members are not admitted to active membership until they have presented papers before the group. The following titles were formally presented and informally discussed: Mathematical tricks; The 200 inch telescope; On the classification and construction of point curves of the third order; Numbers; Extremely low temperatures; The fourth dimension; Television; Relativity; Geometrical arithmetic; Film characteristics; Interrelation between mass, matter and energy; Early mathematics and numerical systems; The classification and construction of plane rational algebraic curves of the fourth class.

President, R. McK. Kiskaddon; Secretary, A. H. Logan; Faculty Advisor, Professor Clyde S. Atchison.

Mathematics Club, University of Nevada

The program for the year included a mathematics play, a Christmas party, a spring picnic at Lake Tahoe, and monthly meetings on the following topics: Mathematical amusements; Slide rule; Evolution of certain mathematical symbols; Biographies of famous mathematicians.

President, Llewellyn Young; Vice-President, Ben Morehouse; Secretary-Treasurer, Martha Holcomb; Faculty Advisor, Mr. Ingo Maddaus.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications concerning Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 345. *Proposed by F. E. Wood, Northwestern University.*

Let $S=a+b+c$, and $T=ab+bc+ca$, where a, b, and c are the sides of a triangle. Show that $3T \leqq S^2 < 4T$. What are the analogous inequalities for a tetrahedron?

E 346. *Proposed by Jack Lorell, Brooklyn, N. Y.*

Circles are constructed with the sides of a quadrilateral as diameters. Prove or disprove that at least one of the opposite pairs intersect.

E 347. *Proposed by A. V. Richardson, Bishop's College, Lennoxville, Que.*

The digits of a nine-place number are 1, 2, 3, 4, 5, 6, 7, 8, and 9, the order being determined by chance. Find the odds against the number being divisible by eleven.

E 348. *Proposed by Cezar Coşniţă, Foeşani, Roumania.*

In the triangle ABC, M is placed on AC, N on AB, and P on MN, in such a manner that $MC/AM = AN/NB = MP/PN = r$, a variable parameter. Show that P moves on a parabola, and determine its position with respect to the triangle.

SOLUTIONS

E 271 [1937, 245] *Proposed by V. Thébault, Le Mans, France.*

Find the base B of a system of notation in which there exists a number of the form *abba*, which is the square of the number *bb*. Show that $b^2+B=aa$.

Solution by the proposer.

The problem requires that we find positive integer solutions to the equation,

$$(B+1)^2 \cdot b^2 = (B^3+1) \cdot a + B(B+1) \cdot b$$

or, after removing the factor $(B+1)$ which is not equal to zero,

$$(B+1) \cdot b^2 = (B^2 - B + 1) \cdot a + B \cdot b,$$

with $a < B$ and $b < B$. This equation may be put in the form,

$(B+1) \cdot b^2 = (B+1)^2 a - B(3a-b)$, or $b^2 = (B+1) \cdot a - B(3a-b)/(B+1)$.

Since B and $(B+1)$ have no common factor, $(3a-b)$ must be a multiple of $(B+1)$. Now $-B<3a-b<3(B+1)$, so that the problem reduces to three cases: (1) $3a-b=0$; (2) $3a-b=B+1$; (3) $3a-b=2(B+1)$.

The discussion of case (1) was given by E. P. Starke [1937, 664].

For case (2) we replace b in the equation

$$b^2 = (B+1)\cdot a - B(3a-b)/(B+1)$$

by $(3a-B-1)$. Since the resulting quadratic equation in B must have rational roots its discriminant must be a perfect square. This gives $13a^2-14a+5-x^2=0$. The positive root of this equation is $a=[7+\sqrt{(13x^2-16)}]/13$, which must be a positive integer. So we set $13x^2-16=y^2$, where y is a positive integer such that $7+y$ is a multiple of 13. The first solution gives $x=218$, whence $a=61$, $b=79$, $B=103$, $N=61\ 79\ 79\ 61=(79\ 79)^2$, and $b^2+B=aa$.

The third case reduces similarly to $25a^2-52a+36-x^2=0$; the positive root of this equation is an integer if $(5x)^2-y^2=224=2^5 7$, where y is a positive integer such that $26+y$ is a multiple of 25. But no integer solutions exist for these last relations, so that case (3) adds nothing to what we already have.

Note that these same three cases (1), (2), and (3), also lead to squares of the more general form, $abba=cc^2$, where c is a digit different from b, in some system of enumeration. For instance, if $B=11$, $5335=77^2$, and if $B=22$, $16\ 2\ 2\ 16=(18\ 18)^2$. In general, $c^2=aa$, or $c^2+B=aa$, or $c^2+2B=aa$. Thus c^2 is written with two digits, of which the second exceeds the first by 0, 1, or 2.

E 293 [1937, 479]. *Proposed by J. H. Butchart, Phillips University, Enid, Okla.*

Construct three circles through a point P so that the sum of the directed segments cut off by the circles on any line through P is zero.

Generalized solution by Wallace Givens and R. J. Walker, Cornell University.

While this problem was solved by W. B. Clarke [1938, 189], the extension to n circles by vector methods is here given.

Let the diameters through P of the n circles be given by vectors $\mathbf{D}_1$, $\mathbf{D}_2$, $\mathbf{D}_3, \cdots, \mathbf{D}_n$, and let $\mathbf{V}$ be a unit vector directed along an arbitrary line through P. Since the scalar product $\mathbf{D}_1\cdot\mathbf{V}$ is equal to the product of the length of $\mathbf{D}_1$ multiplied by the cosine of the angle between it and $\mathbf{V}$, the required condition is

$$\mathbf{D}_1\cdot\mathbf{V} + \mathbf{D}_2\cdot\mathbf{V} + \cdots + \mathbf{D}_n\cdot\mathbf{V} = 0, \tag{1}$$

or

$$(\mathbf{D}_1 + \mathbf{D}_2 + \cdots + \mathbf{D}_n)\cdot\mathbf{V} = 0. \tag{2}$$

Since this must be satisfied for all directions of $\mathbf{V}$, the only solution is

$$\mathbf{D}_1 + \mathbf{D}_2 + \cdots + \mathbf{D}_n = 0. \tag{3}$$

If any $n-1$ circles which do not themselves satisfy this condition are given,

the diameter of the nth circle is uniquely determined by (3). The extension to space of three or more dimensions is immediate.

E 309 [1938, 47]. *Proposed by N. A. Court, University of Oklahoma.*

The orthogonal projections of any point of a sphere upon three diameters determine a plane, the orthogonal projections of whose pole, for the sphere, upon the same diameters, determine a plane which touches the sphere.

Solution by the proposer.

Let A, B, and C be the projections of the point M of the sphere (O) upon diameters OA, OB, and OC. The polar plane of A for (O) passes through the pole P of ABC for (O), and is perpendicular to OA at, say, D. Hence the polar plane of D for (O) is perpendicular to OAD at A, so that this plane passes through the point M.

Thus the orthogonal projection D of the pole P upon OA is conjugate, for (O), to the point M, and the same holds for the projections E and F upon OB and OC. Hence DEF is the polar plane of M for (O), and is thus tangent to (O) at M.

Note. The corresponding proposition in the plane was considered in the *Educational Times*, Reprints, vol. 54, 1891, p. 25, q. 10447.

No other solutions were received for this problem.

E 310 [1938, 47]. *Proposed by V. Thébault, Le Mans, France.*

In a certain system of notation there exists a four-place number of the form $aabb$ which is the square of bb. Show that the numbers, b and $a^2+4(a-1)^2$, are each perfect squares. Determine the base of such a system and the values of a and b, knowing that a is also a perfect square.

No solutions have been received to this problem, as it was originally misquoted, the correction only appearing in the August-September issue of the MONTHLY.

E 311 [1938, 47]. *Proposed by J. S. Robberson, Dallas, Texas.*

The quadrant, AOB, of a circle varies in size and position, but keeps the segment, QA, of one bounding radius, fixed. Find the locus of the point P on the arc AB, if it divides that arc in the same ratio as Q divides the radius AO.

Solution by G. A. Williams, Corvallis, Oregon.

Use polar coördinates with AO as initial line, and A as the pole. Let $AQ=a$, and $AO=R$. Now if we call angle OAP, θ, angle $POA=\pi-2\theta$. Then $a/R=(\pi-2\theta)/\frac{1}{2}\pi=2(\pi-2\theta)/\pi$, and, calling AP, r, we see that $r/R=\sin(\pi-2\theta)/\sin\theta=2\cos\theta$. Eliminating the parameter R from between these two equations gives $r=a\pi\cos\theta/(\pi-2\theta)$, the equation of the locus.

L. M. Kelly points out in his solution, first that the arc AP is of constant length, and second, that in this light the problem reduces to E 41 [1933, 296], proposed by V. F. Ivanoff, and solved by O. J. Ramler, [1933, 609].

Also solved by the proposer.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch side.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3888. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

Let P_1, P_2, P_3, P_4 be the projections of a point P on the faces of a tetrahedron $A_1A_2A_3A_4$. The intersection of the edge A_1A_2 with the plane passing through A_3A_4 and perpendicular to the median corresponding to P in the triangle $P_1P\,P_2$ and the five other similar intersections are in a plane π. The plane π is perpendicular to the straight line joining the isogonal point P' of P to the point having as barycentric coördinates in the pedal tetrahedron of P' the squares of the normal coördinates of P in the tetrahedron $A_1A_2A_3A_4$.

3889. *Proposed by V. Thébault, Le Mans, France.*

A triangle ABC is inscribed in a circle (O) with a fixed diameter Δ, and a transversal Δ', which turns about a fixed point, cuts BC, CA, AB in A_1, B_1, C_1. Let A_2 and A_3, B_2 and B_3, C_2 and C_3 be the orthogonal projections of A and A_1, B and B_1, C and C_1 on Δ. (1) Show that the circles with centers at the midpoints of AA_1, BB_1, CC_1 and passing through A_2 and A_3, and B_2 and B_3, C_2 and C_3 meet in a fixed point on the Euler line of the triangle. (2) Find the locus of the second point of intersection of the three circles.

3890. *Proposed by V. Thébault, Le Mans, France.*

Given four straight lines Δ_1, Δ_2, Δ_3, Δ_4, in a plane; through the orthogonal projections of the vertices of the triangles $T_1 \equiv (\Delta_2, \Delta_3, \Delta_4)$, $T_2 \equiv (\Delta_1, \Delta_3, \Delta_4)$, $T_3 \equiv (\Delta_1, \Delta_2, \Delta_4)$, $T_4 \equiv (\Delta_1, \Delta_2, \Delta_3)$ on Δ_1, Δ_2, Δ_3, Δ_4, respectively, parallels are drawn to the sides opposite the corresponding vertices of the triangle considered: these parallels determine four other triangles T_1', T_2', T_3', T_4' symmetrically equal to the first. (1) Show that the Miquel circles of the quadrilaterals (T_1', Δ_1), (T_2', Δ_2), (T_3', Δ_3), (T_4', Δ_4) are equal to the nine-point circles of T_1, T_2, T_3, T_4 and tangent to the circumcircles of T_1', T_2', T_3', T_4'. (2) Show that the Miquel points of the above quadrilaterals are collinear.

SOLUTIONS

3788 [1936, 377]. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

There are infinitely many pairs of straight lines, each marking on the sides of a given quadrilateral four segments having their midpoints collinear. Show that the locus of the intersections of the straight lines of each pair is a straight line.

Editorial Note. After the preparation of the following two notes a letter was received from the proposer stating that the first line of the problem should read, pairs of straight lines *which are mutually perpendicular*, and that the results below are in accord with his own.

The interesting special case of the problem where the two segment-cutting lines δ and δ' are perpendicular will be considered first. Let ABC be the triangle formed by three sides of the quadrilateral, and suppose first that the fourth side is parallel to BC cutting AB in N and CA in M. The two perpendiculars δ and δ' through a given point P cut from BC and MN two segments whose midpoints lie on a straight line d through P. If the midpoints of the segments cut from AB and CA lie also on d, then P must be at A. Since this case is without interest we now assume that no two sides of the given quadrilateral are parallel, and that the fourth side cuts BC in L. We assume the notation is so chosen that M is within the segment CA and L is within the segment BC. Let the midpoints of the segments cut by δ and δ' from BC, CA, AB, LMN be L', M', N', K, respectively; and we assume that they lie on a straight line d. We consider first the triangle ABC and later the triangle formed by AB, CA, LMN. Through P, the intersection of δ and δ', draw three rays l'', m'', n'' parallel to BC, CA, AB and with these directions; then the order of rotation of the rays is the same as that of ABC, and the angles from m'' to n'', from n'' to l'', from l'' to m'', are $\pi-A$, $\pi-B$, $\pi-C$, where A, B, C are the interior angles of the triangle ABC. The reflections l', m', n' of this system of rays in δ have the sense opposite to that of the original system l'', m'', n''; l', or its extension in the opposite sense, passes through the midpoint L' of the segment cut by δ and δ' from BC; m' passes through M'; n' passes through N'; and L', M', N' are on d. Now reflect l', m', n' in d, obtaining l, m, n passing through F, the reflection of P in d; l goes through L'; m goes through M'; n goes through N'. Let δ cut d in Q; then δ goes into FQ. We may replace these two reflections by the single operation of rotation about Q through the angle $\theta = \angle PQF$. Thus n'', which is parallel to AB and directed in the same sense, is rotated through the angle θ into the position n through N' and F; m'', parallel and with the sense of CA, is by the same rotation carried into m through M' and F. Hence A, N', M', F are concyclic; also B, L', N', F and C, M', L', F. Hence F lies also on the circumcircle of ABC. Considering the triangle ANM in the same way and observing that M' is on AM, and N' is on AN, we have a similar result, that is F lies on the circumcircles of $AN'M'$, NKN', $MM'K$; therefore F lies also on the circumcircle of AMN. It now follows that F is the focus of the parabola touching the sides of the quadrilateral and also d. Hence the projections of F on these five lines lie on the tangent to the parabola at its vertex V; and if we double the lengths of the projecting segments from F, we obtain five points on the directrix, and P is one of these points. We conclude then that P must lie on the directrix of the parabola tangent to the four sides of the quadrilateral.

Conversely, given the quadrilateral above, no two of whose sides are parallel, there is a parabola tangent to its four sides and with its focus F at the common

intersection of the circumcircles of ABC, AMN, MLC, BLN. We may determine the tangent at the vertex and the directrix in the way explained above. Let P be any point on the directrix, and draw the perpendicular bisector d of FP; then d is a tangent to the parabola cutting the four sides in the points L', M', N', K. From the elementary properties of the parabola we see that the three lines FL', FM', FN' determine three rays l, m, n in the same order of rotation as of ABC and such that the angles from m to n, n to l, l to m are respectively $\pi-A$, $\pi-B$, $\pi-C$. The reflection of these rays in d gives the rays l', m', n' through P. Now through P draw the rays l'', m'', n'' parallel to BC, CA, AB and in these directions. There is a common bisector δ of the angle between corresponding rays through P such as l' and l''; and if we reverse the directions of the rays of one of the two systems, there is a common bisector δ' of the angles between resulting corresponding rays, where δ and δ' are perpendicular. Then δ and δ' cut three segments from the sides of ABC whose midpoints lie on d. We have a similar result for triangle ANM, and this merely adds to the rigid system l, m, n a fourth ray k through F and K, and corresponding rays k', k'', where k'' is parallel to NM and K is the midpoint of the segment cut from NM. We now have the theorem:

Given a quadrilateral, no two of whose sides are parallel; then P is a point in its plane such that two perpendiculars can be drawn from P cutting from its sides four segments whose midpoints lie on a straight line d, if and only if P lies on the directrix of the parabola tangent to its sides. The straight line d is the tangent to to the parabola which bisects the segment from the focus to P.

It will now be shown that, if we drop the requirement that δ and δ' shall be perpendicular, there are many other points P satisfying the required conditions, not on the directrix, nor even on any chosen straight line. The theorem of the problem cannot be true unless some restriction is placed upon δ and δ'. Select any point S on the parabola (F), not the vertex; let s be the tangent at S; let a be the line through S parallel to the axis of (F); and let π be the plane of (F) and the given quadrilateral (Q). Through a and s construct two planes, say orthogonal to π; and let π' be a fourth plane cutting two perpendiculars s' and a', intersecting in S', from the two planes on s and a. Then there are a quadrilateral (Q') and its tangent parabola (F') in π' which project, in this case orthogonally, into (Q) and (F), and S', s', a' are the vertex, tangent at the vertex, and axis of (F'). The theorem above applies to (Q') and (F'); and the directrix of (F') projects into a line parallel to s. Since in this projection midpoints project into midpoints, parallel lines into parallel lines, etc., we conclude that any point P on this parallel to s has the desired property, omitting perpendicularity. It is easily seen that in place of the orthogonal projection we may use any parallel projection. We shall add a few remarks on the case where δ and δ' are perpendicular.

Considering only the triangle ABC, we observe that the set of rays from P, l', m', n', determines the points L', M', N' on the line d, and one is led to consider the envelope of $M'N'$, where M' is the intersection of m' with CA and N'

is the intersection of n' with AB, as the set of rays rotates as a rigid system about P. Similarly there is an envelope of $N'L'$, and of $L'M'$. For an arbitrary position of P these three envelopes do not coincide; it will be seen later that they are conics with a common focus P and tangent respectively to corresponding pairs of sides of ABC. The straight line d must be a common tangent. In order for these envelopes to coincide P must be placed so that when M' is at C, N' is at B; the envelope of $M'N'$ will then have BC as an element. The orthocenter H of ABC satisfies this condition, and hence P must lie on a circle (O_1) through H, B, C. Similarly, P must lie on a circle (O_2) through H, C, A, as we see by considering the envelope of $N'L'$. Thus H is the only position that P can have; and we shall show that the envelopes actually coincide in this case in the central conic (H, O) with H and the circumcenter O of ABC as foci, and with the nine-point circle of ABC as the principal auxiliary circle. To prove this we have merely to show, say in the case of the envelope of $M'N'$, that there are six elements such that the projections of H on them lie on the nine-point circle, since we know that the envelope is a conic. This is obviously true of the elements BC, CA, AB. Consider the perpendicular at the midpoint of HA: it will be seen at once that its intersections with CA and AB determine a position for the system m', n', and moreover the midpoint of HA lies on the nine-point circle. The perpendicular bisector of HC gives another element; for its intersection with CA gives the ray m' while the line through H parallel to AB gives the corresponding position of n'. Finally the perpendicular bisector of HB is an element. For the envelopes of $N'L'$ and $L'M'$ we find in the same way the same six elements, and the rest of the proof easily follows. The vertices of the triangle formed by the above three perpendicular bisectors are clearly the centers O_1, O_2, O_3 of the above mentioned circles which determined H: these circles are equal to the circumcircle of ABC; OO_1 is the perpendicular bisector of BC, *etc.*; and O is the orthocenter of $O_1O_2O_3$. The triangles $O_1O_2O_3$ and ABC have the center of similitude $\overline{N}$, the center of the nine-point circle, with the ratio of similitude $1:-1$. From this follows that, if d is any tangent to (H, O) cutting BC, CA, AB in L', M', N', the segments HL', HM', HN' determine three rays l', m', n' of the kind considered; and the reflections of these rays in d give l, m, n passing through F, the reflection of H. Hence F is the focus of the parabola (F) tangent to the sides of ABC and to d, and therefore H must lie on the directrix of (F). If A_0, B_0, C_0 are the points of tangency of (H, O) with BC, CA, AB then $\angle B_0HA = \angle AHC_0 = A$, $\angle C_0HB = \angle BHA_0 = B$, $\angle A_0HC = \angle CHB_0 = C$: there are similar equations for (F). If F lies on an altitude, say AH, then (H, O) and (F) are tangent with BC as the common tangent. See the solutions of 3758 [1937, 668] for the special case where P is at the orthocenter H of triangle ABC; numerous references are given by the proposer of this problem. O. D.

A Note by J. R. Musselman, Western Reserve University.

There is one parabola tangent to any four lines, no two of which are parallel, and no three of which are concurrent. Four triangles can be formed from the

four lines. Through the orthocenter of each triangle one can pass a certain pair of perpendicular lines δ and δ', which possess the remarkable property, that they cut off segments, on every line tangent to the parabola, whose midpoints are collinear.

We write the equation of the parabola as

$$z = 2/(1 - t)^2, \qquad |t| = 1.$$

The focus F of the parabola is $z=0$, the axis $z-\bar{z}=0$. The tangents at the points t_1, t_2, t_3 form a triangle whose orthocenter H_4 is $z=2(1-\sigma_1)/\pi_3$ where $\pi_3=(1-t_1)\,(1-t_2)\,(1-t_3)$ and σ_i is an elementary symmetric functions of t_1, t_2, t_3. (See Musselman, *Four lines and their associated parabola*, this MONTHLY, 1937, p. 513). Any two perpendicular lines δ and δ' through H_4 will have the equations

$$x \pm \alpha\bar{x} = 2(1 - \sigma_1 \pm \alpha\sigma_2 \pm \alpha\sigma_3)/\pi_3. \qquad |\alpha| = 1.$$

These cut the triangle of tangents in segments whose midpoints lie on the line

$$\text{(D)} \qquad \begin{aligned}(\alpha^2\sigma_2 + 1)x - \alpha^2(\sigma_1 + \alpha^2\sigma_3)\bar{x} \\ = 2[1 - \sigma_1 + \alpha^2(1 - \sigma_1)(\sigma_2 - \sigma_3) + \alpha^4\sigma_3(\sigma_3 - \sigma_2)]/\pi_3.\end{aligned}$$

This is the Droz-Farny theorem. Now the equation of the tangent at t_4 is

$$t_4x - \bar{x} = 2t_4/(1 - t_4).$$

The midpoint of the segment on this line cut by δ and δ' is

$$z_4 = \frac{2(1 - \sigma_1)}{\pi_4} - \frac{2t_4[\alpha^2(\sigma_2 - \sigma_3) - \sigma_1 + 1]}{\pi_4(1 - \alpha^2t_4^2)}.$$

$$\pi_4 = (1 - t_1)(1 - t_2)(1 - t_3)(1 - t_4).$$

For $\alpha^2=(\sigma_1-1)/(\sigma_2-\sigma_3)$, $z_4=2(1-\sigma_1)/\pi_4$ and (D) becomes

$$\text{(d)} \qquad (\sigma_2 - \sigma_3)x - (\sigma_1 - 1)\bar{x} = 2(1 - \sigma_1)(\sigma_2 - \sigma_3)/\pi_3.$$

Since z_4 satisfies the equation (d), then for this particular value of α^2, the lines δ and δ' will cut off on the four tangents of the parabola segments whose midpoints are collinear. But this value of α^2 is independent of t_4, and hence the line (d) contains the midpoint of the segment on *every* line tangent to the parabola.

The line (d) is the perpendicular bisector of H_4F, and hence also tangent to the parabola. To construct the lines δ and δ' we draw through H_4 the two lines parallel to the bisectors of the angle formed by (d) with the axis of the parabola. Since δ and δ' cut segments on every line tangent to the parabola whose midpoints are collinear, they will cut any other quadrilateral of tangents in segments whose midpoints are collinear. But for these other quadrilaterals H_4 is simply a point on the directrix; hence through any point of the directrix of the parabola tangent to any given four lines a pair of perpendicular lines can be constructed such that the midpoints of the segments are collinear not only for the given four lines, but for every tangent line of the parabola.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

The Fourth Annual Research Conference of the Cowles Commission for Research in Economics met at Colorado College from Monday, July 5, to Friday, July 29, 1938. Two lectures each morning (Saturdays and Sundays excepted), followed by discussion periods, were features of the program. A total of 38 lectures by 27 lecturers were given; the average attendance at the lectures was 47. About half of the papers were of a mathematical character. A fifth conference is being planned for the summer of 1939, to be held at Colorado College from July 3 to July 28. Persons desiring further information about these conferences may write to the Cowles Commission, Colorado Springs, Colorado.

The second volume of the Collected Works of G. A. Miller has been published by the University of Illinois Press. It contains 537 pages, of which the last 36 pages relate to the general history of mathematics with emphasis on Professor Miller's publications along this line in recent years.

Professor Nola Lee Anderson of Sophie Newcomb College was married July 9, 1938, to Professor E. S. Haynes, chairman of the department of astronomy at the University of Missouri.

Professor E. M. Berry of Bluefield College, Virginia, has been appointed to a professorship at Nebraska State Teachers College, Chadron.

Dr. A. J. Cooley of the University of Idaho has been appointed an associate professor at the University of Tennessee.

Associate Professor J. L. Dorroh of Judson College has been appointed professor of mathematics and physics at Ouachita College.

Assistant Professor F. C. Gentry of the University of Oklahoma has been appointed an assistant professor at Louisiana Polytechnic Institute.

Assistant Professor Lois Griffiths has been promoted to an associate professorship at Northwestern University.

Professor E. G. Harrell of Ouachita College has been appointed head of the department of mathematics at North Dakota State Teachers College, Valley City.

Assistant Professor Ralph Hull of the University of Illinois has been appointed to a professorship at the University of British Columbia.

Assistant Professor W. C. Janes of Kansas State College is on leave of absence for 1938–39 and will study at the University of Nebraska.

Dr. W. J. Kirkham of Oregon State College has had his leave of absence extended a second year to continue statistical research in Washington, D.C.

Professor W. C. Krathwohl of Armour Institute of Technology has been appointed director of the new department of educational tests and measurements.

Dr. Ivan Niven has been awarded a fellowship for postgraduate work at the University of Pennsylvania where he will study under Professor Rademacher.

Captain J. K. Peterson of Nashville, Tennessee, has been appointed head of the department of mathematics at Wentworth Military Academy, Lexington, Mo.

Dr. W. C. Randels of Northwestern University has been appointed assistant professor at the University of Oklahoma.

Assistant Professor Augustus Sisk of the University of Tennessee has been elected head of the department of mathematics and physics at Maryville College, Tennessee.

Professor Gabriel Szegö of Washington University, St. Louis, has been appointed to a professorship at Stanford University.

Associate Professor T. Y. Thomas of Princeton University has been appointed to a professorship at the University of California at Los Angeles.

Assistant Professor Marie J. Weiss of Vassar College has been appointed professor of mathematics and chairman of the department at Sophie Newcomb College.

The following appointments to instructorships are announced:
Cornell University: Dr. F. A. Ficken, G. L. Walker, W. D. Wray
Massachusetts Institute of Technology: A. H. Clifford
University of Nebraska: Dr. D. M. Dribin, Dr. D. L. Netzorg
Purdue University: Dr. R. H. Downing
St. Francis Xavier College: Evelyn R. Garbe
University of Tennessee: W. R. Hanson, H. M. Hughes
Villanova College: Emil Amelotti
University of Wisconsin: Dr. L. R. Wilcox
University of Wisconsin at Milwaukee: M. J. Turner

Dr. J. J. Luck, professor of mathematics for fifteen years at the University of Virginia, died September 15, 1938, at the age of fifty-five. He was a charter member of the Mathematical Association.

Professor Ernest E. Allen, chairman of the mathematics department of Occidental College, died on May 6, 1938, at the age of sixty-eight. He had taught at Occidental College for thirty-six years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

The usual summer meeting of the Mathematical Association of America was omitted in 1938 in deference to the Semicentennial Celebration of the American Mathematical Society. The Trustees of the Association met at New York on Tuesday evening, September 6, 1938.

The following fifty-eight persons were elected to membership on applications duly certified:

L. V. AHLFORS, Ph.D.(Helsingfors) Prof., Univ. of Helsingfors, Helsingfors, Finland.

HARRIET W. ALLEN, Ph.D.(Mass. Inst. of Tech.) Asst. Prof., Hollins Coll., Hollins, Va.

A. A. AUCOIN, A.M.(Rice) Asst., Louisiana State Univ., University, La.

MRS. KATHRYN C. BAKER, M.S.(Chicago) Instr., Baylor Univ., Waco, Texas.

W. A. BARNETT, A.B.(W. Kentucky State Teachers Coll.) Teacher, High School, Burlington, Ky.

EDNA C. BATTIN, A.M.(Southern California) Teacher, Union High School, Colton, Calif.

JACOB BEARMAN, A.B.(Minnesota) Asst., Univ. of Minnesota, Minneapolis, Minn.

E. A. BEITO, A.M.(Minnesota) Asst. Prof., Univ. of Wichita, Wichita, Kans.

SAMUEL BOROFSKY, Ph.D.(Columbia) Asst. Prof., Brooklyn Coll., Brooklyn, N. Y.

J. A. COOLEY, Ph.D.(Illinois) Asso. Prof., Univ. of Tennessee, Knoxville, Tenn.

T. M. EDISON, B.S.(Mass. Inst. of Tech.) President, and research engineer, Calibron Products, Inc., West Orange, N. J.

H. C. T. EGGERS, Ph.D.(Michigan) Asst. Prof., Drawing and desc. geom., Univ. of Minnesota, Minneapolis, Minn.

L. BEATRICE FLEMING, A.M.(Michigan) Instr., Florida A. and M. Coll., Tallahassee, Fla.

M. G. GALBRAITH, M.S.(Rutgers) Asst. Prof., Rutgers Univ., New Brunswick, N. J.

CARMELITA T. GONZÁLEZ, M.S.(Tulane) Instr., William Woods Coll., Fulton, Mo.

H. J. G. GUTEKUNST, M.S.(Illinois) Teacher, Batavia High School, Batavia, Ill.

D. W. HALL, Ph.D.(Virginia) National Research Fellow, Univ. of Pennsylvania, Philadelphia Pa.

H. H. HARTZLER, Ph.D.(Rutgers) Asso. Prof., Goshen Coll., Goshen, Ind.

CORINNE R. HATTAN, A.M.(Kansas) Head of Dept., Northwestern Jr. Coll., Orange City, Iowa.

REV. J. R. HEARN, A.M.(Woodstock) Teacher, Coll. of St. Ignatius, Manhasset, N. Y.

E. R. HEINEMAN, A.M.(Wisconsin) Asst. Prof., Texas Tech. Coll., Lubbock, Tex.

W. N. HERR, Ph.D.(Chicago) Instr., Chemistry, Jr. Coll., Duluth, Minn.

G. B. HUFF, Ph.D.(Illinois) Asst. Prof., Southern Methodist Univ., Dallas, Tex.

J. F. KENNEY, A.M.(Michigan) Instr., Northwestern Univ., Evanston, Ill.

CORNELIUS LANCZOS, Ph.D.(Szeged) Prof., Purdue Univ., Lafayette, Ind.

NATHAN LAZAR, Ph.D.(Columbia) Teacher, Alexander Hamilton High School, Brooklyn, N. Y.

W. G. LEAVITT, A.M.(Nebraska) Grad. asst., Univ. of Nebraska, Lincoln, Nebr.

HERMAN LEVY, A.B.(New York Univ.) 3736 Oceanic Ave., Brooklyn, N. Y.

NEIL LITTLE, A.M.(Michigan) Instr., Purdue Univ., Lafayette, Ind.

OLWEN LLOYD (MRS. GEORGE), M.A.(Cambridge, England) Headmistress, Mount Vernon Seminary, Washington, D. C.

MARGARET P. MARTIN, B.S.(Minnesota) Asst., Univ. of Minnesota, Minneapolis, Minn.

W. R. MCEWEN, B.E.(State Teachers Coll., Duluth) Asst., Univ. of Minnesota, Minneapolis, Minn.

PARRY MOON, M.S.(Mass. Inst. of Tech.) Asso. Prof., Elec. Eng., Massachusetts Inst. of Tech., Cambridge, Mass.

LILLIAN MOORE, A.M.(Pennsylvania) Teacher, Far Rockaway High School, Far Rockaway, N. Y.

FREDERICK MOSTELLER. Senior, Carnegie Inst. of Tech., Pittsburgh, Pa.

P. M. NIERSBACH, A.M.(Southern California) Teacher, Bell High School, Huntington Park, Calif.

IRENE A. NOLAN, B.S.(Marquette) Grad. stu-

dent, Louisiana State Univ., University, La.

LLEWELLYN NOTLEY, A.M.(Texas) Supt. of Schools, Teague, Tex.

T. G. OSTROM, A.B.(Minnesota) Asst., Univ. of Minnesota, Minneapolis, Minn.

C. G. PHIPPS, Ph.D.(Minnesota) Asso. Prof., Univ. of Florida, Gainesville, Fla.

A. R. POOLE, Ph.D.(Calif. Inst. of Tech.) Instr., Univ. of Minnesota, Minneapolis, Minn.

REV. J. G. PRÉCOURT, M.S.(Catholic Univ.) Prof. of Physics, Asso. Prof. of Math., Providence Coll., Providence, R. I.

H. F. PRICE, Ph.D.(Pennsylvania) Prof., Pacific Univ., Forest Grove, Ore.

E. G. SALISBURY, Ph.D. Prof., State Teachers Coll., California, Pa.

H. F. SCHROEDER, M.S.(Louisiana) Asst. Prof., Louisiana Poly. Inst., Ruston, La.

C. S. STUCKEY, A.B.(Hastings) Grad. asst., Univ. of Wyoming, Laramie, Wyo.

W. I. THOMPSON, A.M.(California) Instr., Los Angeles Jr. Coll., Los Angeles, Calif.

A. W. TUCKER, Ph.D.(Princeton) Asso. Prof., Princeton Univ., Princeton, N. J.

ROBERT TUCKER, A.B.(Macalester) Grad. asst., Univ. of Minnesota, Minneapolis, Minn.

J. W. TUKEY, M.S.(Brown), A.M.(Princeton) Grad. student, fellow, Princeton Univ., Princeton, N. J.

L. H. TULLOCH, A.M.(Brown) Instr., Jr. Coll., San Antonio, Tex.

O. D. TYNER, A.M.(Chicago Law School) Teacher, Chicago Tech. Coll., and Lane Tech. High School, Chicago, Ill.

G. C. VEDOVA, A.M.(Columbia) Asst. Prof., St. John's Coll., Annapolis, Md.

R. T. WALLACE, A.B.(British Columbia) Asst. Prof., Victoria Coll., Victoria, B. C., Canada.

AGNES E. WELLS, Ph.D.(Michigan) Prof., Indiana Univ., Bloomington, Ind.

D. E. WHELAN, JR., B.S. in C.E.(Mass. Inst. of Tech.) Dean, Coll. of Science, Loyola Univ. of Los Angeles, Los Angeles, Calif.

MARELENA WHITE, M.S.(Louisiana) Asst., Louisiana State Univ., University, La.

P. M. WHITMAN, A.M.(Harvard) Instr., Harvard Univ., Cambridge, Mass.

It was voted that the invitation to meet at Ohio State University in December 1939 be accepted, and that greetings be sent to the American Mathematical Society on the occasion of their Semicentennial.

W. D. CAIRNS, *Secretary-Treasurer*

THE ANNUAL MEETING OF THE MINNESOTA SECTION

The annual meeting of the Minnesota Section of the Mathematical Association of America was held at St. John's University, Collegeville, Minnesota, on Saturday, May 14, 1938. A morning session was held at 11:00 o'clock and was followed by luncheon and an afternoon session at 2:15 o'clock. Professor G. L. Winkelmann of St. John's University presided at the two sessions.

Eighty-one persons attended the meeting including the following thirty-two members of the Association: Mae R. Anderson, C. J. Blackall, L. E. Bush, W. H. Bussey, E. J. Camp, S. Elizabeth Carlson, Sister M. Claudette, R. W. Cowan, Arthur Danzl, J. H. Daoust, Louis De LaSalle, Bernard Dimsdale, Gladys Gibbens, W. L. Hart, H. E. Hartig, J. S. Hickman, Dunham Jackson, A. L. O'Toole, J. M. Rysgaard, R. B. Saunders, M. G. Scherberg, Ole Schey, C. Grace Shover, R. R. Shumway, A. J. Strane, F. J. Taylor, H. P. Thielman, Ella Thorp, A. L. Underhill, K. W. Wegner, Marian A. Wilder, G. L. Winkelmann.

At the business session officers were elected for the coming year as follows: Chairman, W. H. Bussey, University of Minnesota; Secretary, A. L. Underhill, University of Minnesota; Members of the Executive Committee: E. J. Camp, Macalester College, A. J. Strane, Duluth Junior College; Louis De LaSalle, St. Mary's College.

It was decided to hold a second meeting this year, a joint meeting with the Mathematics Section of the Minnesota Educational Association in Minneapolis, October 28, 1938.

The following ten papers were presented:

1. "On finding the reciprocal of a matrix" by Dr. K. W. Wegner, University of Minnesota.

2. "Note in factor analysis" by Mary Elveback, University of Minnesota, introduced by Professor Dunham Jackson.

3. "Operator isomorphism and equivalence of ideals" by Dr. C. Grace Shover, Carleton College.

4. "Some present policies in French and German universities" by Professor A. L. Underhill, University of Minnesota.

5. "Note on experimental testing in trigonometry" by Dr. Marian A. Wilder, Research Statistician, University of Minnesota.

6. "Some remarks on modular geometries and their relation to Euler's criterion in the theory of numbers" by Professor W. H. Bussey, University of Minnesota.

7. "Note on the summation of Fourier Series" by Professor Dunham Jackson, University of Minnesota.

8. "Beaming definite integrals" by Professor G. L. Winkelmann, St. John's University.

9. "The utility of indeterminate forms in the first course of the calculus" by Professor M. G. Scherberg, University of Minnesota.

10. "The determination of the characteristic exponent for linear differential equations of the second order" by Dr. R. W. Cowan, College of St. Scholastica, Duluth.

Abstracts of some of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. Dr. Wegner described a method of finding the reciprocal of a matrix which is based on the Hamilton-Cayley Theorem. The method is not new, but is little known, and it is interesting if not practical.

2. Miss Elveback showed that the observed standard scores (z_{ik}) are transformed into the independent scores or "factors" (γ_{ik}) by the matrix equation $z = A\gamma$ where A is the unitary matrix of the characteristic vectors of the real symmetric correlation matrix $R = (r_{ik})$ and therefore represents a pure rotation. Such a real orthogonal matrix A always exists and the characteristic numbers λ_k are real and equal the variance of the factors γ_k. The condition that a set of numbers be direction cosines of a principal axis of the surfaces of equal frequency was shown to be satisfied by each characteristic vector of R. Thus the

rotation $Z=A\gamma$ rotates the correlation ellipsoids to their principal axes, the equation in terms of the factors and their variances being $\sum_{i=1}^{n}\gamma_i^2/\lambda_i=D$.

3. Dr. Shover proved that ideals which are equivalent in the sense of Dedekind are operator isomorphic and conversely.

5. The types of questions tried were matching, multiple choice, and indirect, none of which were found significantly different from the usual direct questions. Direct questions broken into small units were easier than the entire problem for the class as a whole but harder for the top group. The placement as explained by Dr. Wilder was found useful for predicting success in the course for students having higher algebra in high school.

6. Professor Bussey showed how the solutions of the linear congruence $ax^2+2hxy+by^2+2gx+2fy+c=0$, modulo p, can be pictured by points in a euclidean plane. He referred to such collections of points as second degree "constellations" in finite geometries, modulo p, and showed how their properties lead to a very simple proof of the theorem that the number -1 is a square or a non-square, modulo p, according as p is of the form $4k+1$ or $4k+3$.

7. The paper of Professor Jackson called attention to an especially simple proof of the well known fact that if $f(x)$ satisfies a Lipschitz condition the error of the Fejér mean of the partial sums of its Fourier series through terms of the nth order does not exceed a quantity of the order of magnitude of $(\log n)/n$. The error is not greater than a constant multiple of an integral which can be interpreted as giving the value of the corresponding error in the case of the particular function $|x|$, and an upper bound for the latter can be immediately deduced from the explicit form of the Fourier coefficients.

8. Space diagrams, shear diagrams, moment diagrams, slope diagrams, and deflection diagrams for beams are presumed to be developed in the usual manner. Professor Winkelmann developed formulas for the shear, the binding moment, the slope of the beam, and the deflection of the beam at a point. For a cantilever beam with a concentrated load at each end similar formulas were presented as well as for a simple beam with a uniformly distributed load over the entire span.

9. Dr. Scherberg showed by giving several examples that the indeterminate form $\lim_{x\to a}(f(x)-f(a))/(g(x)-g(a))$ appears in a number of places in the first course of the calculus in the process of developing formulas for applications (radius of curvature, *etc.*). Also that these forms are readily evaluated by means of the formula

$$\lim_{x\to a}\frac{f(x)-f(a)}{g(x)-g(a)}=\lim_{x\to a}\frac{f'(x)}{g'(x)}.$$

It is believed that the latter method simplifies the development of these formulas and hence also the whole structure of the course.

10. Dr. Cowan showed that the second order differential equation may be transformed to a Riccati equation. Then by requiring that a series consisting of ascending powers of the parameter with undetermined coefficients be a solu-

tion of the Riccati equation, we are led to a set of first order differential equations, the integration of which is easily accomplished. A transformation of variable and integration from zero to $2\pi i$ determine the characteristic exponent within an integer. The solution of the original differential equation for which the characteristic exponent has been determined is shown to converge by setting up an absolutely convergent series which dominates it.

A. L. UNDERHILL, *Secretary*

THE 1938 MEETING OF THE TEXAS SECTION

The 1938 meeting of the Texas Section of the Mathematical Association of America was held at Texas Christian University, Fort Worth, on Friday and Saturday, April 22–23. The meeting was presided over by the chairman, Professor J. H. Binney of the A. and M. College of Texas.

Among the seventy-two persons attending the meeting were the following twenty-three members of the Association: E. F. Beckenbach, J. H. Binney, A. A. Blumberg, H. E. Bray, Myrtle C. Brown, J. E. Burnam, J. G. Chaney, E. L. Dodd, Nat Edmonson, Jr., H. J. Ettlinger, E. H. Hanson, C. M. Howard, A. J. Kempner, H. A. Luther, B. C. Moore, E. D. Mouzon, Jr., M. E. Mullings, G. A. Newton, Maxwell Reade, Ethel A. Rice, C. R. Sherer, Jennie L. Tate, E. V. White.

Local arrangements were in charge of Professor C. R. Sherer, who was assisted by members of the "Parabola," a club composed of those students of Texas Christian University majoring in mathematics. An outstanding feature of the program of entertainment was the dinner on Friday evening for those attending the meeting. Dean E. V. White of Texas State College for Women was the invited speaker at the dinner.

The Section was very glad to have as the guest of honor for this meeting the president of the Association, Professor A. J. Kempner of the University of Colorado. His two addresses were enthusiastically received and were definitely stimulating to the Section membership.

At the business session the following officers were elected for the coming year: Chairman, H. J. Ettlinger, University of Texas; Vice-Chairman, E. D. Mouzon, Jr., Southern Methodist University. At the invitation of Abilene Christian College, Hardin-Simmons University, and McMurray College, all at Abilene, the 1939 meeting was scheduled to meet at Abilene.

The following papers were presented:

1. "The rôle of isomorphism in scientific systems" by Professor A. J. Kempner, University of Colorado.

2. "On continued fractions" by Dr. Walter Leighton, Rice Institute, introduced by Professor Bray.

3. "A highly irregular boundary value problem of order $2n$" by Dr. H. A. Luther, A. and M. College of Texas.

4. "Interior and exterior means obtained by the method of moments" by Professor E. L. Dodd, University of Texas.

5. "A generalization of the theorems of Cauchy and Morera" by Maxwell Reade, Rice Institute.

6. "On the measure of linear point sets" by A. A. Blumberg, A. and M. College of Texas.

7. "On the limit of a sequence of integrals" by J. P. Nash, Rice Institute, introduced by Professor Bray.

8. "Extended expansion theorems" by J. G. Chaney, A. and M. College of Texas.

9. "The Texas Curriculum Revision Movement in the field of mathematics from the viewpoint of the secondary schools" by Bee Grissom, Austin High School, introduced by Professor Ettlinger.

10. "The Texas Curriculum Revision Movement from the college and university viewpoint" by Professor H. J. Ettlinger, University of Texas.

11. "The need for cooperation between high school teachers and college teachers of mathematics" by Professor A. J. Kempner, University of Colorado.

Abstracts of some of these papers follow, numbered as in the list above:

1. Professor Kempner's paper will appear in a later issue of the MONTHLY.

2. Dr. Leighton presented an exposition of continued fractions from a function-theoretic point of view. A general discussion of convergence criteria was given with particular reference to the results of E. B. Van Vleck, A. Pringsheim, O. Szàsz, H. S. Wall, J. Q. Jordan, and the author. Reference was made to the first "gap" theorems for continued fractions, proved recently by W. T. Scott and Walter Leighton.

3. Dr. Luther considered the problem of expanding functions in infinite series whose terms satisfied the equation $d^{2n}u/dx^{2n}+\rho^{2n}u=0$, $n\geqq 2$, and certain highly irregular boundary conditions at the two ends of an interval. It was found that conditions for the expansibility of functions could be derived by finding such conditions for a related problem in which the boundary conditions bore at only one end of the interval.

4. Professor Dodd discussed frequency functions $\alpha^{-1}\Phi(t')$ in x, with reference to estimating by the method of moments the location κ and the scale α associated with x thus: $t'=(x-\kappa)/\alpha$. With k and a as estimates of κ and α, the frequency becomes $y=a^{-1}\Phi(t)$, $t=(x-k)/a$. Now, independent of a and k, $\Phi(t)$ has a mean value μ, and a variance σ^2. For n measurements, X_i, let $\overline{X}$ be the arithmetic mean, and let τ^2 be the variance. By the method of moments, $a=\tau/\sigma$. Thus a is a root-mean-square of $|X_i-\overline{X}|/\sigma$, and $k=\overline{X}-a\mu$, a substitutive mean of X_i. It is indeed an external mean if these measurements are equal numerically, just half of them positive, and the second moment of $\Phi(t)$ is less than twice the square of the first moment. For example, this is the case if $\Phi(t)$ is a Pearson Type III, with origin at the start of the curve, and with the exponent of t positive.

5. Mr. Reade obtained generalizations of the Cauchy and Morera theorems. A typical result is the following: Let $X_j(u, v)$, $j=1, 2, 3$, be a triple of conjugate harmonic functions defined in domain D: (u, v), where D is the interior of the circle $u^2+v^2=1$. Let $X_j(s, t)$, $j=1, 2, 3$, be a conformally equivalent triple de-

fined in $\bar{D}$: (s, t). Then a necessary and sufficient condition that the triple $X_j(u, v)$, $j=1, 2, 3$, map D on a plane is that for all maps of D on domains $\bar{D}$ in the (s, t) plane we have $\sum_{j=1}^{3}[\int_{\bar{c}} \overline{X}_j(s, t)dw]^2=0$, where $w=s+i\cdot t$, and $\bar{c}$ is the map of circle $u^2+v^2=1$ in $\bar{D}$. Further results were obtained by considering the average function defined by

$$A\rho_j(u_0, v_0) = \frac{1}{\pi\rho^2}\int\int_R X_j(u_0 + \rho\cos\phi, v_0 + \rho\sin\phi)dA,$$

where dA is element of area, and R the interior of circle $(u-u_0)^2+(v-v_0)^2=\rho^2$.

6. Mr. Blumberg proved the following theorems: (1) If M and N are bounded linear point sets and $m_i(M\cdot N)=0$, then a necessary and sufficient condition that $m_i(M)+m_i(N)=m_i(M+N)$ is that $H\cdot M$ shall be measurable where H is a measurable subset of $M+N$ such that $m(H)=m_i(M+N)$. (2) If M and N are bounded linear point sets, then a necessary and sufficient condition that $m_e(M)+m_e(N)=m_e(M+N)$ is that for each positive number ϵ there shall exist collections G_ϵ and H_ϵ of mutually exclusive segments, covering M and N respectively, such that $m(G_\epsilon^*\cdot H_\epsilon^*)=0$.

7. Mr. Nash dealt with convergent sequences or families of non-negative summable functions of x, $f(t, x)$ with $\lim f(t, x)=f(x)$, $0\leqq x\leqq 1$, of which the Poisson integral, Fejér sum, *etc.* are typical. It was found that a sufficient condition in order that $f(t, x)$ satisfy the more restrictive condition of Lebesgue is that

$$\int_0^1 g(x)\log\frac{1}{1-x} < \infty,$$

where $g(x)$ is the *monotone* function equivalent to $f(x)$ in the sense that $mE[g(x)<A]=mE[f(x)<A]$. This condition is also necessary if $f(x)$ is itself monotone.

8. Mr. Chaney derived an identity, which he referred to as *the key identity*, which includes Bartky's Identity as a special case. This identity, in conjunction with some fundamental operational formulas, was used to develop some very general extensions of the Heaviside type expansion theorem. The key identity was shown to be the connecting link between the usual scalar type of expansion theorem and the vector-tensor type of expansion theorem. The entire paper was considered from the viewpoint of applications to electric circuit theory.

10. As the University representative of the mathematics section of the Texas State Curriculum Revision Committee, Professor Ettlinger was interested in the following aims: (1) avoiding the dilution or weakening of the high school mathematics courses; (2) strengthening of the fundamentals in these courses; (3) enrichment of essential drill with material taken from everyday experience and from scientific fields; and (4) assurance that those who will pursue studies in scientific work in any field will have adequate preparation to continue their mathematical studies.

11. In this paper Professor Kempner expressed ideas and viewpoints which have already appeared in various papers by him, such as an address printed in this MONTHLY for December 1937.

NAT EDMONSON, JR. *Secretary*

THE FIFTEENTH ANNUAL MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The fifteenth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at Mississippi State College, March 11–12, 1938. The chairman of the Section, Professor Dorothy McCoy of Belhaven College, presided.

The attendance was approximately sixty-five, including the following twenty-five members of the Association: T. A. Bickerstaff, H. E. Buchanan, W. E. Cox, G. F. Cramer, D. S. Dearman, W. L. Duren, Jr., Virginia I. Felder, Elizabeth Freas, Deborah May Hickey, Dorothy McCoy, Janet McDonald, A. C. Maddox, B. E. Mitchell, I. C. Nichols, Arthur Ollivier, W. V. Parker, H. L. Quarles, W. D. Reeve, S. T. Sanders, C. D. Smith, H. L. Smith, P. K. Smith, V. B. Temple, J. F. Thomson, B. A. Tucker.

The following officers were elected for the year 1938–39: Chairman, J. F. Thomson, Tulane University; Vice-Chairman for Louisiana, H. F. Schroeder, Louisiana Polytechnic Institute; Vice-Chairman for Mississippi, Dorothy McCoy, Belhaven College; Secretary, W. V. Parker, Louisiana State University.

The Section was honored by the presence of Professor W. D. Reeve of Teachers College, Columbia University. The value of the meeting was strengthened by the two strong addresses which he gave. In the first he gave a report on the progress made by the Joint Commission on the Place of Mathematics in Secondary Schools. In his address on "Modern curriculum problems in the teaching of mathematics" Professor Reeve considered problems relating (1) to the pupil, (2) to the teacher, (3) to content.

(1) The nature of the high school population has radically changed since 1900. The pupils vary not only in native ability to do high school work, but also in experience and interests. Traditional plans for solving our difficulties have not always been successful; they have had their good points but we need a new philosophy of education to meet modern demands. He considered also the evils of mass education and the injurious effect of emotionalized attitudes.

(2) Higher qualifications, particularly in regard to personality, character, and knowledge of subject matter, are necessary. He spoke also of the reasons for the failure of teachers and of new standards for teacher training.

(3) Professor Reeve emphasized the importance of a six-year course in general mathematics beginning in grade seven and running through gradé twelve, as well as certain fundamental reorganization principles.

The following program was given:

1. "A general theory of limits" by Professor H. L. Smith, Louisiana State University.

2. "Homogeneous diophantine equations" by A. A. Aucoin, Louisiana State University, introduced by the Secretary.

3. "On Tschirnhausen transformations" by W. E. Cox, Mississippi State College.

4. "Book values of a specific bond purchase" by Professor I. C. Nichols, Louisiana State University.

5. "Involutions on a complex line" by Professor B. E. Mitchell, Millsaps College.

6. "The differentiability of an arc" by Professor H. T. Fleddermann, Loyola University, introduced by the Secretary.

7. "The N curvatures of a twisted curve in euclidean N-space" by Gordon Walker, Louisiana State University, introduced by Professor H. L. Smith (read by title).

8. "The determination of the Maya unit of measure" by Professor G. F. Cramer, Tulane University.

9. "Functions analogous to trigonometric functions" by Professor V. B. Temple, Louisiana College.

10. "Some notes on divisibility and direct division" by Professor T. A. Bickerstaff, University of Mississippi.

11. "A certain type of correspondence" by E. P. Coleman, Mississippi State College, introduced by Professor C. D. Smith.

12. "The use of operators in evaluating infinite integrals" by Professor J. F. Thomson, Tulane University.

13. "Mean figures" by Bayliss Shanks, Millsaps College, introduced by Professor Mitchell.

14. "A new approach to the solution of the cubic and quartic" by Eckford Cohen, student, Starkville High School, introduced by Professor C. D. Smith.

15. "The human aspect of mathematics teaching" by Professor S. T. Sanders, Louisiana State University.

16. "Expedient compromise between the traditional viewpoint and current trends in mathematical curricula and instructions" by Professor A. C. Maddox, Louisiana State Normal College.

17. "The Joint Commission on the Place of Mathematics in the Secondary Schools" by Professor H. E. Buchanan, Tulane University.

Abstracts of some of these papers follow, the numbers corresponding to the numbers in the list of titles.

1. In 1922 E. H. Moore and H. L. Smith published in the *American Journal of Mathematics* a general theory of limits, which theory included nearly all the types of limits found in the literature. A notable exception is the approximate limit employed in the advanced work on integration. In the present paper, which has appeared in the *National Mathematics Magazine* (vol. 12, 1938, pp. 371–379), Professor Smith presented a new theory which generalizes the earlier theory and does include the approximate limit as an instance.

2. The Diophantine equation $f(x_1, x_2, \cdots, x_r) = g(y_1, y_2, \cdots, y_s)$, where f and g are homogeneous polynomials of degrees m and n respectively, and m and n are relatively prime, always has an infinitude of integral solutions. Mr. Aucoin showed how these solutions may be obtained in terms of $r+s$ parameters. He further showed that the solutions are general except those for which $f=0$, in which case the trivial solution $x_i=y_j=0$ is obtained.

5. Employing the method of Laguerre in the representation of the imaginary point Professor Mitchell considered the following problem: Given an involution on a complex line determined by two complex point-pairs, to find its center and its double points, the Jacobian pair. The general result, including the real case of course, is: The initial points of the real representation of the Jacobian pair are doubly cyclic and anticyclic with respect to each pair of initial points of the original pairs, with an identical relation of the terminal point-pairs. The six points involved in each case are the vertices of two hexagons which are inversely similar—the condition of collinearity of the four points involved. The construction involved is that employed by Morley.

6. After defining what is meant by the various orders of differentiability of an arc and a vector function, Professor Fleddermann showed that the sufficient condition for an arc to be nth differentiable at a point is that the vector function which represents it be nth differentiable at that point. He also showed that the necessary and sufficient condition for an arc to be first differentiable at a point is that its contingent, in Bouligand's sense, consist of a single half line or a pair of complementary half lines, and that the necessary and sufficient condition for second order differentiability of an arc at a point is that it have contact of order two at least with some circle at that point.

8. This paper appeared in the June-July 1938 number of this MONTHLY.

9. Professor Temple defined functions, based on the solutions of the differential equation $d^nu/dx^n+u=0$, which were analogous to the trigonometric functions sin x and cos x. He also showed that functions based on the solutions of the differential equation $d^nu/dx^n-u=0$ could be defined, and are analogous to the hyperbolic functions sinh x and cosh x.

10. After a review of certain criteria for divisibility by given numbers, in which use was made of appropriate congruences and of sets of remainders resulting from dividing powers of ten by the given numbers, attention was given by Professor Bickerstaff to the question of the multiples of divisor removed by casting out processes.

11. Mr. Coleman considered the relation $t^2+yt+x=0$, where x and y are the usual rectangular coördinates of a point in the plane and t refers to a point on a line. For points in the plane such that $y^2-4x=0$ there exists a (1–1) correspondence between the points of the parabola and the points of the line. The following theorems were proved: (1) The arbitrary choice of a point $P(x, y)$ in the plane fixes points t_1 and t_2 on the line, and their correspondences Q_1 and Q_2 on the parabola. The point P is the pole of the line Q_1Q_2. (2) If two points P_1 and P_2 be chosen in the plane so that the corresponding points on the line are harmonic, then P_1 and P_2 are conjugate points.

12. The relationship between the impedance operator of Heaviside $1/z(p)$ and the indicial admittance function $A(t)$ was shown in several cases. The infinite integral theorem $1/az(t)=\int_0^\infty e^{-ax}A(x)dx$ was then introduced, and this theorem used to evaluate several infinite integrals.

13. Extending the mean division of a line segment arithmetically, geometri-

cally, and harmonically to other geometric figures, Mr. Shanks divided a square by means of parabolas into the same three means. Then employing the right conoid as a sort of general mean figure between a right circular cone and a right circular cylinder, he secured a geometric mean section by a secant plane, and further showed that the conoid is the harmonic mean between the cone and the cylinder in respect to the volume, centroid, and moment of inertia referred to its axis.

14. Mr. Cohen's method was based upon equivalence of forms expressed as combinations of linear and quadratic factors. Each solution was obtained by a resolvent equation, the solution of which leads directly to evaluation of the coefficients in the factors.

W. V. PARKER, *Secretary*

THE TWENTY-FOURTH ANNUAL MEETING OF THE KANSAS SECTION

The twenty-fourth annual meeting of the Kansas Section of the Mathematical Association of America was held in conjunction with the Kansas Association of Mathematics Teachers at the State Teachers College, Pittsburg, on Saturday, April 2, 1938. This was the first meeting of the two organizations to be held in Pittsburg, and there was much appreciation expressed over the cordial welcome of President Brandenburg of the College, the delightful luncheon in the College dining room, and the evidences of careful planning for our comfort and pleasure by the department, under the leadership of Professor J. A. G. Shirk.

The attendance was one hundred and fourteen, including the following twenty-nine members of the Association: Sister Mary N. Arnoldy, R. W. Babcock, Wealthy Babcock, Lois E. Bell, Florence L. Black, R. D. Daugherty, Lucy T. Dougherty, D. D. Driver, W. H. Garrett, A. J. Hoare, Emma Hyde, W. C. Janes, H. E. Jordan, C. F. Lewis, W. H. Lyons, Anna Marm, U. G. Mitchell, Thirza A. Mossman, O. J. Peterson, G. B. Price, C. B. Read, B. L. Remick, J. A. G. Shirk, G. W. Smith, R. G. Smith, E. B. Stouffer, W. T. Stratton, C. B. Tucker, J. J. Wheeler.

In the morning there was a joint session of the two organizations, which was presided over by Professor C. B. Tucker of State Teachers College, Emporia. The program included papers by Professor A. R. Congdon of University of Nebraska, "Mathematical requirements for high school graduation and college entrance"; and by Professor R. W. Babcock of Kansas State College, Manhattan, "Scientific surplus"; also a report by Professor U. G. Mitchell, University of Kansas, on "The work of the Joint Commission on the Place of Mathematics in Secondary Schools." Professor Anna Marm, Bethany College, Lindsborg, was president of the Kansas Association, and arranged the program.

Professor C. B. Tucker, vice-chairman of the Section, presided at the meeting of the Section after the luncheon, the chairman, Professor W. G. Warnock, being on leave of absence for this semester. The officers chosen for the coming

year were: Chairman, C. B. Tucker, Kansas State Teachers College, Emporia; Vice-Chairman, C. B. Read, University of Wichita; Secretary, Lucy T. Dougherty, Junior College, Kansas City.

The following program was presented:

1. "On Bernoulli polynomials and numbers" by Dr. G. C. Munro, Kansas State College, Manhattan, introduced by Professor Stratton.

2. "Vexing minor problems of the mathematics curriculum" by Professor C. B. Read, University of Wichita.

3. "The Pascal configuration in a modular geometry" by C. E. Rickart, University of Kansas, introduced by Professor Mitchell.

4. "A program for the Association" by Professor G. B. Price, University of Kansas.

Abstracts of the papers follow:

1. Three different sets of definitions for the numbers and the polynomials of Bernoulli were shown by Dr. Munro to be equivalent.

2. Professor Read called attention to problems common to Kansas colleges and universities. Among such we find the student deficient in the usual college entrance mathematical subjects, the content of certain college courses, the use of textbooks. A temporary solution was given to the problems arising when the teaching staff is not large enough to handle the load.

3. Following a brief outline of some of the properties of the Pascal configuration Mr. Rickart examined them in the modular plane $PG(2.5)$ and showed that every Pascal configuration reduced to a Desargues configuration which is self-polar as to the fundamental conic. The 15 Salmon points, the 20 Steiner points, 30 of the 45 Pascal points, and the 60 Kirkman points reduce to the 10 points of the Desargues configuration, while the 15 Steiner-Plücker lines, the 20 Cayley-Salmon lines, and the 60 Pascal lines reduce to the 10 lines. The (1,1) correspondence between the above points and lines becomes a correspondence between poles and polars with respect to the fundamental conic.

4. This paper appeared in the October 1938 number of this MONTHLY.

LUCY T. DOUGHERTY, *Secretary*

GEOMETRIES*

MAYME I. LOGSDON, University of Chicago

When in high school you became acquainted with euclidean geometry it seemed completely in accord with your world of experience. When later you learned of the existence of other geometries, it is quite likely that your instinctive reaction was the thought that these were probably creations of the pure mathematician for which there could be no practical use since we live in a euclidean space.

When a physical or geometrical theory has been for a long time used to explain certain observed data, it is not surprising that there is a tendency to believe that this is perforce the correct theory and to expect that henceforward nature will act in the manner predicted by the theory. The danger inherent in this attitude is that the mind is closed against alternative theories which indeed may provide an equally good interpretation of the same data.

Not until the investigator realizes that no single theory has been found or will be found which explains completely the behavior of Nature but that there are numerous theories which explain most of the facts of observation, will he be equipped to study with interest and profit the relation of fact to theory. Only then can he be expected not to make the error of concluding that other situations, not within the range of observation, are necessarily thus and so because his favorite theory so predicts it.

From the year 1826 when Lobatchevsky, to quote Einstein's phrase, *challenged an axiom*, to the present day, much has been heard of the non-euclidean geometries and the Riemannian geometry. Because the latter is the geometry of the general relativity theory, it challenges the interest of all who are trying, sometimes more and sometimes less casually, to understand the developments and implications of this theory.

In order to understand the fundamental properties of Riemann space, let us start with a brief description of the structure of an axiomatic geometry and follow with a discussion of the structure of an analytic geometry. With this preliminary it will be possible to proceed toward the goal which has been set.

Three-dimensional geometry as developed by Euclid is an axiomatic geometry. It starts with three undefined notions called points, lines, and planes for which are assumed certain relations, called postulates or axioms. We need only recall a few of the (according to Hilbert) twenty axioms, *e.g.*, through two distinct points one and only one line can be drawn, a line contains at least two points, *etc.* On the basis of these axioms and his definitions, Euclid built a geometry. An axiomatic geometry is defined as *a set of undefined terms, of axioms and definitions, and their consequences.*

On the other hand *analytic geometry of three-space is a purely arithmetical theory.* We define a point as a triple of numbers (x, y, z) and say that the totality

* Presented for the Slaught Memorial Volume. Read at a meeting of the National Council of Teachers of Mathematics in Atlantic City, February, 1938.

of such triples constitutes a three-dimensional space. In addition we must agree upon a *distance function, i.e.*, we *define* as the distance between two points $P(x, y, z)$ and $P'(x', y', z')$ the *number* obtained by substituting the coördinates of the two points in a chosen formula. Finally we give definitions of plane and line. Our analytical geometry is the logical consequence of this set-up.

You will note that there are no axioms and no undefined terms, but that we can get many analytical geometries by choosing different distance functions, and if we so desire, different definitions of line and plane. As a particular instance of an analytical geometry of three-space let us choose the following as definitions:

(a) The distance PP' is the number $\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}$.

(b) A plane is the totality of points (triples) which satisfy a given linear equation $ax+by+cz+d=0$ where a, b, c, and d (not all zero) are given except for a proportionality factor.

(c) A line is the totality of points (triples) which satisfy two such equations, $ax+by+cz+d=0$ and $a'x+b'y+c'z+d'=0$, where the continued proportion $a:b:c=a':b':c'$ does not hold.

With these agreements it is possible by use of purely algebraic methods to prove all of the axioms of Euclid. And conversely, if we start with our axiomatic geometry, we can prove that points, lines, and planes, hitherto undefined, can be designated by triples of numbers and linear equations. Hence the properties of euclidean three-space are equally well investigated by use of Euclid's axiomatic geometry or by the arithmetical methods of analytical geometry. Recalling our definition of a geometry, we may say that the euclidean geometry of a three-space S_3 is nothing more nor less than the logical consequences of the twenty axioms and that it is nothing more nor less than the logical consequences of defining a point as a triple of numbers and adding the definitions of distance, plane, and line as given above in (a), (b), and (c).

The last statement is equivalent to this: *The two geometries are logically equivalent*, or it is even better to state it thus: *They are two logical representations of one and the same logical system*.

You are doubtless thinking that heuristically the concepts of both structures were motivated by the facts of experience. Indeed the development of analytical geometry by Descartes and Fermat had its origin in the fact that when referred to a rectangular system of coördinates, a point is completely located by three numbers, its coördinates, and that the distance formula comes directly from the theorem of Pythagoras.

This is indeed true but it does not invalidate the contention that an analytical geometry which is logically equivalent to Euclid's geometry may be looked upon as a purely mathematical creation and all of its theorems and their consequences developed without at any time making an application to the world of experience.

Indeed you spare yourself uncomfortable moments if you look upon Euclid's axiomatic geometry as a purely mathematical creation; it is not a simple matter

to attempt to explain any part of it in the world of experience. For in this world there is no point or line or plane, and if we are willing to be guided by our intuition as regards these terms, we may well be puzzled at the necessity for certain of the axioms, for example, according to our intuition surely two distinct points can be joined by one and only one line. Why then did Euclid find it necessary to include this among his axioms? It will take only a cursory examination to convince you of the beauty and perfection of Euclid's axiomatic geometry as a purely mathematical structure.

As we proceed to examine other geometries let us agree first of all to look upon them as logical structures and not let our minds be puzzled as to the possible existence of spaces with the properties described.

You know the history of the famous parallel axiom of Euclid. Given a line g and a point P not on the line, there exists in the plane determined by P and g one and only one line through P which does not intersect g. For approximately twenty centuries mathematicians believed that this statement (you note that it is completely satisfying to your geometric intuition) was capable of being *proved* by use of the remaining axioms but no effort to produce a satisfactory proof was successful. It was Lobatchevsky who first conceived the idea of investigating the logical consequences of postulating all of the relations given in Euclid's axioms except the parallel axiom, for which he substituted one reading "an infinite number of lines can be drawn through P having no point in common with g."

On this foundation he and Bolyai (independently) constructed a geometry which of necessity differs widely from Euclid's geometry. Three of these differences, needed in the sequel, will be stated here. You recall that in Euclid's geometry one of the undefined terms is *line* and there is a theorem that a straight line is the shortest distance between two points. In this non-euclidean geometry the shortest path joining two points, called a *geodesic*, is not a straight line (in the euclidean sense). Furthermore, in a triangle, a figure whose perimeter consists of three geodesics, the sum of the three angles is 180° in Euclid's geometry while in the non-euclidean geometry under discussion in no figure whose perimeter consists of three geodesics is the angle sum exactly 180°; in fact, it is *always less than* 180° differing from this number by very little when the triangle is small but differing more and more as the area of the triangle increases.

Still another notable difference in the two geometries has regard to similar figures. In the new geometry two figures cannot be similar when they have not the same areas.

We may summarize the last paragraphs by formulating a reply to the question, What is a non-euclidean three-dimensional space? The answer is, a non-euclidean three-dimensional space is a system of things called points, lines, and planes with relations which satisfy all of the euclidean axioms except the parallel-axiom which is replaced by a contrary one. Non-euclidean geometry is a set of all logical consequences of this system of axioms.

Whether or not these non-euclidean axioms conform to the so-called space of

observation is of no consequence to the development of the deductive system of non-euclidean geometries.

One question concerning the non-euclidean geometry is certainly pertinent. The novelty of the theorems when compared with the euclidean theorems, may raise doubts in the mind as to whether it may not possibly contain contradictions, *i.e.*, whether from the axioms may not be obtained a theorem which negates another theorem already established. The reply, says Nöbeling,* is that there is just as little likelihood for this in non-euclidean as in euclidean geometry. Indeed Klein succeeded in constructing *within the euclidean three-space* a set of things designated as points, lines, and planes which do not satisfy all of the euclidean axioms.

The non-euclidean geometry which has been described is the so-called *hyperbolic* geometry. A second non-euclidean geometry, called *elliptic* geometry, has been constructed by replacing the parallel axiom by one asserting that through P no line can be drawn which does not meet g. We need not further study the details of these geometries but we raise the significant question: *Are there analytical geometries which are logically equivalent to these geometries*? The answer is affirmative but before they can be described it is desirable to review briefly some of the distinguishing features of a branch of the analytical geometry of euclidean space. This is called *differential geometry* and is used for the study of curves and curved surfaces in euclidean space. It provides methods for the computation of lengths of curves lying on a surface, of angles between such curves, of areas, and other metrical notions. Of especial interest are the geodesic lines on a surface. On a sphere they are arcs of great circles, but on the surface of the earth, for example (not a perfect sphere), if A and B are two points separated by a mountain, the geodesic joining A and B might go around the mountain. The angle sum of a figure made up of three geodesics differs from 180° only if the surface is curved. In this differential geometry a smooth surface F is defined as the totality of all points (triples), (x, y, z), which satisfy a relation

$$F: \qquad f(x, y, z) = 0,$$

where the function f is differentiable as many times as needed with respect to each of the three variables. Such a surface has a tangent plane at every point and the surface can be represented in the neighborhood of a point (stückweise, as the Germans say) by two coördinates, say u and v:

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v).$$

Gauss found that when a piece of a surface is described by two coördinates as above, to each of its points (u, v) can be associated three functions

$$g_{11}(u, v), \qquad g_{12}(u, v), \qquad g_{22}(u, v),$$

whose values, in general, change when the point changes from a given (u, v) to any other (u', v').

* Krise und Neuaufbau in den exakten Wissenschaften, Deutike, Leipzig und Wien, 1933, p. 77.

These functions serve for finding length of arc, angle magnitude, and area. In particular, for two points sufficiently near each other the square of the length of the geodesic arc differs very little from

$$g_{11} \cdot (u - u')^2 + 2g_{12} \cdot (u - u')(v - v') + g_{22} \cdot (v - v')^2.$$

When the equation $f = 0$ is linear, so that its locus is a plane, if we choose for u, v the rectangular cartesian coördinates x, y we find that $g_{11} = g_{22} = 1$ and $g_{12} = 0$, thus giving as the square of the length of the geodesic arc the number

$$(x - x')^2 + (y - y')^2.$$

By use of the g-function we obtain a function $k(u, v)$ whose value for a given point is called the *curvature* of the surface at that point. The value of k differs from zero according as the angle sum of the triangle formed by three geodesic lines differs from 180°. We note the special cases:

1) When F is a plane, $k = 0$.

2) When F is a sphere with radius r, $k = 1/r^2$, *i.e.*, k is constant and *positive.*

3) If F is a surface of *everywhere positive curvature*, the angle sum of any geodesic triangle of F is *greater than* 180°.

4) If F is a surface of *everywhere negative curvature*, the angle sum is *less than* 180°.

Riemann extended these considerations of differential geometry to higher dimensions and it is precisely this extension which gives us the space which we name Riemann space. In R_4 lengths of arcs, angle magnitudes, areas, and volumes of pieces of a three-dimensional solid M can be computed by the use of six functions g_{11}, g_{22}, g_{33}, g_{12}, g_{13}, g_{23}, functions of three coördinates.

Through an arbitrary point P we can construct a surface which contains the geodesic lines through P of our manifold M. This surface F has a curvature at P. If the manifold M has the property that this curvature is constant as the point P varies on M and its geodesic surface F varies with P, the Riemann manifold (or Riemann space) M is said to be a space of constant curvature k. In particular, if M is a euclidean S_3, $k = 0$, since the geodesic surfaces are planes. Also Riemann spaces with constant negative (positive) curvature are hyperbolic (elliptic), *i.e.*, the geodesic lines all satisfy the axioms of hyperbolic (elliptic) geometry.

We can call a Riemann space with constant positive curvature a spherical space. It is the three-dimensional analog of the surface of a sphere. On it, two geodesic lines have *at most one* common point. These Riemann spaces of constant curvature constitute only a narrow special case of the general notion of Riemann space. In general, we call a Riemann space *curved* if triangles formed of geodesic lines have angle sums different from 180°.

To summarize, then, in analytical geometry a four-space is the totality of all quadruples of numbers. A 3-dimensional Riemann space (or Riemann manifold) contained within this space is a triple-infinitude of these points; in particular it may be the totality of points which satisfy a single equation

$$f(x, y, z, t) = 0,$$

with suitable differentiability properties. It can be described in pieces by three coördinates and for it we have given *six* functions of these coördinates which may be used to compute lengths, angle magnitudes, areas, volumes, *etc.*

In an analogous manner a Riemann n-space in a euclidean space of $N>n$ dimensions is defined. The geometry of a Riemann space of whatever dimension is merely the logical consequences of this set-up. In particular, a change in one or more of the g-functions makes a change in the geometry.

Let us now return to a consideration of the relation of these logical systems to our world of experience. In this we have serious handicaps some of which have already been mentioned. If we are to apply our geometries to observations you may make your own choice as to what you are going to look upon as a point, but if we need a straight line of considerable length, perhaps a light ray will be the best choice. We cannot, of course, *observe* any thing which will be analogous to a three-dimensional manifold immersed in a four-space though we know that an event needs a quadruple of numbers for its description, three for its space location and one for the time.

We cannot even derive comfort by attempting to generalize the experiences of a flat-lander whose home is on a plane immersed in a three-space because, granting that the flat-lander does not know that there is a third dimension and consequently does not know any of its properties, *we* have this knowledge and can therefore interpret the things which to him are mysterious.

Let us pursue this a bit further. Imagine a rectangular Cartesian reference frame, say in your home town, and let the flat-landers be our shadows on the ground, $z=0$. Their homes are the shadows of our homes, rectangles or polygons. If a small object is within a certain one of these rectangles, it is not possible for the flat-lander to remove it to a position outside of the rectangle without tearing out a section of the (one-dimensional) wall. But you, a three-dimensional creature, can pick up the object from the plane $z=0$, lift it to the plane say $z=1$, move it in this plane to a suitably chosen new position, then lower it again to $z=0$. As a matter of observation of the flat-lander, the object suddenly disappears from sight and as suddenly and mysteriously reappears in another position.

These motions can be explained quite simply in the language of the analytical geometry of three-space.

A second example is not so simple. Our flat-lander has a glove on his right hand, the shadow of your right-hand glove. He cannot put it on his left hand. If you rotate this glove 180° about an axis through the wrist and any finger, it is now a perfect fit for his left hand though it is certainly not congruent to your left-hand glove. All of these motions can be explained by the analytical geometry of three-space, but the explanation is purely analytical, *i.e.*, arithmetical and algebraic, not physical.

If Sir Oliver Lodge or any of the believers in spiritualism could convince us that our space is immersed in a physical four-space, it would at least divert us

to try to imagine physical explanations for many of the phenomena which are clearly explained algebraically by the analytical and differential geometries of four-space. For example, a chain consisting of two or more solid links cannot be unlinked in three-space without breaking or filing some of the links. A string of finite length in which a simple knot has been tied after which the ends of the string have been fastened together, cannot be unknotted in three space without cutting the string or separating the ends. In the analytical geometry of four-space, the chain can be unlinked without severing any link and the string can be unknotted without separating the ends provided that the chain and the string were actually contained in a three-space within the four-space.

Since there is absolutely no evidence of a four-dimensional physical space, why not be content to accept an analytical explanation of a fact not within the possibility of observation as an *analytical* explanation and not attempt to find a physical explanation?

On the other hand we are certainly justified in seeing how far we can go in fitting observed data to our geometries provided that we agree to be open minded and examine our data in their relation to each of our geometries in turn.

One of the distinguishing features of the euclidean and non-euclidean geometries was found to be the angle sum. If you draw a triangle with a ruler on a sheet of paper and measure the angles with a protractor, while you do not obtain precisely 180° for the angle sum you certainly obtain this number within reasonable limits of error. But in hyperbolic geometry we found that the angle sum differs from 180° very little for small triangles. We raise the question, how large must the triangle be before we can detect a measurable difference between the angle sum and 180°? This question occurred to Gauss (1777–1855) who sought an answer by choosing three mountain peaks as the vertices of his triangle and light rays as the sides. His result was 180° within reasonable limits of error, again an inconclusive result since, while he did not prove that euclidean geometry does *not* hold neither does his result justify the conclusion that hyperbolic geometry does not hold. The triangle which he used must surely be called a small triangle when compared with one whose vertices are fixed stars.

It appears that after this experiment of Gauss, the question was shelved until it became of importance in the general theory of relativity, a theory in which it is assumed that the world of experience is a three-dimensional Riemann space.

We have noted that the properties of such a space depend on the choice of the g_{ik}-functions and you would expect that these functions in the present case would be chosen in such a manner that with a given aggregation of mass points (subject only to the law of gravitation), the geodesic lines would be light rays. A complication arises from the fact that the masses in our universe do not form a static system. Suppose that the functions g_{ik} have certain values in the neighborhood of a point P today but that tomorrow a heavy mass has come quite near to P. The values of g_{ik} cannot be constant but must depend on the time in such a manner that the formulas in use do not fail to determine the geodesics.

Analytically, as already stated, the Riemann space is assumed to be a three-dimensional totality of points, *quadruples*, contained within the totality of all quadruples, *i.e.*, contained within a four-dimensional euclidean space whose general point (quadruple) is given by (x, y, z, t), which in the physical application consist of three position coördinates and one time coördinate.

Moreover, since the geodesic lines and light rays coincide and since the geodesic lines are affected by a change in the distribution of the mass, it follows that light rays are affected by a change in the distribution of mass; in other words, a light ray from P to P' would not describe the same path today and tomorrow if in the meantime a heavy mass had moved close to a point of today's path.

This is a prediction from the differential geometry capable of verification by observation and fortunately for the proponents of the theory they did not have to wait long for an opportunity to test it. At the time of a solar eclipse it was definitely confirmed by observation both quantitatively and qualitatively that the light ray sent out by a fixed star was bent in the force field of the sun. An important conclusion from this is that in our universe there are triangles whose sides are light rays but whose angle sum is not 180°. Hence we conclude that the space of experience is curved. Furthermore, the difference between 180° and the angle sum of a triangle with given vertices is not constant but varies as large masses approach or recede from its sides. Hence in the world of experience, not only does the euclidean geometry not hold, but neither does the hyperbolic nor the elliptic.

A conscientious objector now comes forward with the question: Is it not possible to make an equally plausible explanation of such phenomena by assuming that the space is truly euclidean but that in this space light rays are not straight? Indeed this is a possibility which has been studied. The consequence of this assumption is that since light rays do not move along the shortest lines but along highly complicated curves, we obtain an extremely complicated theoretical physics. Further, we should be left in the world of experience (assumed euclidean) with no analog of a geometrical line, an entity which plays a part of much importance in geometry. Thus we can find in our world of experience absolutely no observed illustrations of the relations given by the twenty axioms. Would it not seem paradoxical to call such a space euclidean?

In contradistinction to this situation, we find that the logical consequence of assuming that our space of observation is a Riemann space is a set of simple and elegant formulas, the formulas of the relativity theory. Since we have agreed in advance that we are not going to permit ourselves to fall into the error of accepting *any* theory as a *correct* theory in the sense that its conclusions are to be accepted without question for situations not capable of observation, it is surely reasonable that as far as it is valid we choose the theory which provides the simplest formulas and the easiest explanations.

There is another phase of the relativity theory which is a stumbling block for the uninitiated. This is the assertion that the Riemann space is closed, *i.e.*, un-

bounded (unbegrenzt) and finite. What is meant by the statement that a space is closed? This is a term whose definition comes to us from the youngest branch of geometry, topology. I shall first give you an example of a closed surface (two-dimensional) and of a surface which is not closed, this for the sake of your intuition, and shall then formulate a precise definition which may be generalized for spaces of arbitrary dimension.

The surface of a sphere is a closed surface, it has no boundary (is unbounded) and is finite since the distance between two arbitrary points (measured along a geodesic) does not exceed a certain fixed number. A circular disk is not closed. It is finite but it has a boundary.

Definition. A surface is called *closed* if it can be cut into a finite number of pieces in such a manner that each bounding arc belongs to exactly two pieces and if the distance between two arbitrary points does not exceed a certain fixed number.

Since after a closed surface has been cut in this manner there is no *free* bounding arc (*i.e.*, no bounding arc which is found on only one piece), the closed surface is called unbegrenzt. We may equally apply the term "closed surface" to a surface made up of a finite number of pieces provided that we can find a scheme for associating the bounding arcs in advance so that each will belong to exactly two pieces.

In the light of this definition let us examine the euclidean plane. It can indeed be cut into pieces in such a manner that every bounding edge belongs to exactly two pieces but there will not be a finite number of these pieces if each piece is to have the property that the distance between two arbitrary points has an upper bound. Hence the euclidean plane is not a closed surface.

The same definition with appropriate change in wording applies to a three-dimensional space. It is closed if it can be cut into a finite number of pieces (of three-space) in such a way that every bounding surface appears on exactly two space pieces. If it is closed, it is unbegrenzt and finite, but of course not conversely.

The relativity theory assumes that the astronomical space of experience is closed. Since each of the finite number of pieces of this space has a finite diameter (*i.e.*, two arbitrary points have a finite distance), it follows that the same is true of two arbitrary points in the space and, consequently, the number of stars is finite, and the total mass is finite.

Again the conscientious objector comes forward with the question: Since you assert that space is finite, should it not be possible to come to the end? He is forgetting that the space is *closed*. Let us transfer our two-dimensional flat-lander to the surface of a sphere and see what experiences might befall him which are not possible on the open euclidean plane. Let him draw a small triangle with arcs of great circles as sides (they appear to him to be straight lines on the surface, since they are short). He finds that the angle sum is approximately 180° and thinks that he is living on a two-dimensional euclidean plane, provided that the flat-lander is very small as compared with the surface of the sphere.

But suppose that these flat-landers are venturesome and go on extensive exploring trips. If they measure large triangles they will find that they are dwelling on a curved surface because the angle sum of a large triangle is greater than 180°. Suppose that they build telescopes—two dimensional ones, of course. All light rays entering the telescopes will come from two-dimensional objects on the sphere and will travel from the object to the lens on a geodesic. Suppose that they succeed in building a telescope sufficiently refined to bring in a light ray from the most distant object in their world. Our flat-lander is amazed to find that he is looking at the other end of himself.

Do you wish to extend the analogy and ask if such a thing is possible for you? Could you imagine a telescope sufficiently powerful to bring to you a view of the other side of the earth, a view carried by a light wave completely around our universe? The answer to that is that the estimated diameter of our universe is 150 milliards of light years so that if you see the other side of the earth you will see it as it was 150 milliards of light years ago.

And now for fear that the preceding paragraphs have made you forget some earlier remarks I shall quote some remarks by G. A. Bliss published in 1933 in the AMERICAN MATHEMATICAL MONTHLY and reprinted in *A Mathematician Explains*.

> The theory of general relativity as applied to the solar system is already threatened, even before its usefulness has been completely established. Recently, when Einstein and DeSitter were sojourning in the United States, the newspapers reported them as discussing still newer theories from which it might be concluded that the universe is finite. This conclusion should never be made. All that we could be justified in saying is that the data which we have concerning the distances and motions of the stars are in closer accord with a theory which is finite than with others which have not this property. The finiteness which can be demonstrated is a property of the theory and not of the universe.

If it has amused you to have the imagined experiences of the flat-lander used to demonstrate the properties of curvature, of being curved and of finiteness, you might wonder what would happen to him if he were placed on a surface whose properties are topologically different. Let us move him to a Möbius band, and let us assume that he is a right-handed creature and wears his heart on the left side just as we do. As stated earlier, as long as he stays near home and measures small triangles, he may well think that he dwells on a euclidean plane and that euclidean geometry holds in his world. But when he ventures farther afield, he will soon learn that angle sums are not always 180°. The amazing experience however will come to him who ventures to walk, say that he remains constantly within sight of the edge of his world and keeps it always on his right. If he travels long enough he will return to his home and family, but what a return. To those who have remained at home, he appears to be left-handed and to wear his heart on the right side. He himself knows full well that he has undergone no anatomical change and it seems to him that all of the stay-at-homes are now left-handed and have their hearts on the right side.

We have an analogy to this on the earth. The man who goes around the

world without living through one day twice or else omitting a day can have quite an argument with those who stayed at home about the calendar. We avoid all such trouble by establishing an International Date Line whose use you understand. On the Möbius band, uniformity would be attained if at some point on the band a line should be drawn across the band and if it were decreed that any person crossing this line should from that moment interchange the words "right" and "left."

THE PERIODIC FUNCTION OBTAINED BY REPEATED ACCUMULATION OF A STATISTICAL SERIES

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1. Introduction. It has been noted as a remarkable fact that in treating a random statistical series by a certain iterative process a periodic function results.* This conclusion was based on calculations with certain series which were assumed to be random. In the calculations discrete series were used but the conclusion was stated for the case of a function of a continuous variable.

It is the object of the present paper to show why this periodic function is obtained. For simplicity the case of a continuous variable will be treated, so that the theory of Fourier series is available. It will be shown that the iterative process leads to a multiple of the first non-vanishing term of a Fourier series development of the function. It will be found that the process has no special significance for a *random* series, leading, as it does, to similar results for a wider class of functions. The conclusion appears more as a characterization of the iterative process than of the statistical series. To show that the iterative process is particularly valuable one would need to show that it gives the first term of the Fourier series more readily than do other methods. Its practical significance in an application of statistics to economics seems doubtful since the periodic function obtained depends so largely on the choice of the interval over which the data extend.

2. The iterative process. Let $x(s)$ be a function (statistical series) defined on the interval $-a \leqq s \leqq a$. We assume that the function (series) is referred to its arithmetic mean, that is, that

$$\int_{-a}^{a} x(s)ds = 0.$$

Instead of assuming that $x(s)$ is a random series we assume that it is representable by a Fourier series which converges at every point of the interval $(-a, a)$:

* First pointed out by Dr. Herbert E. Jones in a paper entitled, "The theory of runs applied to time series," presented at the Third Annual Research Conference of the Cowles Commission. Noted by H. T. Davis, Mathematical adventures in social science, this MONTHLY, vol. 45, 1938, p. 102; see note on page 105 by the present writer.

$$x(s) = A_1 \cos\frac{\pi s}{a} + A_2 \cos\frac{2\pi s}{a} + \cdots$$

$$(1) \qquad + B_1 \sin\frac{\pi s}{a} + B_2 \sin\frac{2\pi s}{a} + \cdots$$

$$= C_1 \cos\left(\frac{\pi s}{a} - \alpha_1\right) + C_2 \cos\left(\frac{2\pi s}{a} - \alpha_2\right) + \cdots,$$

where for $j = 1, 2, 3, \cdots$,

$$C_j = \sqrt{A_j^2 + B_j^2}, \quad \cos\alpha_j = A_j/C_j, \quad \sin\alpha_j = B_j/C_j.$$

By definition the *first accumulation* of $x(s)$ on $(-a, a)$ is

$$(2) \qquad y_1(s) = \int_{-a}^{s} x(s)ds - \frac{1}{2a}\int_{-a}^{a}\int_{-a}^{t} x(s)dsdt = S(x(s)),$$

where S is the linear operator

$$S = \int_{-a}^{s} - \frac{1}{2a}\int_{-a}^{a} dt \int_{-a}^{t}.$$

We observe that $y_1(s)$ is referred to its arithmetic mean, since

$$\int_{-a}^{a} y_1(s)ds = 0.$$

We now form a sequence of accumulations $y_1(s), y_2(s), \cdots$ by repeated application of the operator S, which we call *the iterative process*:

$$(3) \qquad y_{k+1}(s) = S(y_k(s)), \qquad k = 1, 2, 3, \cdots.$$

3. The conjecture and the theorem. It has been conjectured by others that when $x(s)$ is a random series

$$y_k(s) \sim A \cos\left(\frac{\pi s}{a} + \alpha_n\right),$$

that is, that $y_k(s)$ approaches a cosine form with period $2a$ when k is increased indefinitely. We shall prove the following

THEOREM. *If $x(s)$ is a function representable by a Fourier series convergent at every point of the interval $(-a, a)$ the first non-vanishing term of which is $C_n \cos(n\pi s/a - \alpha_n)$, then*

$$y_k(s) \sim A \cos\left(\frac{n\pi s}{a} - \alpha\right),$$

that is, the accumulations approach a cosine form with period $2a/n$ when k is increased indefinitely.

4. Proof of the theorem. One readily finds from the definition of the operator S that for any positive integer q

$$S\left(C_q \cos\left(\frac{q\pi s}{a} - \alpha_q\right)\right) = \frac{a}{q\pi} C_q \sin\left(\frac{q\pi s}{a} - \alpha_q\right)$$

$$= \frac{a}{q\pi} C_q \cos\left(\frac{q\pi s}{a} - \left(\alpha_q - \frac{\pi}{2}\right)\right). \tag{4}$$

It follows that

$$y_1(s) = \frac{a}{\pi} C_1 \cos\left(\frac{\pi s}{a} - \left(\alpha_1 - \frac{\pi}{2}\right)\right) + \frac{a}{2\pi} C_2 \cos\left(\frac{2\pi s}{a} - \left(\alpha_2 - \frac{\pi}{2}\right)\right)$$

$$+ \frac{a}{3\pi} C_3 \cos\left(\frac{3\pi s}{a} - \left(\alpha_3 - \frac{\pi}{2}\right)\right) + \cdots,$$

and for $k = 1, 2, 3, \cdots$

$$y_k(s) = \left(\frac{a}{\pi}\right)^k C_1 \cos\left(\frac{\pi s}{a} - \left(\alpha_1 - \frac{k\pi}{2}\right)\right) + \left(\frac{a}{2\pi}\right)^k C_2 \cos\left(\frac{2\pi s}{a} - \left(\alpha_2 - \frac{k\pi}{2}\right)\right)$$

$$+ \left(\frac{a}{3\pi}\right)^k C_3 \cos\left(\frac{3\pi s}{a} - \left(\alpha_3 - \frac{k\pi}{2}\right)\right) + \cdots \tag{5}$$

$$= \left(\frac{a}{\pi}\right)^k \left\{C_1 \cos\left(\frac{\pi s}{a} - \alpha_{1k}\right) + \frac{1}{2^k} C_2 \cos\left(\frac{2\pi s}{a} - \alpha_{2k}\right)\right.$$

$$\left. + \frac{1}{3^k} C_3 \cos\left(\frac{3\pi s}{a} - \alpha_{3k}\right) + \cdots\right\},$$

where

$$\alpha_{1k} = \alpha_1 - \frac{k\pi}{2}, \quad \alpha_{2k} = \alpha_2 - \frac{k\pi}{2}, \cdots.$$

It seems fairly obvious that the factors $1/2^k, 1/3^k, \cdots$, which multiply the second, third, $\cdots$ terms respectively will for large values of k render the sum of those terms negligible compared with the first term if this term is not zero, so that if $C_1 \neq 0$,

$$y_k(s) \sim \left(\frac{a}{\pi}\right)^k C_1 \cos\left(\frac{\pi s}{a} - \alpha_{1k}\right),$$

or if $C_1 = C_2 = \cdots = C_{n-1} = 0$, $C_n \neq 0$, then

$$y_k(s) \sim \left(\frac{a}{n\pi}\right)^k C_n \cos\left(\frac{n\pi s}{a} - \alpha_{nk}\right).$$

A proof may be given by showing that if we write for the latter (general) case

$$y_k(s) = \left(\frac{a}{n\pi}\right)^k C_n \left\{ \cos\left(\frac{n\pi s}{a} - a_{nk}\right) + R_k \right\},$$

then for k sufficiently large R_k is uniformly small on the interval $(-a, a)$. Now we may write

$$R_k = \frac{C_{n+1}}{C_n}\left(\frac{n}{n+1}\right)^k \cos\left(\frac{(n+1)\pi s}{a} - \alpha_{n+1k}\right) + \frac{C_{n+2}}{C_n}\left(\frac{n}{n+2}\right)^k \cos\left(\frac{(n+2)\pi s}{a} - \alpha_{n+2k}\right) + \cdots .$$

Since the Fourier series (1) is convergent, the C's have a maximum absolute value C, and hence throughout the interval $(-a, a)$

$$|R_k| \leqq \frac{Cn^k}{|C_n|} T,$$

where

$$T = \frac{1}{(n+1)^k} + \frac{1}{(n+2)^k} + \cdots \leqq \int_0^\infty \frac{dx}{(n+x)^k} = \frac{1}{(k-1)n^{k-1}}.$$

It follows that uniformly on the interval $(-a, a)$

$$\lim_{k\to\infty} |R_k| \leqq \lim_{k\to\infty} \frac{Cn}{|C_n|(k-1)} = 0,$$

and our theorem is a consequence.

FRACTIONS*

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Perhaps the author owes an apology to the reader for asking him to lend his attention to so elementary a subject, for the fractions to be discussed in this paper are, for the most part, the halves, quarters, and thirds of arithmetic. But the fact is that the writer has, for some years, been looking on these entities in a somewhat new way. Here will be found a geometric picturization which will be novel to the reader and which will supply a visual representation of arithmetical results of diverse kinds.

The idea of representing a fraction by a circle is one at which the author arrived by an exceedingly circuitous journey. It began with the Group of Picard. In the geometric treatment of this group as carried out by Bianchi in accordance with the general ideas of Poincaré certain invariant families of spheres appear. These spheres, which are found at the complex rational fractions, are mentioned later in this paper. They suggest analogous known invariant families of circles at real rational points in the theory of the Elliptic Modular Group in the com-

* Some of the material of this article was presented in an address before the American Mathematical Society at Lawrence, Kansas, November 28, 1936. Other parts have been given in lectures at the Rice Institute, the University of Texas, Northwestern University, and the Armour Institute.

plex plane. Finally, it became apparent that this intricate scaffolding of group theory could be dispensed with and the whole subject be built up in a completely elementary fashion. It is this treatment that is undertaken here.

1. The geometric representation. We begin with real fractions. These are usually represented, along with real irrationals, by points on a line. Let this line be the x-axis in an xy-plane of rectangular coördinates.

Through each rational point $x=p/q$, where p and q are integers and the fraction is in its lowest terms, we construct a circle of radius $1/(2q^2)$ tangent to the x-axis and lying in the upper half-plane. It is this circle, which touches the x-axis at the point usually taken to represent the fraction, which will be the geometric picture of the fraction. The integers are represented by circles of radius $1/2$; the fractions $1/3$, $2/3$, $4/3$, etc., by circles of radius $1/18$; and so on. Every small interval of the x-axis contains points of tangency of infinitely many of these circles.*

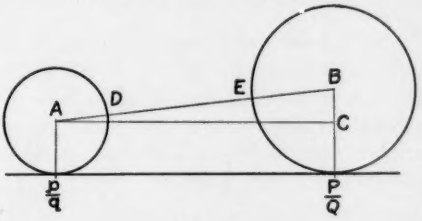

FIG. 1

Let p/q and P/Q be two different fractions in their lowest terms. Consider the distance between the centers of their representative circles. In the figure the horizontal distance AC is $\left|(P/Q)-(p/q)\right|$ and the vertical distance CB is the difference of the radii, $\left|(1/2Q^2)-(1/2q^2)\right|$. We have

$$\begin{aligned} AB^2 &= \left(\frac{P}{Q}-\frac{p}{q}\right)^2+\left(\frac{1}{2Q^2}-\frac{1}{2q^2}\right)^2 \\ &= \left(\frac{1}{2Q^2}+\frac{1}{2q^2}\right)^2+\frac{(Pq-pQ)^2-1}{Q^2q^2} \\ &= (AD+EB)^2+\frac{(Pq-pQ)^2-1}{Q^2q^2}. \end{aligned}$$

* Families of circles tangent to the real axis at the rational points are intimately involved in the geometry of the modular group and in the theory of quadratic forms. In these connections they have been used for some time. Circles defined as in the text except for the more general radius $1/(2hq^2)$ were used, with various values of h, by the present author in papers in the Proceedings of the Edinburgh Mathematical Society in 1917. They were probably used earlier by others.

These circles are called "Speiser circles" by some writers. This name appears to be due to a note of a dozen lines by A. Speiser in the Actes de la Société Helvétique des Sciences Naturelles in 1923. Mention should also be made of a very interesting ninety-page booklet by Züllig, Geometrische Deutung unendlicher Kettenbrüche (1928).

If $|Pq-pQ|>1$, then $AB>AD+EB$, and the two circles are wholly external to one another.

If $|Pq-pQ|=1$, then $AB=AD+EB$, and the two circles are tangent.

If $|Pq-pQ|<1$ we have, since $Pq-pQ$ is an integer, that $Pq-pQ=0$. From this $P/Q=p/q$, which is contrary to the assumption that the fractions are different. So this last inequality is not possible. We may state the following result:

THEOREM 1. *The representative circles of two distinct fractions are either tangent or wholly external to one another.*

2. Adjacent fractions. We shall call two fractions p/q and P/Q *adjacent* if their representative circles are tangent. The condition for this is that $|Pq-pQ|=1$. (We assume here and henceforth that fractions appear in their lowest terms.) We shall prove

THEOREM 2. *Each fraction p/q possesses an adjacent fraction.*

That integers P and Q exist satisfying the equation $|Pq-pQ|=1$ (or, arranging signs suitably, satisfying $Pq-pQ=1$) is a fundamental proposition of the theory of numbers. A great many proofs have been given. For the sake of completeness we prove the theorem here. The proof given is so arranged as to cover the case of complex fractions which will appear subsequently in this paper.

The proof is by induction. Clearly the theorem holds for $|q|=1$, since $p/1$ then has the adjacent fraction $(p+1)/1$. To prove the theorem in general we assume that all fractions whose denominators are less in absolute value than $|q|$ possess adjacent fractions and prove it for the fraction p/q.

Let n be the integer nearest to p/q, whence we can write, with m integral,

$$\frac{p}{q}=n+\frac{m}{q}=\frac{nq+m}{q}, \qquad 0<|m|<|q|.$$

Since $|m|<|q|$, then q/m has an adjacent fraction r/s, so $|sq-rm|=1$. Then the fraction

$$\frac{P}{Q}=n+\frac{s}{r}=\frac{nr+s}{r}$$

is adjacent to p/q; for

$$|Pq-pQ|=|(nr+s)q-(nq+m)r|=|sq-rm|=1.$$

This establishes the theorem.

We can now give a formula for all fractions adjacent to p/q.

THEOREM 3. *If P/Q is adjacent to p/q then all fractions adjacent to p/q are*

$$\frac{P_n}{Q_n}=\frac{P+np}{Q+nq},$$

where n takes on all integral values.

We find readily that the fractions given are adjacent to p/q, for

$$|(P+np)q - p(Q+nq)| = |Pq - pQ| = 1.$$

We find that P_n/Q_n and P_{n+1}/Q_{n+1} are adjacent to one another, since

$$|[P+np][Q+(n+1)q] - [P+(n+1)p][Q+nq]| = |Pq - pQ| = 1.$$

The circles corresponding to these fractions form a ring around the circle of p/q, all tangent to the circle of p/q, and each tangent to the circles which precede and follow it in the sequence (Figure 2). That this ring completely sur-

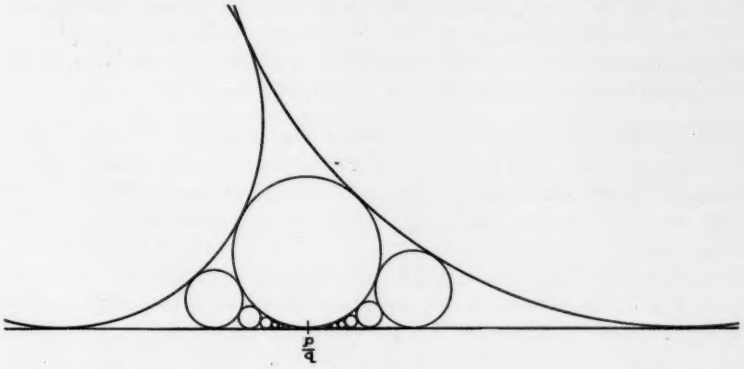

FIG. 2

rounds the circle of p/q we see from

$$(a) \qquad \frac{P_n}{Q_n} = \frac{p}{q} + \frac{Pq - pQ}{q(Q+nq)} = \frac{p}{q} \pm \frac{1}{q^2(n+Q/q)}.$$

When $n \to +\infty$, P_n/Q_n approaches p/q from one side; when $n \to -\infty$, P_n/Q_n approaches p/q from the other side.

It is obvious from the geometric picture that it is not possible to draw a circle lying in the upper half-plane, touching the x-axis, and tangent to the circle of p/q but not intersecting the circles of the ring surrounding the circle of p/q. It follows that there are no further fractions adjacent to p/q.

THEOREM 4. *Of the fractions adjacent to p/q ($|q| > 1$) exactly two have denominators numerically smaller than q.*

That two circles of the preceding ring about the circle of p/q are larger than that circle is fairly evident from the geometrical picture. We see also that

$$|Q+nq| < |q|, \qquad \text{or} \qquad |n+Q/q| < 1$$

for exactly two values of n; namely, those integers between which $-Q/q$ lies.

For one of these two values of n, $n+Q/q$ is positive, for the other, negative. We see then from (a) that one of the fractions of Theorem 4 is greater than p/q, the other is less.

We shall add one further circle to our collection. The fraction $1/0$ is formally adjacent to the integers $p/1$, since $1\cdot 1-p\cdot 0=1$. We take as its circle the line $y=1$, which touches the circles of all the integers. We consider as the interior of the circle that part of the plane above the line.

3. The mesh triangles. The parts of the upper half plane exterior to all the circles of the system consist of an infinite number of circular arc triangles to which the name of "mesh triangles" will be given. Any two sides of a mesh triangle lie on circles belonging to adjacent fractions. Its angles are zero.

A further study of the mesh triangle will be made in a later section, where its properties will be used in the theory of approximation.

4. Farey's series. Let a curve be drawn across the set of circles that we have just defined. We start with a point A_0 of the upper half-plane and trace a continuous curve L which remains in the upper half plane except that its terminal point, if any, may possibly lie on the x-axis. We consider the fractions whose circles are passed through in succession by L. Each circle K is surrounded by mesh triangles. If L issues from K into one of these triangles and if it does not return to K it will on leaving the triangle pass in general into a circle tangent to K. We agree that K shall not be counted twice if L passes out of K and then returns to K again without entering another circle. We also make the convention that if L touches two circles at their point of tangency without entering either then one or the other shall be considered as crossed by L (the larger circle, say, or the one on the left, or the one whose fraction is the greater). We can then state as an established proposition the following:

PRINCIPLE. *If two circles of the system are penetrated in succsssion by L then the corresponding two fractions are adjacent.*

As a first illustration let L be a line, $y=k$, parallel to the x-axis, starting say at a point on the positive y-axis and stopping at $x=1$. The points of tangency with the x-axis of the circles through which L passes are arranged in order from left to right; that is, the corresponding fractions are arranged in numerical order. If $1/(n+1)^2<k<1/n^2$, L intersects the circles of all fractions in the interval $0\leqq x\leqq 1$ whose denominators do not exceed n and the circles of no other fractions. These fractions arranged in numerical order constitute what is known as a Farey's series of order n for the interval.

Certain theorems concerning Farey's series are now readily established.* Let $p/q<p'/q'$ be successive fractions of the series, all the numbers being positive.

* See, for example, Landau, Vorlesungen über Zahlentheorie, 1927, Band I, pp. 98–100. Farey discussed these fractions more than a century ago.

(1) $p'q-pq'=1$. This results from the Principle stated earlier in this section.

(2) $q+q' \geqq n+1$. For $(p+p')/(q+q')$ is a fraction between p/q and p'/q'; since it does not belong to the series its denominator is greater than n.

(3) For any number ω of the interval there is a fraction p/q of Farey's series of order n such that

$$\left| \frac{p}{q} - \omega \right| \leqq \frac{1}{(n+1)q}.$$

The number ω lies in an interval formed by a Farey's fraction p/q and an adjacent fraction $(p+p_1)/(q+q_1)$ not of the series (p_1/q_1 being the Farey's fraction preceding or following p/q). The length of this interval is $1/(q+q_1)q$, where $q+q_1 \geqq n+1$.

If we take other simple forms for the curve L we get series analogous to Farey's. Thus if we take a line with a positive slope starting above $y=1$ and terminating at the origin we have all non-negative fractions such that

$$pq \leqq n,$$

where n is suitably chosen. If these be arranged in order of numerical magnitude then successive fractions satisfy (1) above; (2) is replaced by $(p+p')(q+q') \geqq n+1$.

The series such that $0<p \leqq n$ may be got by taking for L the arc of a suitable circle tangent to the x-axis at the origin.

5. The problem of approximation. Dirichlet showed by elementary means that if ω is a real irrational number then the inequality

$$\left| \frac{p}{q} - \omega \right| < \frac{k}{q^2} \tag{1}$$

is satisfied by infinitely many fractions p/q if $k=1$. If, however, $\omega=r/s$, a rational, then the inequality is satisfied by only a finite number of rationals p/q however large k be chosen. For

$$\left| \frac{r}{s} - \frac{p}{q} \right| = \left| \frac{rq-ps}{sq} \right| \geqq \frac{1}{|sq|} = \frac{\left| \frac{q}{s} \right|}{q^2} > \frac{k}{q^2}$$

except for a finite number of values of q for which $|q| \leqq |ks|$; also for each q there is clearly only a finite number of fractions satisfying the inequality.

Various suites of fractions approximating to ω satisfy (1) with suitable k. The convergents in the ordinary continued fraction for ω satisfy the inequality with $k=1$. A suite of Hermite admits the smaller value $k=1/\sqrt{3}$.

The determination of the best value of k is due to Hurwitz,* who proved the following theorem.

* Mathematische Annalen, vol. 39, 1891, pp. 279–284. See also Borel, Journal de Mathématiques, 5th ser., vol. 9, 1903, pp. 329 ff; L. R. Ford, Proceedings of the Edinburgh Mathematical Society, vol. 35, 1917.

THEOREM 5. *If $k=1/\sqrt{5}$, then for each irrational ω there are infinitely many fractions p/q satisfying the inequality* (1).

If $k<1/\sqrt{5}$, then there are irrationals ω such that only a finite number of fractions p/q satisfy (1).

Let the curve L of the previous section be a vertical line which terminates at the irrational ω. Then L intersects infinitely many circles of the system we are considering and is tangent to none. If L cuts the circle of p/q then the distance from ω to p/q is less than the radius:

$$\left|\frac{p}{q}-\omega\right|<\frac{1}{2q^2}.$$

The curve L thus provides us with an infinite suite of all those fractions satisfying (1) when $k=1/2$. If p/q and p'/q' are successive fractions of this suite then, from the Principle of the preceding section, we have $p'q-pq'=\pm 1$. Also, we see geometrically that $|q'|>|q|$. The convergence of the suite to ω is then immediately proved.

We propose now to prove Hurwitz' theorem by a study of the circles of our system. This elementary proof will be free of continued fractions on the one hand and of the theory of the Modular Group on the other.

The line L cuts across infinitely many mesh triangles. The first part of the theorem of Hurwitz is a consequence of the following result which will now be proved:

THEOREM 6. *Of the three fractions whose circles form the boundary of a mesh triangle which L crosses, at least one satisfies* (1) *with $k=1/\sqrt{5}$.*

We consider the mesh triangle in some detail. Let P/Q, p/q, p_1/q_1 be the fractions whose circles bound the area, where

$$0<Q\leqq q<q_1=q+Q,\quad p_1=p+P.$$

We suppose, to fix the picture, that the largest of the three circles is on the right; that is, $P/Q>p/q$, whence $Pq-pQ=+1$. If not, we could make suitable changes of sign in the following; or, more simply, we could reflect the entire figure in the y-axis by changing the signs of the numerators and of ω and at the conclusion of our analysis we could reflect again.

The vertex A (see Figure 3) is the point of contact of the circles of p/q and P/Q. It divides the line of centers of these circles in the ratio of their radii, $1/2q^2:1/2Q^2$, or $Q^2:q^2$. The abscissa of this point is found by an early formula of analytic geometry, or by the methods of high school plane geometry, to be

$$a=\frac{q^2\cdot\dfrac{p}{q}+Q^2\cdot\dfrac{P}{Q}}{q^2+Q^2}=\frac{pq+PQ}{q^2+Q^2}.$$

The abscissas of B and C, the other vertices, may be written down by an interchange of letters,

$$b = \frac{pq + p_1q_1}{q^2 + q_1^2}, \qquad c = \frac{PQ + p_1q_1}{Q^2 + q_1^2}.$$

In order that L cross the mesh triangle under consideration it is necessary and sufficient that ω lie in the interval whose right end is c and whose left end

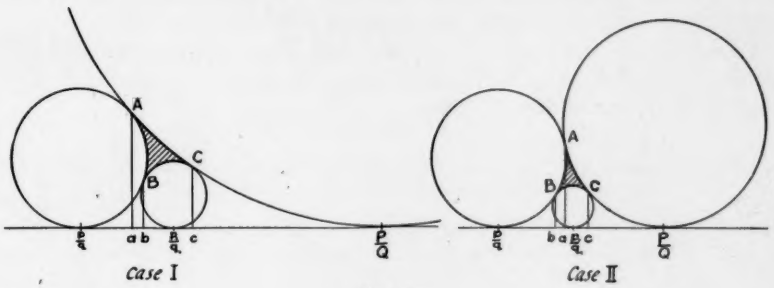

FIG. 3

is the lesser of the two quantities a and b. We find

$$a - \frac{p}{q} = \frac{Q}{q(q^2 + Q^2)}, \qquad b - \frac{p}{q} = \frac{q_1}{q(q^2 + q_1^2)},$$

whence subtracting and remembering that $q_1 = q + Q$, we have

$$b - a = \frac{q^2 - qQ - Q^2}{(q^2 + Q^2)(q^2 + q_1^2)}.$$

We put $q/Q = s \geqq 1$. The sign of $b - a$ is (dividing the preceding numerator by Q^2) the same as the sign of $s^2 - s - 1$. Now, factoring,

$$s^2 - s - 1 = \left(s + \frac{\sqrt{5} - 1}{2}\right)\left(s - \frac{\sqrt{5} + 1}{2}\right),$$

and the square root of 5 first enters the picture. Here the first factor is positive and the sign is determined by the second factor. We have two cases to consider.

Case I. $a < b$, and * $s > \frac{1}{2}(\sqrt{5}+1)$. We shall show that P/Q satisfies the inequality (1) with $k = 1/\sqrt{5}$. We have

$$\left| \frac{P}{Q} - \omega \right| < \frac{P}{Q} - a = \frac{q}{Q(q^2 + Q^2)} = \frac{s}{s^2 + 1} \cdot \frac{1}{Q^2}.$$

* Here and in many subsequent relations the sign of equality is not possible, for one member is rational and the other is irrational.

If now

$$\frac{s}{s^2+1} > \frac{1}{\sqrt{5}}$$

we have

$$s^2 - \sqrt{5}\,s + 1 < 0, \qquad \left(s - \frac{\sqrt{5}+1}{2}\right)\left(s - \frac{\sqrt{5}-1}{2}\right) < 0,$$

which is impossible since both factors in the first member are positive. It follows that $s/(s^2+1) < 1/\sqrt{5}$, and the inequality is satisfied.

Case II. $b < a$, and $s < \frac{1}{2}(\sqrt{5}+1)$. We shall show that p_1/q_1 satisfies the inequality (1) with $k = 1/\sqrt{5}$. It is clear that c is nearer p_1/q_1 than b is, since C is higher on the circle than B. Then

$$\left|\frac{p_1}{q_1} - \omega\right| < \frac{p_1}{q_1} - b = \frac{q}{q_1(q^2+q_1^2)} = \frac{s(s+1)}{s^2+(s+1)^2}\cdot\frac{1}{q_1^2}.$$

If

$$\frac{s(s+1)}{s^2+(s+1)^2} > \frac{1}{\sqrt{5}},$$

then

$$(\sqrt{5}-2)s^2 + (\sqrt{5}-2)s - 1 > 0,$$

or, dividing,

$$s^2 + s - \sqrt{5} - 2 > 0.$$

Factoring,

$$\left(s - \frac{\sqrt{5}+1}{2}\right)\left(s + \frac{\sqrt{5}+3}{2}\right) > 0,$$

which is impossible since the first factor is negative and the second is positive.

We have proved that one, at least, of the three fractions whose circles bound the mesh triangle satisfies (1) with $k = 1/\sqrt{5}$. Since L passes through infinitely many mesh triangles it follows that infinitely many rational fractions satisfy the inequality with $k = 1/\sqrt{5}$.

It remains to exhibit an irrational ω for which (1) holds for only a finite number of fractions for $k < 1/\sqrt{5}$. We take, in fact, $\omega = \frac{1}{2}(\sqrt{5}+1)$ and any $h < 1$ and show that there is a finite number of fractions for which

$$\left|\frac{p}{q} - \frac{\sqrt{5}+1}{2}\right| < \frac{h}{\sqrt{5}q^2}. \tag{2}$$

For a fraction satisfying this we may write

$$\frac{p}{q} - \frac{\sqrt{5}+1}{2} = \frac{\theta}{\sqrt{5}q^2},$$

where $|\theta| < h < 1$. Writing this

$$\frac{p}{q} - \frac{1}{2} = \frac{\sqrt{5}}{2} + \frac{\theta}{\sqrt{5}q^2},$$

squaring, and rearranging, we get

$$5q^2[(p^2 - pq - q^2) - \theta] = \theta^2.$$

The expression in square brackets must be positive, whence the integer $p^2 - pq - q^2$, which cannot be zero (since then p/q would turn out irrational) must equal or exceed 1. We have

$$q^2 = \frac{\theta^2}{5[(p^2 - pq - q^2) - \theta]} < \frac{h^2}{5(1 - h)}.$$

This limits q to a finite number of values. For each q, the inequality (2) then limits p; and the number of fractions is finite.

Other irrationals might have been used here; for example, $\omega = (r\sqrt{5} + s)/t$, where r, s, t are integers. These last irrational numbers are found in every interval of the x-axis.

6. Continued fractions. One of the most successful interpretations supplied by our system of circles is the geometric picture of the continued fraction

$$(3) \qquad a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}}.$$

or, as often written,

$$a_0 + \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \cdots$$

Here the quantities a_n are integers. If all the integers are positive, except possibly a_0, the continued fraction is *simple.**

The nth *convergent*, p_n/q_n, is the quantity that remains when all that part of the expression following a_{n-1} is erased. We have

$$\frac{p_1}{q_1} = \frac{a_0}{1}, \qquad \frac{p_2}{q_2} = a_0 + \frac{1}{a_1} = \frac{a_0a_1 + 1}{a_1},$$

$$\frac{p_3}{q_3} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} = \frac{(a_0a_1 + 1)a_2 + a_0}{a_1a_2 + 1} = \frac{a_2p_2 + p_1}{a_2q_2 + q_1}.$$

* So called by Chrystal in whose Algebra, Part II, pp. 396ff, will be found one of the best elementary treatments of continued fractions in English.

Here and in the following we are using the p's and q's to represent the actual numerators and denominators of the fractions in the second members and not merely numbers proportional thereto.

The last equation exhibits for $n=2$ the recurrence formula from which we calculate the convergents step by step,

$$\frac{p_{n+1}}{q_{n+1}} = \frac{a_n p_n + p_{n-1}}{a_n q_n + q_{n-1}}. \tag{4}$$

This formula is readily established by induction. It holds, we have just seen, for $n=2$; assuming it up to any later n we prove it for $n+1$. We get p_{n+2}/q_{n+2} from p_{n+1}/q_{n+1} by replacing a_n by a_n+1/a_{n+1};

$$\frac{p_{n+2}}{q_{n+2}} = \frac{(a_n + 1/a_{n+1})p_n + p_{n-1}}{(a_n + 1/a_{n+1})q_n + q_{n-1}} = \frac{a_{n+1}(a_n p_n + p_{n-1}) + p_n}{a_{n+1}(a_n q_n + q_{n-1}) + q_n} = \frac{a_{n+1}p_{n+1} + p_n}{a_{n+1}q_{n+1} + q_n}.$$

This is the required formula with n replaced by $n+1$ and (4) holds for $n \geqq 2$. It holds also for $n=1$ if we take $p_0/q_0 = 1/0$.

If the continued fraction terminates, its value is the last convergent. If there are infinitely many a's the value to be assigned to the fraction is $\lim p_n/q_n$ if this limit exists.

We investigate the suite of convergents

$$\frac{p_0}{q_0}\left(=\frac{1}{0}\right),\ \frac{p_1}{q_1}\left(=\frac{a_0}{1}\right),\ \frac{p_2}{q_2},\ \frac{p_3}{q_3},\ \ldots.$$

THEOREM 7. *Successive convergents, p_n/q_n and p_{n+1}/q_{n+1}, are adjacent fractions.*

From (4)

$$p_{n+1}q_n - q_{n+1}p_n = (a_n p_n + p_{n-1})q_n - (a_n q_n + q_{n-1})p_n = -(p_n q_{n-1} - q_n p_{n-1}).$$

Since $p_1q_0 - p_0q_1 = a_0 \cdot 0 - 1 \cdot 1 = -1$ we have that

$$p_{n+1}q_n - q_{n+1}p_n = (-1)^{n+1},$$

which establishes the theorem.

Given the tangent circles of p_{n-1}/q_{n-1} and p_n/q_n and the integer a_n, how do we find the circle of p_{n+1}/q_{n+1}? For all integral a_n we get from (4) (see Theorem 3, Section 2) the ring of tangent circles around the circle of p_n/q_n. For $a_n=0$ we get the circle of p_{n-1}/q_{n-1}; for $a_n=1$ we get a circle next to this; for $a_n=2$ we get the next around the ring; and so on. For $a_n=-1, -2,$ *etc.* we pass around in the opposite direction.

To determine the direction consider

$$\frac{p_{n+1}}{q_{n+1}} - \frac{p_n}{q_n} = \frac{(-1)^{n+1}}{q_{n+1}q_n} = \frac{(-1)^{n+1}}{(a_n q_n + q_{n-1})q_n} = \frac{(-1)^{n+1}}{q_n^2\left(a_n + \dfrac{q_{n-1}}{q_n}\right)}.$$

For a_n large and positive the last denominator is positive. If $n+1$ is even then p_{n+1}/q_{n+1} is to the right of p_n/q_n and we have counted around the circle of p_n/q_n in a clockwise direction; if $n+1$ is odd p_{n+1}/q_{n+1} is less than p_n/q_n and we have counted in a counter-clockwise direction.

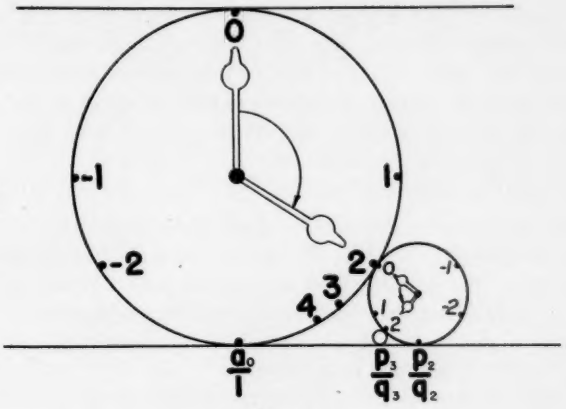

FIG. 4

We may liken the circle of p_n/q_n to the face of a clock (Figure 4). Set the hand to point toward the center of the circle of p_{n-1}/q_{n-1}. It is now in the zero position. If we seek an even convergent turn clockwise to the a_nth point of tangency; if we seek an odd convergent turn counter-clockwise to the a_nth point of tangency. If a_n is negative the turning is in the opposite direction in each case. The hand now points to the center of p_{n+1}/q_{n+1}.

Initially ($n=1$) we set the hand to point vertically upward to the point of tangency of $y=1$ with the circle of $a_0/1$, as in the figure. The positive direction is clockwise. On the next clock the positive direction is counter-clockwise; on the next, clockwise; and so on. In the figure $a_1=2$, $a_2=2$, and if $a_0=0$ the first four convergents, starting with p_0/q_0, are

$$\frac{1}{0}, \quad \frac{0}{1}, \quad \frac{1}{2}, \quad \frac{2}{5}.$$

We see then that *the circles of the convergents, in order, of a continued fraction form a sequence, or chain, of circles. The chain begins with the circle $y=1$ and each circle is tangent to the circles preceding and following it in the sequence.*

Conversely, any such chain has a unique corresponding continued fraction. We determine a_0 as the integer whose circle touches the first circle, $y=1$, in the chain; and a_2, a_3, etc., are got uniquely by counting points of tangency, as already explained.

A chain can be defined in many ways and with much arbitrariness. A simple

way is to use a curve L, starting above $y=1$, as explained in Section 4. We shall not object to $a_n=0$, which gives $p_{n+1}/q_{n+1}=p_{n-1}/q_{n-1}$, nor to $q_n=0$, which means that $y=1$ reappears as a circle of the chain.

The preceding picture answers various questions about convergence. We see that it is possible so to choose the chain that p_n/q_n approaches a prescribed rational or irrational limit in a great variety of ways, or so that p_n/q_n becomes positively or negatively infinite, or so that p_n/q_n wanders more or less arbitrarily over the x-axis. We may even choose the chain so that each interval of the x-axis contains infinitely many convergents. Such a chain is the set of circles passed over by the broken line L formed by joining in succession the points $A_1, A_2, \cdots$ where A_n has the coördinates $[(-1)^n n, 1/n]$. In this example each real rational number appears infinitely often among the convergents.

7. Simple continued fractions.* The situation is quite different if all $a_n (n>0)$ are positive integers. The first clock runs right (see Figure 4) and makes p_2/q_2 greater than p_1/q_1; the next runs left and places p_3/q_3 between p_1/q_1 and p_2/q_2. Each convergent falls between the two immediately preceding it. Certain facts then appear:

(a) The fraction converges to a value ω.

(b) The odd convergents increase monotonically to ω, the even convergents decrease monotonically to ω.

(c) ω lies between each pair of successive convergents.

(d) $q_{n+1} > q_n \qquad (n > 1)$.

(e) $$\left| \frac{p_n}{q_n} - \omega \right| < \frac{1}{q_n q_{n+1}} < \frac{1}{a_n q_n^2} \leqq \frac{1}{q_n^2}.$$

Here the second expression is the distance between the convergents p_n/q_n and p_{n+1}/q_{n+1}, and $q_{n+1}=a_n q_n+q_{n-1}>a_n q_n \ (n>1)$.

(f) $$\left| \frac{p_n}{q_n} - \omega \right| > \frac{a_{n+1}}{q_n q_{n+2}},$$

for the second member is the distance between p_n/q_n and p_{n+2}/q_{n+2}.

We see geometrically how to find the simple continued fraction for a given ω. We select for a_0 the integer next below ω (see Figure 4). We then turn the clock to the right until p_2/q_2 lies as far to the left as possible without passing ω, thus finding a_1. We turn the second clock to the left until p_3/q_3 is as far to the right as possible without passing ω, whence we have a_2; and so on.

That the development is unique is apparent. The first a_n's that differ in two developments throw the values of the two continued fractions into different intervals thereafter. There is an unimportant exception in the case of a fraction which terminates. A last $a_n>1$ may be written $(a_n-1)+1/1$, thus inserting an additional convergent.

* Züllig, *loc. cit.*, has the chain of circles given here with figures for many numerical cases.

It is not difficult to identify this process of forming the continued fraction for ω with the usual arithmetic process: (1) taking a_0 as the largest integer in ω; (2) forming ω_1, the reciprocal of the remainder $\omega - a_0$, and taking a_1 as the largest integer in ω_1; (3) taking a_2 as the largest integer in ω_2, the reciprocal of $\omega_1 - a_1$; and so on.

We see geometrically that p_n/q_n is a good approximation to ω if a_n is very large, for our clock has then turned far toward the bottom. This is clear, of course, from (e) above. Thus in the continued fraction

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}}}}}},$$

the second convergent 22/7 and the fourth 355/113 (here $a_4 = 292$) are especially good and are, in fact, much used.

Suppose we take $a_n = 1$ for all n in order to avoid a good approximation as far as we can. We have

$$\omega = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cdots}} = 1 + \frac{1}{\omega},$$

or $\omega^2 - \omega - 1 = 0$. From this, since $\omega > 1$,

$$\omega = \tfrac{1}{2}(\sqrt{5} + 1).$$

This is the irrational quantity we used in the latter part of the proof of Hurwitz's theorem.

8. Complex fractions.* Complex numbers $x + iy$, where x and y are real and $i = \sqrt{-1}$, are commonly represented by points (x, y) in an xy-plane using rectangular coördinates. The complex integers, $n = n' + in''$, where n' and n'' are real integers, form a square lattice in the plane. A fraction is the quotient of two integers

$$\frac{p}{q} = \frac{p' + ip''}{q' + iq''} = \frac{p' + ip''}{q' + iq''} \cdot \frac{q' - iq''}{q' - iq''} = \frac{p'q' + p''q''}{q'^2 + q''^2} + i\,\frac{p''q' - p'q''}{q'^2 + q''^2}$$

and is represented by the point with the coördinates shown in the last member. The complex fractions thus consist of the numbers $x + iy$, where x and y are real fractions. We think of the xy-plane as being horizontal, and we introduce a third or z-axis perpendicular to the x- and y-axes.

The analogue of the circle of Section 1 is a sphere. Through the point in the plane which represents the fraction p/q (in its lowest terms) we construct a sphere touching the complex plane there, lying in the upper half-space, and having the radius $1/(2q\bar{q})$. Here $\bar{q}$ is the conjugate of q, $\bar{q} = q' - iq''$; whence

* For the matters treated in this section see L. R. Ford, Transactions of the American Mathematical Society, vol. 19, 1918, pp. 1–42.

$q\bar{q}=q'^2+q''^2=|q|^2$. This sphere will be the geometric representation of p/q. Every small area in the complex plane contains points of tangency of infinitely many of these spheres.

Interpreting Figure 1 as a pair of spheres at complex fractions, we have

$$AB^2=\left|\frac{P}{Q}-\frac{p}{q}\right|^2+\left(\frac{1}{2Q\bar{Q}}-\frac{1}{2q\bar{q}}\right)^2=\left(\frac{1}{2Q\bar{Q}}+\frac{1}{2q\bar{q}}\right)+\frac{|Pq-pQ|^2-1}{Q\bar{Q}q\bar{q}}.$$

If $|Pq-pQ|>1$, then $AB>AD+EB$, and the spheres are wholly external to one another. If $|Pq-pQ|=1$ the spheres are tangent. In no case do the spheres intersect. We call the fractions *adjacent* if the spheres are tangent.

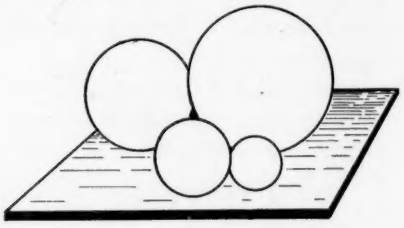

FIG. 5

That p/q has an adjacent fraction P/Q we already know, for we have designed the proof of Theorem 2 to cover this case. Then the fraction

$$\frac{P_n}{Q_n}=\frac{P+np}{Q+nq}$$

of Theorem 3, where n is any complex integer, is adjacent to p/q. We note that the four fractions P_m/Q_m, where $m=n+1$, $n-1$, $n+i$, $n-i$, are adjacent to P_n/Q_n; for $|(P+np)(Q+mq)-(P+mp)(Q+nq)|=|m-n|\cdot|Pq-pQ|=1$. That is, each sphere tangent to a given sphere touches four other spheres tangent to the given sphere. It is not difficult to show that the fractions P_n/Q_n constitute the complete set of fractions adjacent to p/q.

How many of the spheres touching the sphere of p/q are larger than that sphere? We require the number of solutions of

$$|Q+nq|<|q|, \qquad \text{or} \qquad |n+Q/q|<1.$$

Now $|n+Q/q|$ is the distance from the integer n to the point $-Q/q$. The problem takes the form: If a unit circle is drawn in the complex plane, how many complex integers does it contain? For various positions of the center $(-Q/q)$ the answer is 2, 3, or 4, as is readily seen. We put aside the case $|q|=1$, which puts the center at an integer, when clearly there are no larger tangent spheres. We introduce the plane $z=1$ to touch the spheres of the integers and be the sphere of their adjacent fraction 1/0.

The use of complex integers in the continued fraction (3) offers no difficulty. We are led to the geometric picture of the continued fraction as a chain of spheres, beginning with the plane $z=1$, and proceeding thence from sphere to tangent sphere. We have here the same possibilities of convergence or divergence as before. At the worst we can set up continued fractions whose convergents are present in every small region of the complex plane. There is no immediate generalization of the simple continued fraction to the complex case. There are, however, schemes for developing a complex number in a continued fraction, the best processes being due to Hurwitz.*

The analogue of Hurwitz's theorem on rational approximations to an irrational number was first discovered by the present author.† The inequality (1) is replaced by

$$\left| \frac{p}{q} - \omega \right| < \frac{k}{q\bar{q}},$$

and the minimum k for which infinitely many fractions always satisfy the inequality is $k=1/\sqrt{3}$. The proof of this, however, is not elementary.

MATHEMATICAL EDUCATION

EDITED BY C. A. HUTCHINSON, University of Colorado

This Department of the MONTHLY *has been created as an experiment to afford a place for the discussion of the place of mathematics in education. With this topic will naturally be associated other matters emphasizing the educational interests of those who teach mathematics. It is not intended to take up minute details of teaching technique. The columns are open to those who have thoughtful critical comment to make, be it favorable or adverse to the cause of mathematics. The success of this department obviously will depend upon the cooperation of the readers of the* MONTHLY. *Address correspondence to Professor C. A. Hutchinson, University of Colorado, Boulder, Colorado.*

MATHEMATICAL EDUCATION IN GERMANY BEFORE 1933

RICHARD COURANT, New York University

At the suggestion of the editors I shall try to give a brief account of trends in the teaching of mathematics in Germany, particularly at German universities, in the period from the World War until 1933.

The situation of mathematics in Germany, since the early part of the 19th century, has been pivoted around a definite connection between university and high school. In a sweeping reform following the French Revolution, the institution of the German "humanistisches Gymnasium" was established. Teachers at these institutions were required to undergo a very thorough academic preparation, and the task of preparing them was entrusted to the philosophical faculties of the universities. The teachers' training was in no way of an elementary char-

* Acta Mathematica, vol. 11, 1887, pp. 187–200.

† L. R. Ford, Transactions of the American Mathematical Society, vol. 27, 1925, pp. 146–154.

acter. First rate mathematical scholars—following the example of classical philologists—gathered a crowd of enthusiastic students and lectured on the most advanced subjects on the boundary line of research. The pace was set by Jacobi, who discussed in his class, ten hours a week, his newest discoveries on elliptic functions.

The requirements of the state board examination for high school teachers were excessively high; they called for full-fledged scholars and probably could never be enforced even approximately. Still, published yearly reports of the better high schools sometimes contained important mathematical contributions from the staff, as for instance the famous paper on Abelian functions by Weierstrass. A considerable number of the famous German mathematicians of the 19th century started their careers as more or less successful teachers in high schools.

For the development of mathematics at universities it certainly was fortunate that a rather stationary and responsive audience was assured. But the young teacher, coming from the university to the school, which he had left five or six years ago, found little use for his specialized knowledge of Abelian functions or other sophisticated subjects, while he had more or less forgotten the material to be covered in the high school curriculum.

However, on the whole, the system worked quite satisfactorily as long as the number of teachers and of high school students was comparatively small, and the latter somewhat restricted to a group seriously striving for a higher scholastic education. But gradually this situation changed with the progress of democratization in education. New types of high schools sprang up, drawing away from the rigorous humanistic ideal and tending more to useful lines. After the war of 1870 this process led to an enormous increase in the demand for teachers, not all of whom were able to attain high scholastic standards. But nothing was changed in the fundamental system of four to six years of graduate study as preparation for school teachers. The number of mathematics chairs at universities increased considerably; even more the number of students in the courses. The contradictory character of the relationship between university and high school became a burning question. What could be done to preserve the university standards and still meet the real needs of a broadening school system?

Under Felix Klein's inspiring leadership progress was made in bridging the gap from both sides. The curriculum of the high schools was modernized. The concepts of function and of limit, the elements of calculus, analytic geometry, projective geometry, *etc.*, conquered much space in the high school curriculum, superseding antiquated routine material. On the other hand, in the university, survey courses were introduced, Felix Klein's lectures on elementary mathematics from a higher viewpoint being the outstanding example. It was very fortunate that also other academic teachers of highest rank, *e.g.*, Hilbert, followed this example enthusiastically. Hilbert's courses in foundations of geometry and in intuitive geometry, and his lectures on the problem of squaring the circle are famous instances.

The process of mutual adaptation was interrupted by the World War, the aftermath of which created intensified problems. The number of high school students leaped to unknown heights. A desperate demand for large numbers of teachers was voiced officially. Thus the classrooms of the universities became overcrowded with students who were not driven by thirst for higher learning but attracted by the prospect of a secure job. Soon the teaching jobs were filled, but the inert stream of students in the universities preparing for the teaching profession continued.

At the University of Berlin in some years more than 600 students registered for a single course in number theory, and about as many in calculus and analytic geometry. At Göttingen courses on the theory of functions of a complex variable had an attendance of up to 330 students. In highly specialized and advanced courses 50–100 students could often be found, many of whom were also preparing for a teaching job. While at the high schools the reform work proceeded organically, the universities were facing a more complex problem. How could they give the masses a really worth-while instruction without hopelessly lowering the general level and without giving up the ideals of academic freedom?

The complexity of the situation was enhanced by different factors: While the great majority of mathematics students still was preparing for the state board examination for teachers (incidentally at the same time studying two minor subjects, usually physics and another science), an increasing number of students was considering other professions, *e.g.*, the actuarial career, and at the principal mathematical centers there was always a group of deeply interested young scholars, partly from abroad, with outspoken academic ambitions. This smaller elite group was apt to form a separate stratum in closer contact with the staff and rather separated from the bulk of the student body. Naturally the professors were more interested in this top group than in the crowd.

Another difficulty arose from the traditional academic freedom in learning. There were no official examinations and no marks all through the years between high school and the state board examination (except examinations for tuition scholarships). It happened that candidates, after five years of attending advanced classes and having had little contact with instructors and fellow students, came up for examination, and to the dismay of both sides it turned out that they knew literally nothing about mathematics and had stupidly wasted their years at the university. The absolute freedom for the student to select his courses and to conduct his studies to his own taste was sometimes a handicap for the majority, although of the highest benefit to the top group. A similar effect came from the academic freedom in teaching. The professor was absolutely free in the selection and presentation of his material, not bound to any curriculum. Thus it actually happened that elementary calculus was taught in a class of several hundred students on the basis of Brouwer's intuitionism, a very interesting experiment for the lecturer and three or four students but less useful for the others.

During the period after 1919 the technological institutes ("Technische Hochschulen") began to compete with the universities in the task of training high school teachers in mathematics and physics. The mathematics classes at these institutions were terribly overcrowded by the enormous number of prospective engineers. But the professors of mathematics were anxious to have also some students interested in mathematics as such, and the technological institutes succeeded in being opened also for teacher training. However, this competition of the technological institutes did not amount to much.

Since there was little uniformity in the different universities, I shall, in some detail, describe only the situation which had developed at Göttingen.

The beginner, entering the university and facing a confusing variety of possibilities, found some advice in a printed guide and, if he desired, could consult with assistants assigned to advisory work. He was not bound to heed suggestions, but usually in the first year he took differential and integral calculus, 4 hours a week, analytic geometry, also 4 hours, and the *Anfängerpraktikum*, practical exercises for beginners, 3–5 consecutive hours a week. (In addition a 5 hour lecture course on experimental physics, and other courses.) The courses never followed closely a definite textbook. The students were supposed to take and work out notes or to study the official record of the lectures in the reading room. The level of the calculus courses corresponded more or less to an advanced calculus course at American universities.

It was a strict rule in the science faculty that these beginners' courses be given by experienced, responsible professors and not by young instructors. But a course was hardly ever repeated by the same professor in consecutive years. Thus entirely different, even antagonistic tendencies in presentation were offered to the student, who was urged to supplement his information by independent study of the literature easily accessible in the large mathematical reference library. While the lecture courses followed more or less the old tradition of separation between the talking professor and the passive audience, the five hour period of practical exercises was an important innovation for coping with the problem of mass instruction. The students, more than 200, were given in advance a mimeographed sheet with problems, some mere exercises, some requiring a certain amount of inventive thinking. They were supposed to come to class after having already given some thought to the problems, while the professor in charge, prior to the class meeting, held a conference with a group of assistants (about one for every 15–20 students) concerning the problems, the different methods and aspects of the solutions, and the probable difficulties of the students. The class room consisted of two large halls with big tables and blackboards, and the 5 hours (or less if the student preferred) were spent not only in finding the solutions, but mainly in personal discussion between students and assistant instructors. There was ample opportunity for anyone who cared to clarify any subject connected with the beginners' courses in discussion with competent people. The solutions were later written up by the students, handed in, criticized in writing by a conference of the assistants, and returned

and individually discussed at the next meeting. Records with marks were taken. This institution proved astonishingly efficient. A considerable percentage of the students really learned something; others realized that it was best for them to give up mathematics, and soon the few with talents were discovered and given particular consideration. Forming of study groups among the students was encouraged, and collusion in solving the problems was not discouraged.

After the beginners' courses, which were sometimes supplemented by a course on descriptive geometry, the period of the normal full courses, "Kursusvorlesungen," started. A great variety was offered here: Theory of functions of a complex variable, differential equations, functions of real variables, potential theory, partial differential equations of physics, analytic dynamics, algebra, elementary number theory, differential geometry, projective and non-euclidean geometry, calculus of variations, group theory, mechanics of continuous bodies, and many other subjects were covered. The average student took two or three courses a term, altogether often more than 12 of these before his examination. Theory of functions of a complex variable was always included, but generally there was much leeway left in the selection.

In the same category belonged the full courses on elementary mathematics from a higher viewpoint, and other survey courses. Many of these "Kursusvorlesungen" again were attended by hundreds of students, and again an attempt was made to reach at least a part of them personally in additional two hour periods of exercises ("Übungen"), often held by one or more assistants. Here problems of a more advanced type were discussed, sometimes in writing, but not on such a systematic basis as in the beginners' "Praktikum."

At Göttingen this method of supplementing lecture courses by "Übungen" for larger groups of students was made possible by the institution of the mathematical assistant. Before the war, assistantships in mathematics hardly existed except for a few such positions at Göttingen and Berlin. Future professors were recruited from the ranks of the "Privatdozenten," voluntary academic teachers admitted to teaching only after a very careful process of selection, but not paid except by student fees. They came mostly from the moderately well-to-do middle class and were more or less able to support themselves. Only at some larger institutions, like Göttingen and Berlin, did income from student fees suffice to support a few of the less wealthy "Privatdozenten," and, in certain exceptional cases, government fellowships, *etc.*, made academic careers possible for promising students without means. After the destruction of middle class wealth by war and inflation, it became imperative to put the financial care for the next academic generation on a more systematic basis. Thus most "Privatdozenten" were paid some modest salary by making them assistants or giving them some official duties in the teaching process, and it was only by this definite innovation that the problem of instruction for larger masses could be tackled. At the university of Göttingen, for example, where, before the war, only one full-time and one or two part-time assistants in mathematics existed, there were in 1929 more than 10 full-time assistants, all actual or prospective "Privatdozenten," and in

addition an indefinite number of part-time assistants according to the needs of the "Praktikum" and "Übungen." Generally also these part-time assistants were fully trained young scholars with the doctor's degree and some record of independent research.

After a student had taken a few of the "Kursusvorlesungen," he usually started attending some courses of the highest category, so-called special lectures and seminars. These were mostly two hour courses, and covered an enormous variety of subjects according to the personal taste and interest of the professor or instructor. Very often these special courses concerned the particular field in which the instructor was doing research and led directly to or beyond the boundary line of general knowledge. It was mostly in connection with these special courses and seminars that actual research was done by teachers and students. Little special consideration was given to students who were merely sitting in and did not cooperate. Also in these courses and seminars, a considerable number of prospective school teachers were active. The state board examination required written theses and very often these theses originated in such advanced courses or seminars and were the nuclei of doctoral theses. Subjects and standards of these state board theses varied greatly. A few examples: Reciprocity theorems in number theory; Morse's relations for stationary points; Morse's theory of conjugate points in the calculus of variations; the role of semicontinuity in the calculus of variations; theory of characteristics for hyperbolic systems of differential equations; characteristic values of partial differential equations in their dependence on the domain; finding explicit formulas for the conformal mapping of given Riemann surfaces on plane domains; critical survey of certain current high school textbooks of mathematics from the viewpoint of mathematical correctness; and so on.

Not infrequently these theses contained some modest original contribution, and in most cases they at least proved that the student had studied some field with real understanding. Sometimes, it is true, results were very poor and unsatisfactory.

At other universities different attempts were made to cope with the mass problem; *e.g.*, at the University of Berlin there was a systematic organization of many smaller study groups of students under the guidance of advanced students and younger instructors. At smaller universities with a much smaller number of students and in the absence of a coherent group of ambitious future scholars, it was easier to maintain personal contact between responsible professors and a large portion of the student body. But almost nowhere was it possible to restore the idyllic pre-war situation, where the whole mathematics group at a university was somewhat like a family with mutual personal confidence and contacts. Where this mathematical "family life" was preserved and even developed after the war, as in Göttingen, it was really restricted to the comparatively small, though by no means exclusive, top group. This class division could not fail to create among the less fortunate majority the psychological atmosphere of an inferiority complex, which so easily becomes dangerous for scientific in-

stitutions and which was, in connection with the hopeless outlook regarding jobs, an important factor in the events at the universities in 1933.

While the period between the war and 1933 did not, in spite of honest attempts, solve burning problems of mathematical instruction at the universities, it certainly produced more mathematical progress than most previous periods of similar length.

Since 1933, the total revolution within the system of German universities has so drastically curtailed the number of students that the mass problem of the previous period simply no longer exists. It seems premature to forecast the trend of the future development of mathematics at German universities.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University, Providence, Rhode Island

The department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

A LINKAGE FOR DESCRIBING CURVES PARALLEL TO THE ELLIPSE

R. C. YATES, University of Maryland

Any point P of a bar AB with its ends moving along two perpendicular lines describes an ellipse. If this bar be joined at its midpoint to the origin (the intersection of the lines) by another bar, the half of the original bar not containing P may be discarded.

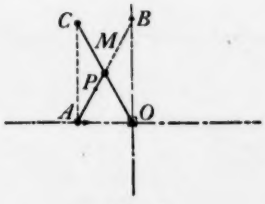

FIG. 1

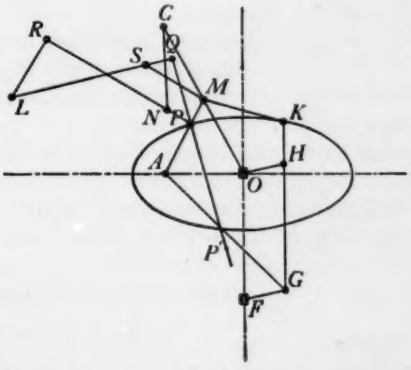

FIG. 2

In any position of the jointed bars the direction of motion of M in Fig. 1 is perpendicular to OM while A has always a horizontal direction. Thus the instantaneous center of rotation of the bar AM is the point C where the vertical line through A meets OM extended. The normal to the ellipse at P then must

always pass through C. Note that C is carried around at the extremity of the bar OC which is twice the length OM; also that $OM = MA$.

If we now add two further bars CN and PN such that $MP = NP$ and $MC = NC$ we have only to arrange for another bar attached at P to bisect the angle MPN. This may be done by a special arrangement of two Hart contraparallelograms as shown in Fig. 2.

The linkage $AGKMOH$ was devised by Kempe in 1874. Here $AG = KG = 4$; $AM = KM = OM = 2$; $KH = OH = FG = 1$. If O and F be fixed to the plane so that FG is parallel to OH then A moves along a line at right angles to the fixed line OF. Thus P describes the ellipse. One contraparallelogram $RLQP$ is attached to another $QSMP$ such that $RL = QP = MS = \lambda$; $QL = PR = \mu$; $QS = MP = \nu$; and $\lambda^2 = \mu \cdot \nu$.

For this arrangement QP bisects angle NPM and is directed therefore to C. It is in all positions normal to the ellipse and any fixed point of it such as P' describes a parallel curve.

This set of curves is of the eighth order and fourth class and has been studied extensively by Catalan, Cauchy, Breton, Cayley, and others. If we let $MP + MO = a$, $AP = b$ be the semi-axes of the ellipse and c be the parallel distance PP' (where P' is on the "inner" side of the normal) then the curves have the following characteristics:*

1. $c < b^2/a$, oval-shaped.
2. $c = b^2/a$, two triple points on major axis.
3. $b > c > b^2/a$, four real cusps and two double points on major axis.
4. $c = b$, four cusps and contact point at center with major axis as double tangent.
5. $a > c > b$, four cusps.
6. $c = a$, four cusps and minor axis as double tangent.
7. $a < c < a^2/b$, four cusps and two double points on minor axis.
8. $c = a^2/b$, two triple points on minor axis.
9. $c > a^2/b$, oval-shaped.
10. If $2a > c > 2b$, the parallel curve cuts the ellipse; if $c = 2a$, it is tangent; and for $c > 2a$, the curve lies wholly outside.
11. Two curves generated by points P' and P'' such that $PP' = PP'' = c$ have for the difference of their lengths the quantity $4\pi c$.†

CONCERNING A THEOREM ON MATRICES

W. V. Parker, Louisiana State University

I have shown (*Bulletin of the American Mathematical Society*, vol. 38, 1932, pp. 259–262.) that if M is a real symmetric matrix of order greater than four in which all principal minors of orders *one* and *four* are zero, then M is of

* See, for example, G. Loria: Ebene Kurven, Teubner, 1902.

† Dienger, Über die Rektification und Quadratur der Toroide, Archiv der Mathematik, vol. 9, 1847.

rank three or less. One may naturally inquire if the corresponding theorem is true for Hermitian matrices. The purpose of this note is to show by an example that the theorem is not true for Hermitian matrices in general.

In the Hermitian matrix

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 4 \\ 1 & 1 & 0 & 1-\sqrt{3}\,i & 7+2\sqrt{3}\,i \\ 1 & 1 & 1+\sqrt{3}\,i & 0 & 3+2\sqrt{3}\,i \\ 1 & 4 & 7-2\sqrt{3}\,i & 3-2\sqrt{3}\,i & 0 \end{pmatrix}$$

all principal minors of order four are zero but the matrix is of rank five.

RECENT PUBLICATIONS

EDITED BY TOMLINSON FORT, Lehigh University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

Theory of Equations. By J. M. Thomas. First Edition. New York and London, McGraw-Hill Book Company, 1938. 10+211 pages. $2.00.

Introduction to the Theory of Equations. By Louis Weisner. New York, The Macmillan Company, 1938. 9+188 pages. $2.25.

Einführung in die neueren Methoden der Differentialgeometrie. By J. A. Schouten and D. J. Struik. Zweiter Band: Geometrie von D. J. Struik. Zweite vollständig umgearbeitete Auflage. Groningen-Batavia, P. Noordhoff N.V., 1938. 12+338 pages. Fl. 11.50, geb. Fl. 12.50, RM 16, geb. RM 17.50.

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Funktionentafeln mit Formeln und Kurven. By E. Jahnke and F. Emde. Third Revised Edition. Leipzig and Berlin, Teubner, 1938. 12+306 pages. RM 15.

Tests of Significance in Reversal or Switchback Trials. By A. E. Brandt. (Research Bulletin 234, Iowa Agricultural Experiment Station.) Ames, Iowa, Iowa State College, 1938. 27 pages.

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Descriptive Geometry. By F. A. Smutz and R. F. Gingrich. (Essential Principles and Applications for Students of Engineering and Architecture.) Second Edition. New York, D. Van Nostrand Company, 1938. 10+227 pages. $2.50.

Intermediate Algebra. By W. E. Brooke and H. B. Wilcox. New York, Farrar and Rinehart, Inc., 1938. 8+323 pages. $1.90.

Teaching Arithmetic in the Elementary School. By Robert Lee Morton. Vol II, Intermediate Grades. New York, Silver Burdett Company, 1938. 12+538 pages. $2.72.

Analytic Geometry. By Clyde E. Love. Third Edition. New York, The Macmillan Company, 1938. 14+279 pages. $2.75.

Solid Mensuration. By W. F. Kern and J. R. Bland. Second Edition. (With proofs.) New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1938. 7+172 pages. $2.00.

A Short Course in Trigonometry. By J. G. Hardy. Revised Edition. 9+152 pages. *Logarithmic and Trigonometric Tables.* Revised Edition. (Prepared under the direction of E. R. Hedrick. Entirely re-set in a new type face.) 16+143 pages. New York, The Macmillan Company, 1938. $2.25.

Advanced Analytic Geometry. By A. D. Campbell. New York, John Wiley and Sons, Inc.; London, Chapman and Hall, Ltd., 1938. 10+310 pages. $4.00.

The Ringstrom Three Dimensional System of the Atoms (With Atomic Musical Pitches) and the Theory of Atomic Integration. By A. B. Ringstrom. Chicago, Vega Publishing Company, 1938. 87 pages.

A Preface to Mathematics. By C. E. Van Horn. Boston, Chapman and Grimes, Inc., 1938. 12+124 pages.

The Importance of Certain Concepts and Laws of Logic for the Study and Teaching of Geometry. By Nathan Lazar. New York, The Mathematics Teacher, 1938. 66 pages. $1.00.

Tafels van e^x. By H. W. Holtappel, Samengesteld door. Groningen-Batavia, P. Noordhoff N.V., 1938. 31+132 pages. fr. 6.

Approximate Computation. By Aaron Bakst. (The National Council of Teachers of Mathematics, the Twelfth Yearbook.) New York, Columbia University Teachers College Bureau of Publications, 1937. 16+287 pages. $1.75.

Some Notes on Least Squares. By W. E. Deming. Washington, D.C., The Graduate School, The Department of Agriculture, 1938. 4+181 pages. $1.50.

REVIEWS

Urphänomene der Geometrie. Erster Teil. By Louis Locher. Zürich-Leipzig, Orell Füssli Verlag, 1937. 16+164 pp. R.M. 3.60.

A few years ago, a major prophet among American mathematicians—whether seeing visions with the young men or dreaming dreams with the old is not quite clear—pronounced a Jeremiad entitled *The rise and fall of projective geometry.** During the past decade, however, numerous books have appeared on the subject both here and abroad, among which indeed might properly be counted the prophet's own treatise on algebraic curves. Comes now the author of the little book under review, a purely synthetic treatment of the elements of

* This MONTHLY, vol. 41, 1934, p. 217.

projective geometry, designed for the wider circle of lovers of geometry, to declare his conviction that "More people than one might think yearn for a kind of activity that guarantees that man's essential being participates in a purely intellectual world." His judgment seems justified by the inclusion of *Euclid's Elements* in the Everyman series, that *rara avis* a book on mathematics among the best sellers, not to mention the success of the popular books of E. T. Bell and Dantzig and the Liebers' modernistic versions of classical theories. The author feels that the *Elements of Euclid* may justly be regarded as a significant civilizing influence, which has become a part of the cultural inheritance of the race. He believes that no less may be expected of projective geometry, which, he laments, is familiar to a very narrow circle of specialists. Were such ideas as the principal of duality thoroughly assimilated he thinks that projective geometry might "suggest to the natural scientist entirely new types of conceptual patterns."

A word about the arresting title of the book may be appropriate. The author decries the abstract formulation of mathematics when carried so far that it loses its connection with reality. To emphasize his view he refers to the properties of space as *phänomene*, a word which with the prefix *ur*, signifying primitive or original, makes up the term which seems to be new to geometry. He says in the preface: "The word axiom will be avoided in the following—it smacks strongly today of a wholly arbitrary, capricious human invention. Since for me the mathematical concept-structure is not a mere figment of the mind, but represents an abstract echo of a genuine world of a physico-mental origin, it seems to me that *Urphänomen*, the word introduced into color theory by Goethe, is more expressive, more suggestive of reality." Thus the word is not used as a mere substitute for axiom, although he lists as *Urphänomene* numerous propositions, together with their duals, which might ordinarily be included in a set of axioms. Again after exhibiting the Möbius band as a one-sided but bounded surface, he presents the projective plane as one-sided but unbounded. He considers a sphere of constantly increasing radius which is metamorphosed into the plane at infinity, one side of the sphere (the outside) completely covering the plane and concludes: "This metamorphosis (*grenzübergang*) of the sphere into the plane at infinity represents the *Urphänomen* for the one-sidedness of the plane."

While borrowing from Goethe, he also appropriates another word, polarity, which he uses throughout in place of the usual duality or reciprocity. This seems to me unfortunate both because the terms replaced are well established and satisfactory, and furthermore polarity is already used in geometry in another sense.

The first chapter deals chiefly with Desargues's theorem on perspective triangles in the plane and the related 10_3, 10_3 configuration. The theorem is proved only experimentally, illustrated by numerous figures.

In Chapter 2 are introduced the three geometric elements, point, line, and plane, which in combination yield the seven primitive forms.

Chapter 3 is concerned with "The Phenomena of Connection," *i.e.*, junction and intersection of the elements and primitive forms in three dimensional space. Axioms (*Urphänomene*) of existence, of uniqueness and some combining both ideas under the "one and only" terminology are presented in parallel columns, leading up to a formal enunciation of the principle of duality, already foreshadowed in the previous chapter.

The longest chapter of the book—about one fourth of the whole—is devoted to "The Infinitely Distant in Geometry." He discusses at length and with profuse illustration the surprising effects on figures of two major projective phenomena—that the line is unbounded and closed, the plane unbounded and one-sided. Much space is given to the regions into which sets of lines and planes respectively divide point spaces of two and three dimensions. The infinite elements act as unifying agents, sometimes reducing, sometimes increasing the cases to be considered. Thus three lines divide the projective plane into only four regions, an instance of simplification. But we find the opposite effect when enumerating the "triangle-figures" which are determined by three points. In elementary geometry we have a single figure, consisting of the three finite segments joining the points, whereas here we get eight—each of the others omitting at least one of the finite segments.

The fifth chapter takes up the simpler complete polygons and polyhedrons, culminating in the proof of Desargues's triangle theorems by means of the complete 5-point in space.

Chapter 6 considers the planar Möbius net—and the space analogue—the configuration arising from the quadrangle-quadrilateral configuration which finds its metrical version in any one of the several lattices. It concludes with a proof of the fundamental theorem of projective geometry, the Pappus-Pascal theorem on triply perspective triangles and a property of the quadrangle published as recently as 1935.

It seems odd that a book which omits the general Pascal theorem should include a discussion of Cassinian ovals, which occupies the major part of the final chapter. But it furnishes at once a tie with the first chapter and the basis for some concluding philosophical remarks. Such observations, as well as comments on the beauty, significance, and novelty of the results, which are interspersed throughout the book seem appropriate in a work of this kind in which the author plays the rôle of guide and companion no less than master.

On the whole the book is well done. I noticed only two trivial misprints: Fig. 125 is labeled 521, while Fig. 170 is cited in the text as 169. It is abundantly illustrated, the number of figures exceeding the number of pages. The figures moreover are exceptionally well executed, clearly and accurately drawn on pure white paper, which is free from glare. The esthetic qualities of the book set a high standard—the printing is clear, the binding attractive. The book is comparable in size and purpose to the Carus Monographs, which however in my judgment yield to this in general artistic merit.

R. M. WINGER

A Course of Pure Mathematics. By G. H. Hardy. Seventh Edition. Cambridge, The University Press; New York, The Macmillan Co. 12+498 pages. $3.75.

This is the seventh edition of a book first published in 1908. The subject matter, fundamental in analysis, is largely that of the elementary calculus. The thorough competence with which the author gives discussions and definitions, and the sensible rigor with which he gives proofs, have made this book one of the "best sellers" among mathematics books. It is lamentable that this book is "too hard" for a text in an ordinary first course in calculus. For a superior student in mathematics, it should be more intelligible and inspiring than is the conventional mediocre elementary calculus text based on a definition of limit so vague as to make a researcher in analysis shudder at the suggestion that he use the definition of limit to prove that if $\lim_{x\to 0} f(x) = a$ and $\lim_{x\to 0} g(x) = b$, then $\lim_{x\to 0}[f(x)+g(x)] = a+b$.

The new edition contains a few new sections, some sections which have been rewritten, and numerous additional problems of which many are recent Tripos examination questions. These Tripos problems are sufficiently difficult to sharpen the wits of a superior undergraduate student. Errors and misprints are almost totally absent from the book. One oversight which persists through the various editions can be corrected by interchanging "negative" and "positive" in the proof of Theorem D of section 124.

R. P. AGNEW

La Cryptarithmie, Extrait des Comptes Rendus. By M. Pigeolet, Brussels, Librairie du Sphinx, 1935. 16 pages. 1 belga.

The author defines Cryptarithmie as the decoding of numerical computations in which letter symbols have been substituted for various digits. He then briefly outlines the history of this subject, classifies different types of such problems and indicates methods of attack. Bases for classification are: whether or not one or more of the digits remain uncoded; whether the distinct digits are represented by distinct or identical letters; whether the computations involved are those of addition, subtraction, multiplication, division, extraction of roots, determination of greatest common divisors, *etc.*; whether the basis of enumeration is ten or some non-decimal base; and whether or not the problem is dependent on primes, powers, logarithms, or other functions.

While M. Pigeolet solves some of the many problems he cites, he leaves enough unsolved to fascinate the average reader for several hours. He concludes by pointing out that there is a spiritual satisfaction to be had from dealing with pure numerical relationships and wresting truth out of intangibles.

Although the name, Cryptarithmie, is but seven years old, many representative problems in this field are somewhat older. Several appeared in the MONTHLY in 1921, and are cited by W. W. R. Ball in the tenth edition of his *Mathematical Recreations and Essays*, pages 248–253.

W. F. CHENEY, JR.

Statistical Methods. By G. W. Snedecor. Ames, Iowa, The Collegiate Press, 1937. 13+341 pages. $3.75.

The mathematician who opens this book with the expectation of being entertained in mathematics will be greeted with disappointment; but if he is desirous of learning how statistics is being used today, at least in some quarters, he will have found a gem. The work is a lucid and delightful exposition of R. A. Fisher's developments. Intricate notions associated with experimental design are very cleverly explained. The book deserves widespread adoption as a text. It can be used in classes where the mathematical attainments are little or nothing beyond college algebra; yet there is plenty of material for a year's course for more matured students, in which event supplementary reading in the way of derivations should be provided.

Statistics, whatever the word originally meant, is today the symbol for thought regarding observed phenomena; what can be inferred from an experiment, and how will further data behave? All branches of science meet in statistics. No one has ever drawn a conclusion empirically without using statistical methods in one form or another, good or bad. As Mr. Snedecor says in his preface, "Statistics is a mode of thought." It is only natural that a subject so vast should present itself in different aspects to different people; Mr. Snedecor's treatment is to be regarded as an exposition of the methods that he himself has put to use at Ames. The same methods in the hands of anyone but a natural scientist of experience can do great harm. The reader will be impressed throughout by the fact that the author is not interested primarily in the subject for itself, but in the fundamental problems for which statistical calculations may be made. Illustrative examples abound, with clear and easily followed directions for machine calculation—enough detail, yet not too much.

The language of statisticians remains ambiguous, owing to a failure to differentiate between what is mathematics and what is the actual behavior of nature. For instance, on page 51 we read, "It is from this that we must estimate the population standard error. . . . In practice, the experiment is seldom repeated," and then on page 55, "The sample value far exceeds even the 1% level, 2.977, and this latter would be expected to occur only once in 100 samples if m were really zero." If experiments are not usually repeated, how can such statements be confirmed? It can not be denied that there is something useful in the calculations on which such statements are based, but only so for the man who first of all knows his wheat, pigs, fertilizers, or electrons: others must expect to be misled.

In the table of chi-squares on page 10, the author might have pointed out that with 20 items, chi-square can take only the 11 values 0, .2, .8, 1.8, $\cdots$, 16.2, 20; the discontinuous nature of chi-square, not chance, is responsible for the empty cell of class 1–1.49, as it is also elsewhere. In the treatment of least squares, the author fails to point out that in his formulas the error is assumed to be all in the y coördinate.

In the way of other minor weaknesses, the reviewer would express regret at

the lack of perspective that the book may create: not that what the author says could be told in less than 341 pages, but that the reader could now and then have been furnished references to other sides and arguments, or at least warned that this is not all. There is no mention, for example, of any criterion for the choice of test that a person should use. Closely associated with such a criticism is the fact that what little history is given is in most instances unreliable: the first calculation of a pair of confidence or fiducial limits was made by E. B. Wilson in 1927, rather than by Fisher in 1930 (p. 60); the analysis of variance (p. 180), not in name but in thought, goes back at least to 1912, as is evident from pages 66–71 of Palmer's *Theory of Measurements* (1912). Indeed, the idea goes back to the introduction of the correlation ratio by Karl Pearson in 1905. On page 30 Mr. Snedecor attributes the factor $n/(n-1)$ to Student in 1908, and on page 51 he apparently attributes it to Bessel, no mention being made of the fact that Gauss in 1823 very clearly subtracted the number of unknowns evaluated from the number of observations to get the corrected denominator (number of "degrees of freedom") in an estimate of the population variance. It was Helmert in 1876 who first derived the distribution of s^2; Student, so far as I can see, was not concerned in 1908 about the average value of his s^2, and made no mention of $n-1$.

Many people have been favorably impressed with the attractive cover; the paper and printing also do credit to the publisher, except that the setting of some of the mathematical formulas does not always do justice to the author's genuine and usually successful attempt at clarity.

W. E. DEMING

Calculus. By Edward S. Smith, Meyer Salkover, and Howard K. Justice. New York, John Wiley and Sons, 1938. 12+558 pages. $3.25.

This text is not just another book on calculus; in some respects it is unique. For instance, the order of arrangement of the topics departs not a little from the traditional scheme of two parts, *viz.*, differential and integral. The reader is introduced to both phases of the subject early in order that he may appreciate it as a science, a philosophy, and not just a body of formulas and rules. In the first three chapters there is developed a general notion of the calculus, using only polynomial functions. Sufficient rigor is employed, however, to establish sound foundations but not a discouraging amount for the beginner. All of the definitions are couched in language tending to impart to the reader (beginner) a rational perspective of the science. The first chapter is written in a form to allure the student and yet to challenge his thinking.

And I suspect that the authors have purposely avoided the title "Four Step Rule" in arriving at the ratio $\Delta y/\Delta x$ on page 15, since the aim is to put in the background the notion of "formulas and rules." Perhaps not too much space has been devoted to the delta ratio in chapter II. At any rate if the student does all of the problems by the delta ratio which are listed, he will have mastered the technique of the process. The completion of chapter III should furnish the stu-

dent with a fair notion of the meaning and use of integral calculus. Chapters IV to X, inclusive, apply the principles learned in chapters I to III, but using now algebraic and transcendental functions. The student is employing in these chapters the calculus and is learning more of its outreach. The analysis and rigor exhibited in article 51, "The Limit Defining e," is a fitting illustration of the scientific study of a question and yet is not beyond the ability of a student who can successfully reach this point in the text. The discussion is not found in very many texts, but the reader will find it challenging and stimulating. Many of the more difficult topics are treated toward the end of the book. Indeterminate forms are first encountered in chapter XII, infinite series in chapter XIII, and expansion of functions in chapter XIV. The remaining three chapters treat of hyperbolic functions, partial differentiation, and multiple integrals.

The authors have made a very reasonable and lucid approach to the calculus, introducing special devices occasionally which serve to enkindle a keener interest in the subject; see page 259, article 92, as an illustration. The drawings are well executed, a table of integrals is found at the end, and there are answers to all the problems.

RICHARD MORRIS

Nouvelle Méthode pour construire et dénombrer certains Carrés magiques d'ordre 4m avec application aux Parcours magiques, Extrait du Sphinx. By Dr. Erich Stern. Brussels, Librairie du Sphinx, 1937. 20 pages. 1 belga.

This treatise, translated from the German into French by General E. Cazalas of Versailles, proposes and explains methods by which the 64 numbers from 0 through 63 may be arranged in a magic square in more than 8×10^{20} essentially different ways. The author proceeds to generalize his result to apply to any magic square of order $4m$.

The method consists in the formation of two auxiliary squares, U and V, such that the numbers in one quadrant of each square may be arranged arbitrarily (except for minor restrictions), with the other quadrants of each square formed from the arbitrary quadrant by certain rules of symmetry. Then these two squares, U and V, are combined in a simple linear relationship to form a magic square.

The author determines the number of ways in which a quadrant of a square may be filled out consistently with the rules he lays down, and the number of variations which may be introduced through changing the linear relationship for combining U and V, and by altering the type of symmetry between the quadrants of each square, and thus arrives at the prodigious total of 800,708,749,625,917,440,000 essentially different, 8×8, magic squares. He sketches an extension of his methods to apply to larger magic squares of order $4m$, and concludes by pointing out untouched possibilities attainable through the pursuit of these methods for magic squares of *any* order.

W. F. CHENEY, JR.

MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, State Teachers College, Upper Montclair, N.J.

CLUB TOPICS

The following play was written and produced by Professor A. Marie Whelan of Hunter College of the City of New York. It would seem that many clubs might be interested in including it in their year's program, hence its publication at this time.

It Can't Happen Here

A mathematical musical farce

Time: *The present.*

Scenes: *All action takes place in a mathematics class-room.*

Note: *Professor Math slips from his rôle of Professor into the rôle of Pure Mathematics, to suit his convenience, and without warning or apology.*

Prologue

(*Prof. Math strides onto the stage. He wears a crown, and sings the following song. Tune: I'm Called Little Buttercup.*)

Math: I'm King Mathematics, Engaged in dramatics.
Your pardon I humbly request.
Your patience I ask for;
Don't take me to task for
A deed that I do by request.
I've strayed from my limits,
And for these few minutes,
I fear I am not at my best.
My rôle is not writing, or acting, or fighting.
My kingdom's a haven of rest.
Through infinite spaces, Through far distant places,
On wings of Logic I climb.
No body to stay me, No object to sway me.
My kingdom's the reach of the Mind.

(*Exit*)

Act I.

Scene I. (*The Mathematics class is gradually assembling. The last bell has not yet rung. One student is sitting at a desk, looking dejectedly at a test paper. A second student approaches her.*)

Helen: What is the matter, Jane? You look like your last hope is gone.

Jane: So it is! Look at that, Helen! Another "F"! I don't know what I am going to do.

Helen: Don't let it discourage you too much. I understand that practically everybody flunked this time. It was a pretty hard test.

Jane: *All* tests are hard for me. I have taken this course three times already, and I don't know any more about it *now* than I did the first time I took it. I am a graduating senior, but I won't graduate if I don't pass this course. And

I haven't a Chinaman's chance. But what is the use of talking to you? You'll never understand. I suppose you got "A" as usual?

Helen: Yes, I did.

Jane: I don't know how you do it.

Helen: Perhaps I work harder than you do. Why don't you try working a little harder?

Jane: It wouldn't do me a bit of good—and I don't believe in wasting my work. I like proper return for my labor, thank you. Once in a blue moon, I think I have caught on to something—but when I try to use it, it seems that it doesn't apply. I gave up long ago. I think it is all nonsense anyway. What do you see in it? Why do you work so hard?

Helen: I don't mind the work. I love it.

Helen sings: (*Tune: Tit Willow.*)

Of work in Math'matics, I'll never complain.
For I love it! I love it! I love it!
And the thrill that it gives me, I'll never explain.
But I love it! I love it! I love it!
If a problem is difficult, all the more fun,
And the pleasure I get, when at last it is done.
OH, the thrill that I feel, when the battle is won!
I love it! I love it! I love it!

Jane: From work in Math'matics, I'll always refrain,
For I hate it! I hate it! I hate it!
Of the pain that it gives me, I'll ever complain.
And I hate it! I hate it! I hate it!
And the silly old battles of A, B, and C!
Does it matter who wins, When the loser is ME?
OH, the thrill that I'll feel,
When at last I am free!
I hate it! I hate it! I hate it!

Chorus: (*The entire class repeats Jane's song.*)

Jane: Well, Helen, it seems that the majority opinion is with me. (*Reflectively.*) Majority Opinion. Say, that gives me an idea!

(*The bell rings. The students hurry to their seats, as Professor Math enters the room.*)

Prof. Math: As usual, I must take the attendance. Will you please sign your names on the slip of paper, that I shall pass around?

(*Prof. Math hands a slip of yellow paper to a student in the first row. The student signs it and passes it along. In the meantime, Jane cautiously slips a similar piece of paper to her neighbor. Both papers are seen to pass from student to student, until Jane's paper reaches the front row. Prof. Math, thinking that it is the attendance slip takes it from a student, who is obviously reluctant to let him have it. The words on it catch his attention and he reads them aloud.*)

Prof. Math: A meeting is called of the students in this class to discuss a plan to ensure that everyone in the class shall pass the course.

Room: 1-125 Time: 12:15
DON'T FAIL TO COME!!!!!

(*Addressing the class*) Very interesting, indeed! No signature, unfortunately. Who wrote this notice? (*No one answers.*) Come now, don't be back-

ward. My only reason for wanting to know is to congratulate the author. I am delighted to find out that someone is taking this serious problem seriously. I assure you that nothing would give me greater pleasure than to pass every student in this class. But I must admit that the probability of that happening is very small—considerably less than one-twentieth, in fact. (*Pauses a few seconds.*) Well, it seems that my curiosity is not to be satisfied. So let us get down to work. Judging from the last test, I have decided that a little review of elementary algebra is in order. Mary, will you go to the board, and solve this equation;

$$x^2 - 2x = 0.$$

(*Mary goes to the board and solves the equation, using the Quadratic Formula.*)

Prof. Math: Why didn't you solve it by factoring? Here, let me show you.

(*Prof. Math solves equation by factoring.*)

Mary: Those are the same answers I got, Professor.

Prof. Math: I did not say that your solution was *wrong*, Mary. See here. Suppose that you wanted to go some place, and that you could get there by bus or subway train. Suppose also that you would have to wait one half-hour for the bus, but could get the subway train immediately. What would you do?

Mary (*innocently*): I would wait for the bus, Professor. I don't like the subway.

(*General laughter.*)

Prof. Math (*joining in the laughter*): Well, you certainly flattened my point! Incidentally no wonder you are late so often. Anne, go to the board, now, and solve this equation:

$$x^2 - 2x - 2 = 0.$$

(*Anne presents the following solution:*)

$$\begin{aligned} x^2 - 2x - 2 &= 0 \\ x^2 - 2x &= 2 \\ x(x - 2) &= 2 \\ x &= 2 \\ x - 2 &= 2 \\ x &= 4 \end{aligned}$$

Prof. Math (*sarcastically*): I suppose *you* never heard of the Quadratic Formula?

Anne: I *do* know the Quadratic Formula, Professor. (*Rattles the formula, parrot-like.*) That is right, isn't it?

Prof. Math: Quite right! May I ask why you did not use it?

Anne: I thought you didn't *like* it. So I solved it exactly the way you did *that* one. (*Pointing to the preceding problem.*) I only wanted to please you.

Prof. Math: Very obliging of you, I'm sure. For your information, you did *not* please me. Far from it! There ought to be a law against that kind of algebra. What am I saying? There is a law against it. You look like a failure to me. I've certainly got *you* on my list.

Prof. Math sings: (*Tune: They'll None of 'Em Be Missed.*)

Since someday it will happen that my failures must be found,
I've got a little list! I've got a little list!
Of Math'matical offenders, that forever do abound—
And who never will be missed.

Who never will be missed.
There's the ever present nuisances, who dare be late for class
All those who murder Algebra, and cannot draw their graphs.
All students who neglect their work, and cut my quizzes flat.
All students who, in finding roots, find roots, like *you*, like *that!*
All students who use standard forms, where standard forms don't fit.
They'd none of them be missed! They'd none of them be missed!

Chorus: He's got us on his list! He's got us on his list!
And we'll none of us be missed! We'll none of us be missed!

(*Curtain*)

Scene II. (*Time: 12:15. Several students are discussing the proposed meeting.*)

Ruth: Who called this meeting, anyway? Do you know, Anne?

Anne: I don't really *know*—but I think it was Jane. The paper seemed to start from her direction. It was stupid of you, Mary, to let Prof. Math get his hands on it.

Mary: Go ahead and blame me. What else could I do? I bet you wouldn't have done any better. Anyway no harm is done. He did not find out who wrote it.

Anne: No, he didn't. It might have been a bit exciting if he had found out that Jane started it—that is—if she did start it. Here she comes now. Let's ask her.

Ruth: Hello, Jane. Are you the one who called this meeting?

Jane: Yes, I did. But I don't want Prof. Math to find out that I did. I do think you might have been more careful, Mary.

Mary: I tell you that I couldn't help it. Anyway, I don't see why you didn't admit writing it. He said he only wanted to congratulate you. (*Laughter.*)

Ruth: You *are* a simpleton, Mary! Congratulate her indeed! He'd have taken her apart. What is it all about, anyway, Jane? I am having a Physics test, next hour, and I really should be studying now—but my curiosity got the best of me. I warn you. It had better be good. Get on with it—everybody is here, I think.

(*Jane goes to the desk, and addresses the meeting.*)

Jane: The meeting will please come to order! As you see, I have taken it upon myself to call this meeting. You shall know my reasons. I understand that nearly everybody in this class flunked the last test in Mathematics. I want the exact figures. How many got "F" on the last test? Let's have a show of hands.

(*Everyone, except Helen, raises her hand.*)

Jane: It is worse than I thought!

Ruth: I don't think I agree with you. I think the fact that so many flunked is a good sign. After all, he can't very well fail the whole class.

Jane: Don't kid yourself! You don't know him! He'd fail us all without a qualm, just to uphold his silly old standards. I tell you the case is desperate. And desperate cases require desperate remedies. I know exactly what the trouble is. Prof. Math has too much power! Take my case, for instance. He has the power, and he'll undoubtedly use it, to fail me in this course.

And what is the result? I don't graduate.—All my plans for the future are destroyed. My whole life, perhaps, is ruined. And all for what? To uphold his inhuman standards. I ask you: Is it *right?* Should any one person have that much power? Again, I ask: *Is it right?*

Cries from all directions: NO! NO! NO! It isn't right!

Anne: But what are you doing to do about it?

Jane: I'll tell you. This autocratic procedure must be abolished! It is upheld by ancient custom, *that,* I know—But TIME MARCHES ON!—And I tell you the time has come for the students to take into their own hands the power to regulate their own assignments, and to determine the mark they shall receive for their work. Time does march on! And the time is HERE—TO-DAY—To abolish the AUTOCRAT OF THE CLASS-ROOM! (*Cheers.*)

Jane: And what shall we substitute for him? I'll tell you—The answer is—MAJORITY RULE! (*More cheers.*) And how shall we establish Majority Rule? I'll tell you that too—We'll form a—UNION!

Excited cries: Yes, yes, let's form a Union!

Jane (*continues*)*:* Yes, we will form a Union. I'll be the leader—After all it is my idea and I'll draw up a contract. With you 100% behind me, Prof. Math will be compelled to sign it. But I must have *100% support!* The contract will specify that our assignment shall consist of only one problem a day; and that the grades in the course shall be determined by a Majority Vote of all the students in the class. Of course (*laughing*) by mutual agreement among ourselves, that will mean that we will all get "A."

Anne: I think it is a wonderful plan—but if we can go that far, why not go a little farther, and abolish assignments, altogether?

Jane (*with absolute sincerity*)*:* Don't you think that would be a little one sided?

Anne: Perhaps it would—after all a contract does imply something mutual.

Ruth: Jane, you said that you would have to get 100% support. I don't think you'll get it. Helen, there, won't join our Union, I'm sure of that.

Jane (*grimly*)*:* She won't, eh? We'll see about that. (*Marches up to Helen, and grips her firmly by the shoulder.*) How about it, Helen? Will you come in with us?

Helen: Of course I won't. Why in the world should I? What would I get out of it? I always get an "A" anyway—an honest "A." And as far as the work is concerned, well I like to work.

Jane: Do you hear that, Girls? Nice public-spirited citizen, she is! She likes to work, and as long as she gets her "A," what does she care if we all flunk—Well, we have a name for her kind, haven't we? (*Hisses and boos.*)

Mary: I do think you are a little selfish, Helen.

Ruth: Selfish?—That is too mild a word for her. She is a Public Enemy! Public Enemy Number One! (*More boos.*)

Anne: I'll tell you what she is—She is a (*Pausing*) REACTIONARY!

(*The crowd takes up the word "Reactionary," dinning it into Helen's ears. Helen shows signs of weakening.*)

Jane: I'll tell you what she really is—an INTELLECTUAL ROYALIST! (*Boos and hisses.*)

(*Jane chants dramatically, always with menacing gestures toward Helen.*)

Jane: She's an intellectual royalist,

A mathematical loyalist,
She'd block us if she could.

Anne (*taking up the chant*):
Selfish-hearted adversary,
Dyed-in-the-wool Reactionary
Opposed to the public good.
Opposed to the public good.

Chorus (*repeats above adding*):
Down with all Reactionaries,
Down with all our adversaries,
Opposed to the public good!
Opposed to the public good!

Helen (*tearfully*): OH, STOP! Please STOP! Don't hurl those dreadful words at me. Those words, those awful words, I can't stand up against them. I give in—I'll join your Union.—What do you want me to do?

Jane: Exactly what you are told. I am the Leader.—You back me up in every action.—Well, Girls, now that we've taken care of Helen, we are ready for a vote.—All in favor of forming a Union, say "AYE." (*All say "AYE."*) Opposed? (*None.*) The motion is carried. Well the first step is accomplished—very successfully accomplished too—100% Unionized! Girls, we're going places!

Song by Jane: (*Tune: Funiculi, Funicula.*)

Jane: I think it well indeed, to form a Union.
Chorus: And so do I, and so do I.
Jane: I think our failure lies in our disunion.
Chorus: And so do I, and so do I.
Jane: And why, why should I spend my time in slaving,
The whole day long, the whole day long?
To strike a valiant blow for Student Freedom.
Is far from wrong!
Chorus: Is far from wrong!
Hearken, Hearken, Union wins the day.
Hearken, Hearken, Union points the way,
To get an "A," to get an "A," to get an "A," to get an "A"!
Union points the way, we'll Unionize, we'll Unionize!

(*Curtain*)

Act II

Scene I. (*Professor Math is standing at his desk, considering the contract which has been presented to him. The students are lined up behind Jane, who is standing to the left of the desk.*)

Jane: Sign the contract, Professor Math. You have no other choice. You have got to sign the contract!

Chorus: (*Tune: For He's Going to Marry Yum-Yum.*)
You have got to sign the contract!
Resistance is idle, your anger, pray bridle.
I think you had better succumb-cumb-cumb,
And sign on the dotted line.

In this matter, we'll brook no delay, delay.
It's no use to fight it, for now we're united,
And bound to have our way, our way.
You've got a good bargain at that.
In this matter, we pray you admit, admit;
There's nothing to do but submit, submit.
For now we're united, there's no use to fight it,
There's no use to fight it. And you've got a good bargain at that.
For we'll give you one vote in the class.
We'll give you one vote, one vote in the class.
We'll give you one vote, one vote in the class.
In the class, in the class, in the class, in the class!

(*Prof. Math signs the contract, and hands it to Jane.*)

Prof. Math: Here is your contract, signed on the dotted line. It seems I had no other choice. I bow to your greater numbers. Let me congratulate you on the the success of your plan—that is, on its apparent success. You must forgive me, if I quote some old proverbs; "Things are not always what they seem" and "He who laughs last, laughs best." I warn you that your Union is not going to accomplish all that you hope for—Now, don't misunderstand me. I am not opposed to Unions. Far from it. For some problems, they seem to be the only solution. I might say that they are a standard form of solution—but, like all standard forms, they don't fit all cases. Incidentally, that is something that I have tried to teach you—not very successfully, I confess. Well, Experience is said to be a very effective teacher. In the meantime, if it is not opposed to the will of the Majority, perhaps you will be seated. I understand that you have done one problem for today.—Jane, will you be kind enough to put your solution on the board?

(*Jane goes to the blackboard and presents a solution that is obviously wrong. Her errors compensate, however, and she obtains the correct answer.*)

Jane: My problem is to solve the system of equations:

$$\frac{y-1}{x-2}=\frac{3}{5} \qquad (1) \qquad\qquad \frac{y+1}{x-5}=\frac{5}{2} \qquad (2)$$

I did not *need* the second one. Using the first

$$\frac{y-1}{x-2}=\frac{3}{5}.$$

First "*you say,*" $\frac{y}{x}-\frac{1}{2}=\frac{3}{5}$;

then "*you say,*" $\frac{y}{x}=\frac{3}{5}+\frac{1}{2}$; $\frac{y}{x}=\frac{3+1}{5+2}$;

then "*you say*": $y=4, \quad x=7.$

Prof. Math: Well, the answer is right, to be sure, but that is the only thing right about it. I never saw such a ridiculous piece of Algebra in all my life! You certainly get zero for that recitation!

(*Prof. Math picks up his roll-book. Jane stops him as he is about to record a grade.*)

Jane: Just a minute, Professor—You forget our contract. I call for a vote. How about it, girls? Was my problem right or wrong?

(*Chorus of cries*): Right! Right! Entirely Right!

(*Jane takes the roll-book from Prof. Math's hands and records her own grade.*)

Prof. Math: (*Picking up the book and reading aloud.*) 100%! Yes, (*violently*) 100% WRONG! AH, Truly, I am deposed!

(*Sings: Tune: A Wandering Minstrel.*)

Unhappy Monarch, I. I bow my head in sorrow.
I never dreamed a morrow, would see me to students submit.
They solve all problems wrong, through every error ranging,
My noble laws they're changing, to please their feeble wit.
To please their feeble wit.

(*Sits down and drops his head on the desk.*)

Chorus: Unhappy Monarch, Aye! He bows his head in sorrow,
He never dreamed a morrow, would see him to students submit.
We solve all problems wrong, through every error ranging.
His noble laws we're changing, to please our nimble wit,
To please our nimble wit.

(*Curtain*)

Scene II. (*Students are standing in groups about the room. Others are seated, apparently working.*)

Jane: Well, Girls—Today is the BIG DAY! Think of it, we'll all get "A"! You can thank me for that.

Mary: I can't think of anything else. I feel a bit nervous, to tell you the truth,—like I was standing on a volcano that might blow up any minute.

Jane: Nothing is going to blow up here. Not a chance of it! Don't worry. Everything is under control.

(*The bell rings. Everyone is seated as Prof. Math enters the room.*)

Prof. Math: As today is the first school day of the month, the first thing on the program is the distribution of grades. As I call your names, will you please come forward to receive your mark for last month's work. To relieve your suspense—I will tell you in advance that every student will receive an "A." It may seem a bit ungracious not to offer congratulations on this remarkable achievement, but I offer none.

(*Prof. Math calls each student by name, and presents each with a large cardboard "A". Each student holds up her "A" so that the audience can see it. On the back of the cardboard is a large "F" not visible to the audience.*)

Jane: Come on, Girls, I think this calls for a celebration! Let's sing our song!

Chorus: I thought it well indeed to form a Union.
And so did I, and so did I.
I thought our failure lay in our disunion.
And so did I, and so did I.
For why, why should we spend our time in slaving,
The whole day long, the whole day long?
To strike a valiant blow for Student Freedom,
Is far from wrong, is far from wrong.
Hearken, Hearken, Union won the day
Hearken, Hearken, Union showed the way,

We got an "A," we got an "A," we got an "A," we got an "A"!
Union showed the way, we got an "A," we got an "A"!

(*While the students are singing, Prof. Math goes to the board, and writes in large letters:* $Y = X - 100$. *As the students sing the last phrase, they hold up their cards, which they have cautiously reversed, so that the large "F" is now visible to the audience.*)

Mary: Anne! Anne! you *didn't* get "A". You got an "F"!

Anne: It *is* an "F"! You got an "F" yourself! Look, look! So did Jane—Everybody got an "F." Even Helen got "F"!

Jane: We'll see about this! He's let us down! Are we going to let him get away with this? He has broken his signed contract!

Ruth: But he didn't break the contract—He gave us all "A." I know he did.

Anne: Ruth is right. He certainly gave us all "A." That is all he contracted for.—We've got to be fair!—But, I would like to know what happened to our "A's"?

Helen: If you would really like to know—I'll tell you. He worked a transformation. I saw him do it.

Prof. Math: (*Tune: Henceforth All the Crimes.*)
It still remains true, that whatever you do,
The record is bound to reveal it.
Unite, if you must, but don't put your trust,
In Union's might to conceal it.
If you really want "A"—There is only one way
You can get it and part with it never.
For the record won't lie, though you try till you die.
And Truth is triumphant, forever!
No, the record won't lie, though you try till you die.
And Truth is triumphant forever!

(*Prof. Math gets his old crown from the desk and puts it on his head.*)

Prof. Math (*holding up the contract*): I now declare this contract abrogated—and tear up this scrap of paper. (*Pauses a few seconds.*) The class is dismissed for today. You are not prepared to get any value out of it. AND—The assignment for tomorrow is—(*Another pause*) Page 125, the first FIFTY problems! (*Groans from the class.*)

(*Curtain*)

Scene III. (*Jane and Helen are talking. The rest of the class is seated.*)

Helen: Nice mess you got us into! I hope you are feeling proud of yourself! First "F" I ever got in my life!

Jane: It will do you good.—Give you some understanding about the way the other half lives (*laughs*). Other half? I mean the other nine-tenths.

Helen: Well, I warn you, if you have any other crazy schemes to try, you can count me out, right from the start. Once is enough for me, more than enough. The next time I'll have more backbone. I'll keep in mind that old truism: "Sticks and stones may break my bones, but words can never hurt me."

Jane: Oh, yes? That may be old, all right. But it certainly is no truism. Whoever wrote it, did not know much, if you ask me. Words hurt all right—worse than sticks and stones—They hurt you right down where you live!

Helen: Hurt, or not, they are not going to make me participate in any more of

your wild plans. If you'll take my advice, you will settle down to work in earnest. Prof. Math is too smart for you. You might as well admit it.

Jane: He was too smart for me this time, I'll admit that—But it may be different the next time. I'll get him yet!

(*Sings:* Tune: *Toreador.*)

Hark to my warning, I will get him yet!
I'll get him yet! I'll get him yet!
I, another plan will devise. I will take his crown!
Perhaps, this very day, I'll bring him down!
AH! I will get him yet!

Anne: (*As Jane finishes last bar*) Look out, Jane, or he'll get you now. Here he comes!

(*They take their seats. The bell rings and Prof. Math calls the class to order.*)

Prof. Math: Before I go on with the advance work, I will devote a little time to answering questions on today's assignment. Are there any questions? Yes, Jane, what is your question?

Jane: It isn't about the assignment, Professor,—It's a more general question. Would you mind telling us what good this course is. What practical value it has for us?

Prof. Math: How often have I heard that question!

(*Sings:* Tune: *Yankee Doodle, with one slight change in the music.*)

'Tis absolute futility, to question my utility.
I never have to work. I never have to work.
I have no reality, in sense or sensuality,
And mundane tasks, I shirk, and mundane tasks I shirk.
The stupid questions, students ask!
So sure they're taking me to task:
What good I do? And am I true?
Such woeful ignorance!
I am pure non-SENSE—(*Interrupted by Jane who springs to her feet shrieking dramatically.*)

Jane: He's NONsense! He admits it himself! He's NONsense! I knew it all the time!

(*Chorus of excited cries NONsense! NONsense!*)

Prof. Math (*making his voice heard above the clamor*): I said "Non-SENSE." (*His words make no impression on the excited students.*)

Jane (*chanting dramatically*):

He's nonsense! He's nonsense! I knew it all the time!

Anne: But he handed such a line!

Ruth: And such awful condescension—
We thought our comprehension
Was in error.

Jane: He's worked us to the limit.
Every hour, every minute,
Spent in terror
That we would flunk the course.

Chorus: He's worked us to the limit
Every hour, every minute,
Spent in terror

That we would flunk the course,
That we would flunk the course.

Chorus: Tune: Farmer in the Dell.

But we won't flunk the course, No, we won't flunk the course.
HI, HO, we'll lay him low. There won't be any course!

(*The group crowds around Math, concealing him from the audience. When they spread out again, Math is seen lying motionless, on the floor.*)

There isn't any Math. There isn't any Math.
HI, HO, we've laid him low. There isn't any Math!

We've won the victory! We've won the victory!
HI, HO, we've laid him low, and set all students free!

We've set all students free, we've set all students free!
HI, HO, away we go. We've set all students free!

(*Serpentine off the stage, still singing.*)

(*Math comes to life, and rises slowly to his feet.*)

Math: Dead, am I? So they think! Poor deluded creatures,—will they never learn that I cannot die—I, who am immortal? Crushed to the earth, I rise again, as always I shall rise. (*Pauses, and walks up and down, meditating.*) So they have abandoned me.—Well, let them go. They never spoke my language! *MY* language? They can't speak the English language. That was the trouble. Yes, they've gone—but they will be back. As soon as they find that they need my help, one by one back they'll come, these bread and butter students. But not one will come seeking me for my own sake. Not one? Ah, I am wrong—I forgot, always there are EXCEPTIONS! I won't be deserted long.—But for the present, I am alone.

(*Sings: Tune: Tit Willow.*)

In this world of my own, I am never alone.
And time does not drag unduly.
My beautiful laws admit of no flaws,
And no one is ever unruly.
Though wars wage without, and the battle cries shout,
Here Order holds sway, undisputed.
No factional strife makes a discord of life,
For each has his limits computed.

(*As Math finishes his song, the students are heard approaching, cheering.*)

Math: They are coming back! I'll play dead a little longer. It would be a pity to spoil their fun.

(*The students serpentine on to the stage, shouting and cheering.*)

Chorus: (*Tune: Whistle While You Work.*)

We've done away with Math. We have swept him from our path.
We've laid him low, and now we go, to celebrate in song.
Oh, sing the whole day long! Oh, sing a joyful song!
Good news we bring. Rejoice and sing,
And let the echoes ring!
No Algebra to do, no Trig to bother you,
No Differential Calculus
To spoil the lives of all of us—

We've done away with Math. We have swept him from our path.
OH, HO, HO, HO— Away we go.
We've done away with Math.

(*As the students finish singing, an excited group rushes upon the stage. They are: The Dean, Physics, Chemistry, Business, and the Bursar.*)

The Dean: Stop this noise, immediately, and take your seats. What is this all about? It can't be true—that nonsense Helen told me—that Math is dead! Oh, (*catching sight of Math, lying on the floor*) it *is* true! Math *is* dead! What are we going to do?

Physics: I don't know what *you're* going to do. Speaking for myself, I might as well shut up shop, and go home. I can't get along without Mathematics.

Chemistry: Don't expect much work from *me*, either. I need Mathematics to help me in my line.

Business: And me—What am I going to do? I tell you—I'll go bankrupt without Math!

The Dean (*wringing her hands*)*:* Oh, what are we going to do? Jane, you wicked girl,—you are responsible for this terrible situation. You are expelled—pack up your things, and go home!

The Bursar: You might as well face facts, Dean—we'll all have to pack up and go home—I can't run the college without the help of Mathematics.

Anne: Oh, look, look! Who is this coming?

Ruth: Why it is the World—and he is lame! He's wearing crutches!

Jane: Oh, what have I done—what have I done! Why didn't someone tell me Math was so important?

(*The World reaches the center of the stage. He wears a large globe over his head, and supports himself on crutches.*)

Jane: (*Tune: Robin Adair.*)

Who has done this to thee? Poor crippled World.
Thy wings are torn from thee, poor crippled World.
Oh weep and mourn with me, because all hope is fled.
The World on crippled feet, for Math is dead!

(*Math rises to his feet. Excited cries, and cheers as the group realizes that he is alive.*)

Prof. Math: Cheer up, World, I am not dead. The report of my death was greatly exaggerated. I am very much alive, as you see. I must admit that it makes me very happy to see you all so happy about that fact.—Jane, you have done me a service. Thanks to you, Mathematics comes into his own, at last. I thank you—and by the way I'll thank you to surrender my crown.

Jane: Take your crown, King Mathematics. Truly you are a King.

Chorus: (*Tune: Melody in F.*)

King Mathematics comes into his own.
Our hearts enthrone him, Our voices own him,
Beautiful dreamer, whose dreams oft come true,
Moulding the World anew.
So great his powers, the gifts that he brings,
Because of him, the World has wings.
Marvels of Nature forever unfurled,
Truly He rules the World.

The End.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

E 349. *Proposed by R. H. Bardell, University of Wisconsin.*

Show that for all positive integral values of k,

$$\sum_{n=0}^{k} (-1)^n C_n^k/(2n+1) = \frac{2^k \cdot k!}{(2k+1)(2k-1)(2k-3)\cdots 5\cdot 3\cdot 1}.$$

E 350. *Proposed by V. Thébault, Le Mans, France.*

If from the feet of the bisectors of the interior angles of a triangle perpendiculars erected to the respective sides are concurrent, prove that the triangle is isosceles.

E 351. *Proposed by Virgil Claudian, Bucharest, Roumania.*

Determine b as a function of a such that 4 may be the sum of the squares of the roots of the equation, $x^3+ax^2+bx=2a$, and discuss the equation, considering a as a variable parameter.

SOLUTIONS

E 312 [1938, 47]. *Proposed by D. L. MacKay, Evander Childs High School, N.Y.*

If the scalene triangle ABC has its external angle bisectors at B and C equal, show that $(s-a)/a$ is the geometric mean of $(s-b)/b$ and $(s-c)/c$.

Solution by E. P. Starke, Rutgers University.

Let the external bisector of angle C meet AB in M. Either A lies between B and M, or B lies between A and M. In the first case, angle M will equal $\frac{1}{2}\pi - (B+\frac{1}{2}C)$. In the second case, angle M equals $B+\frac{1}{2}C-\frac{1}{2}\pi$. Thus $\sin M = \pm\cos(B+\frac{1}{2}C)$. The bisector CM could not be parallel to AB, for then we would have $\frac{1}{2}C+B=\frac{1}{2}\pi$, and angles B and A would be equal, contrary to hypothesis.

In triangle BCM we have, $CM/BC=\sin B/\sin M=\pm\sin B/\cos(B+\frac{1}{2}C)$. Similarly, from triangle BCN, N being the intersection of AC with the external bisector of angle B, we have $BN/BC=\pm\sin C/\cos(C+\frac{1}{2}B)$. Since CM is given equal to BN, we must have

$$\sin C\cos(B+\tfrac{1}{2}C) \mp \sin B\cos(C+\tfrac{1}{2}B) = 0.$$

When reduced to functions of half angles, this becomes

$$(A) \qquad 2[\sin \tfrac{1}{2}B \mp \sin \tfrac{1}{2}C][\cos^2 (\tfrac{1}{2}B + \tfrac{1}{2}C) \pm \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C] = 0.$$

If both bisectors fall under the same category or case, the upper signs in (A) will apply. Since the second factor of (A) is then the sum of the two positive quantities, it cannot vanish, so that $\sin \frac{1}{2}B = \sin \frac{1}{2}C$, and the triangle ABC is isosceles. But if the two bisectors fall in opposite categories or cases, the lower signs in (A) apply, the first factor of (A) cannot vanish, and we must have $\sin \frac{1}{2}B \sin \frac{1}{2}C = \cos^2 (\frac{1}{2}B+\frac{1}{2}C) = \sin^2 \frac{1}{2}A$. This is equivalent to the proposed relation, as may be shown by substituting $[(s-b)(s-c)/bc]^{1/2}$ for $\sin \frac{1}{2}A$, and analogous formulas for the other sines. Furthermore, this condition is sufficient as well as necessary.

Also solved by L. M. Kelly and the proposer.

E 313 [1938, 47]. *Proposed by J. Rosenbaum, Bloomfield, Connecticut.*

Find four rational numbers such that their sum, the sum of their squares, and the sum of their cubes, each equals 2.

Solution by A. G. Clark, Fort Collins, Colorado.

The symmetric functions, $\sum a$, $\sum a^2$ and $\sum a^3$ each being equal to 2 gives us also that $\sum ab = 1$, and $\sum abc = 0$. Hence all rational solutions of the problem will be given by sets of rational roots of the quartic equation,

$$x^4 - 2x^3 + x^2 = R^2,$$

where R is rational, since the left member of this equation is a perfect square. But this quartic degenerates into the two quadratics,

$$x^2 - x \pm R = 0.$$

The necessary and sufficient conditions that all roots be rational are evidently that $1+4R = p^2/r^2$ and $1-4R = q^2/r^2$, where p, q, and r are integers. If the above equations are added, we have $p^2+q^2 = 2r^2$, which is a well known equation whose solutions are all given by

$$p = k(m^2 + 2mn - n^2), \qquad q = k(m^2 - 2mn - n^2), \qquad r = k(m^2 + n^2).$$

Here m and n are relatively prime integers, and k is any integer. Substituting these values above gives us the four fractions, $(m^2+mn)/(m^2+n^2)$, $(m^2-mn)/(m^2+n^2)$, $(n^2+mn)/(m^2+n^2)$, and $(n^2-mn)/(m^2+n^2)$, as representing all rational solutions of the problem.

Also solved by F. A. Alfieri, Fred Discepoli, Daniel Finkel, A. D. Maxwell, W. C. Rufus, E. P. Starke, and Samuel Zuckerman.

E 314 [1938, 47]. *Proposed by Cezar Coşniţă, Roumanian Mathematical Institute.*

Find the locus of the center of a variable sphere which cuts each of two (or three) fixed planes in a circle of constant size.

No solutions have been received for this problem.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3891. *Proposed by J. R. Musselman, Western Reserve University.*

The Apollonian circles of a triangle meet in two points, the Hessian points h_1 and h_2. Show that the two Beltrami points (inverses in the circumcircle of the Brocard points) form with either Hessian point an equilateral triangle. Hence each Beltrami point is the center of the circle passing through h_1 and h_2 and the other Beltrami point. Naturally a Brocard point lies on each circle.

3892. *Proposed by George Rutledge, Massachusetts Institute of Technology.*

Show that, for all even values of n,

$$\int_0^1 \frac{1}{u}\left\{\left(\log u + \log\frac{1+u}{1-u}\right)^n - (\log u)^n\right\} du = 0.$$

3893. *Proposed by Norman Anning, University of Michigan.*

From the vertices of a regular n-gon three are chosen to be the vertices of a triangle. Prove that the number of essentially different possible triangles is the integer nearest to $n^2/12$.

3894. *Proposed by Walter Leighton, The Rice Institute.*

Given a polynomial equation

$$(1) \qquad f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0, \quad a_0 \neq 0,$$

where the a_i are rational integers, find all roots of the form $(a \pm \sqrt{b})/c$, where a, b, c are integers, $c \neq 0$, and $\sqrt{b}$ is irrational (possibly imaginary).

3895. *Proposed by V. Thébault, Le Mans, France.*

Four spheres (ω_1), (ω_2), (ω_3), (ω_4) pass respectively through the vertices A_1, A_2, A_3, A_4 of a tetrahedron and intersect in pairs on the corresponding edges. Straight lines A_5A_1, A_5A_2, A_5A_3, A_5A_4 which join an arbitrary point A_5 to the vertices of the tetrahedron cut the respective spheres again in α_1, α_2, α_3, α_4 and these four points with A_5 lie on a sphere (ω_5). Continuing in this way, the lines $A_nA_1, A_nA_2, \cdots, A_nA_{n-1}$ which join an arbitrary point A_n to the points of a preceding set $A_1, A_2, \cdots, A_{n-1}$ cut the respective spheres (ω_1), (ω_2), $\cdots$, (ω_{n-1}) in the points $\alpha_1, \alpha_2, \cdots, \alpha_{n-1}$ which with A_n lie on a sphere (ω_n). (1) Show that the spheres (ω_1), (ω_2), $\cdots$, (ω_n) meet in a point P. (2) Show that the points

diametrically opposite to $A_1, A_2, \cdots, A_n$ on the corresponding spheres lie in a plane through P.

Note by the Proposer. The part (2) is an extension of a theorem by Bouvaist-Delens, *Annales de la Société Scientifique de Bruxelles*, 1937, p. 159.

SOLUTIONS

3780 [1936, 245]. *Proposed by J. M. Feld, New York, N. Y.*

In triangle $A_1A_2A_3$ the transversal A_iD_i divides A_jA_k in the ratio $A_jD_i : D_iA_k = p_i : q_i$, where ijk is a cyclic permutation of 123. The transversals A_iD_i and A_jD_j intersect in P_k. Find the value of the cross ratio

$$\frac{P_3P_2}{P_2A_1} \Big/ \frac{P_3D_1}{D_1A_1}$$

in terms of the p's and q's. Show that Ceva's theorem is a special case. (*An omission supplied.*)

I. *Solution by Maud Willey, Long Beach, Miss.*

Let A_2P_2 cut A_1A_3 in B. The theorem of Menelaus applied first to triangle $A_1A_2D_1$ and the transversal $D_3P_2A_3$, and then to $A_1D_1A_3$ and transversal A_2P_2B, gives $A_1D_3 \cdot A_2A_3 \cdot D_1P_2 = D_3A_2 \cdot D_1A_3 \cdot P_2A_1$ and $A_1P_2 \cdot A_2D_1 \cdot A_3B = P_2D_1 \cdot A_2A_3 \cdot BA_1$. Hence $A_1B/BA_3 = p_1p_3/q_1q_3$. The point ranges $A_1P_2P_3D_1$ and $A_1BD_2A_3$ are in perspective from A_2; and hence $A_1P_2 \cdot P_3D_1/P_2D_1 \cdot A_1P_3 = A_1B \cdot D_2A_3/BA_3 \cdot A_1D_2 = p_1p_2p_3/q_1q_2q_3$. This gives for the ratio in the problem $1 - (q_1q_2q_3/p_1p_2p_3)$.

If two of the P points coincide this last ratio is zero and we have Ceva's theorem.

II. *Solution by J. H. Butchart, Phillips University, Enid, Okla.*

Let the vectors of A_1, A_2, A_3 be $0, \mathbf{a}_2, \mathbf{a}_3$, and set $p_i + q_i = 1$. Then the vectors of D_1, P_2, P_3 are respectively $q_1\mathbf{a}_2 + p_1\mathbf{a}_3 = \mathbf{d}$; $p_3\mathbf{d}/(p_1p_3 + q_1)$; $q_2\mathbf{d}/(q_1q_2 + p_1)$. The cross ratio in the problem reduces to $1 - (q_1q_2q_3/p_1p_2p_3)$ after certain simplifications. If P_2 and P_3 coincide, this ratio vanishes and we have Ceva's theorem.

Solved also by Elsie J. McFarland, J. Rosenbaum, F. Underwood, William Wernick, and the proposer.

3802 [1936, 580]. *Proposed by D. H. Lehmer, Lehigh University.*

Let $0, u_1, u_2, u_3, \cdots$ be a sequence of numbers satisfying the recurrence $u_{n+1} = au_n + bu_{n-1}$, and consider the function

$$f(x) = \sum_1^\infty u_n x^n/n!.$$

Show that $f(x) = -e^{ax}f(-x)$.

Solution by M. T. Bird, Utah State Agricultural College

The function $y = f(x)$ is the unique solution of the linear differential equation

$$y'' - ay' - by = 0$$

with the initial conditions $y(0)=0$, $y'(0)=u_1$.

Direct substitution reveals that $y=-e^{ax}f(-x)$ is likewise a solution of the indicated differential equation and initial conditions. Consequently, we must have

$$f(x) \equiv -e^{ax}f(-x).$$

The problem can also be solved by exhibiting $f(x)$ in terms of exponentials. However, this method requires a division into two cases.

Solved also by T. S. George, B. P. Hoover, F. Underwood, and W. M. Whyburn.

Editorial Note. In the direct substitution referred to above it may aid the reader to note first that $f(-x)$ satisfies the differential equation $y''+ay'-by=0$, where $f(x)$ is a solution of $y''-ay'-by=0$. The latter equation and $u_{n+2}-au_{n+1}-bu_n=0$ have the same characteristic equation $x^2-ax-b=0$; and there are two cases, one for unequal roots and the other for equal roots. Whyburn set $w'=f(x)$ with $w(0)=0$ and found that w is the unique solution of $w''-aw'-bw=u_1$, $w(0)=0$, $w'(0)=0$. After obtaining the solution of this equation by standard methods he verified the desired result. The remaining solutions were essentially as follows for the two cases:

$$u_n = u_1(\alpha^n - \beta^n)/(\alpha - \beta), \qquad u_n = u_1 n\alpha^{n-1},$$

where α and β are the roots of the characteristic equation. The verification of the desired result for the two cases is then easy. Whyburn obtained

$$f(x) = \frac{2u_1 e^{ax/2}}{d} \sinh\left(\frac{dx}{2}\right), \qquad d = \alpha - \beta,$$

which covers both cases, if we take the limit for $d\to 0$ when the two roots are equal.

3803 [1936, 580]. *Proposed by Frank Ayres, Jr., Dickinson College.*

Prove that

(a) $$\sum_{s=0}^{n} (-1)^s 2^{2s} \binom{n}{s}\binom{2n-2s}{n-s} = (-1)^n \binom{2n}{n},$$

(b) $$\sum_{s=0}^{n-1} (-1)^s 2^{2s} \binom{n}{s}\binom{2n-2s}{n+1-s} = (-1)^{n-1} \binom{2n}{n+1}.$$

Solution by Emma Lehmer, Bethlehem, Pa.

Parts (a) and (b) of this problem can be thought of as special cases for $k=n$ and $k=n+1$ of the following identity.

(1) $$\sum_{s=0}^{m} (-1)^s 2^{2s} \binom{n}{s}\binom{2n-2s}{k-s} = (-1)^k \binom{2n}{k},$$

where m is the lesser of n and $2n-k$.

To establish (1) we start with the obvious algebraic identity

$$(1+x)^{2n}\left[1-\frac{4x}{(1+x)^2}\right]^n=(1-x)^{2n},$$

and first expand the binomial in square brackets by the binomial theorem. Then multiplying the expansion by $(1+x)^{2n}$, we obtain

$$\sum_{\nu=0}^{n}(-1)^\nu 2^{2\nu}\binom{n}{\nu}(1+x)^{2n-2\nu}x^\nu=(1-x)^{2n}.$$

We next expand the remaining binomials and get

$$\sum_{\nu=0}^{n}(-1)^\nu 2^{2\nu}\binom{n}{\nu}x^\nu\sum_{\mu=0}^{2n-2\nu}\binom{2n-2\nu}{\mu}x^\mu=\sum_{n=0}^{2n}(-1)^n\binom{2n}{m}x^m.$$

It now remains to equate the coefficients of x^k in the above identity. To do this let $\nu=s, \mu=k-s$, and $m=k$ and sum over s running from 0 to n if $k\leqq n$ and from 0 to $n-2k$ if $k>n$. This gives (1).

3804 [1936, 581]. *Proposed by N. A. Court, University of Oklahoma.*

Given four spheres with non-coplanar centers, their six spheres of similitude are grouped into three pairs so that each pair involves all four given spheres. If the spheres in two of these pairs are orthogonal, the two spheres of the third pair are also orthogonal to each other. (See, for instance, *Modern Pure Solid Geometry* by the proposer. The Macmillan Co., 1935.)

Solution by the Proposer.

The centers of the six spheres of similitude of four given spheres taken in pairs lie in a plane (P) and on the six edges of the tetrahedron $ABCD$ having for vertices the centers, A, B, C, D of the given spheres (*ibid.*, p. 204, art. 635); hence these six centers are the vertices of a complete quadrilateral (Q). The six spheres of similitude are orthogonal to sphere $(O)=ABCD$, hence the plane (P) cuts these spheres of similitude along circles orthogonal to the circle of intersection (c) of (P) with (O). Now if the two spheres of similitude having their centers on the two opposite edges AB, CD of the tetrahedron $ABCD$ are orthogonal to each other, their centers are conjugate with respect to the sphere (O) (*ibid.*, p. 181, art. 576) and therefore also with respect to the circle (c).

Thus, by hypothesis, two pairs of opposite vertices of the complete quadrilateral (Q) are conjugate with respect to (c), hence the same is true about the third pair of vertices, according to Hesse's theorem. Consequently the two spheres of similitude having these two vertices for centers are orthogonal to each other (*ibid.*, p. 181, art. 573).

Editorial Note. A solution by vectors can be given. Let $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{a}_4$ be the vectors of the vertices of a tetrahedron with the origin O at the circumcenter,

and let r_1, r_2, r_3, r_4 be the radii of spheres having their centers at the respective vertices. If R is the radius of the circumsphere, $R^2=\mathbf{a}_i^2$, $i=1, 2, 3, 4$; and then the equations of two spheres and their sphere of similitude are respectively

$$\mathbf{x}^2+R^2-2\mathbf{a}_1\cdot\mathbf{x}-r_1^2=0; \qquad \mathbf{x}^2+R^2-2\mathbf{a}_2\cdot\mathbf{x}-r_2^2=0;$$
$$(r_1^2-r_2^2)(\mathbf{x}^2+R^2)-2(r_1^2\mathbf{a}_2-r_2^2\mathbf{a}_1)\cdot\mathbf{x}=0. \tag{1}$$

The sphere of similitude for 3, 4 is then easily written. A necessary and sufficient condition that the spheres of similitude 1, 2 and 3, 4 be orthogonal is the vanishing of

$$\phi(12, 34)\equiv(r_1^2\mathbf{a}_2-r_2^2\mathbf{a}_1)\cdot(r_3^2\mathbf{a}_4-r_4^2\mathbf{a}_3)-(r_1^2-r_2^2)(r_3^2-r_4^2)R^2. \tag{2}$$

It is easily verified that

$$\phi(12, 34)+\phi(13, 42)+\phi(14, 23)\equiv 0; \tag{3}$$

hence, if the two spheres in two of these pairs are orthogonal, the two spheres of the remaining pair must also be orthogonal.

If the radii of the four spheres are given, the orthogonality requirement of the problem imposes two conditions upon the relative positions of the centers. The simplest case is that for which all the radii are equal: it then follows from (2) and (3) that $(\mathbf{a}_i-\mathbf{a}_j)\cdot(\mathbf{a}_k-\mathbf{a}_l)=0$ for all sets of values of the subscripts no two of which are equal. This says that the tetrahedron of centers is orthocentric.

3799 [1936, 500]. *Proposed by E. G. Olds, Carnegie Institute of Technology.*

From an urn containing ten balls, numbered from one to ten, balls are drawn one by one and placed in a row of holes, numbered from one to ten, each ball being placed in the proper hole. What is the probability that there will not be an empty hole between two filled ones at any time of the drawing?

I. *Solution by R. F. Rinehart, Case School of Applied Science.*

There is a one to one correspondence between the orders of filling a board of n holes, without the occurrence of a vacant hole between two filled ones at any point in the proceedings, and the orders of emptying a full n hole board under the same restriction. The number of orders of so emptying an n hole board is clearly 2^{n-1}. Hence the required probability is $2^{n-1}/n!$.

A second method is as follows: The number of distinct ways of filling a board with $n+1$ holes is twice the number of ways for an n hole board, since there are precisely two distinct ways of having a last hole to be filled under the required condition. For a one hole board the number of ways is 2^0. Hence the number of ways for a n hole board is 2^{n-1}.

II. *Solution by Michael Goldberg, Washington, D. C.*

Let there be n balls and n holes. Suppose the first ball is placed in the kth hole. Then if no gap is to be formed in the array, each successive ball must be placed directly adjacent, on one side or the other, of the unbroken sequence of balls in their holes. This is the same as the number of permutations of $n-1$

objects of which $k-1$ are of one type and the remaining $n-k$ are of another type. This number is $(n-1)!/(k-1)!(n-k)!$. Hence the total number of successful drawings is

$$\sum_{k=1}^{n} (n-1)!/(k-1)!(n-k)! = 2^{n-1}.$$

The probability of making a complete drawing without gaps is $2^{n-1}/n!$. For $n=10$ it is $2^9/10!$ or $2/14175$.

Solved also by J. P. Ballantine, W. B. Campbell, C. Eisenhart, H. M. Gehman, H. O. Hanson, G. B. Lang, Philip McCarthy, Joseph Noback, Ellis R. Ott, and the proposer.

3801 [1936, 580]. *Proposed by D. H. Lehmer, Lehigh University.*

Show that

$$\text{arccot } 1 = \text{arccot } 2 + \text{arccot } 5 + \text{arccot } 13 + \text{arccot } 34 + \cdots,$$

where these integers constitute every other term of the Fibonacci series and satisfy the recurrence $u_{n+1}=3u_n-u_{n-1}$.

Solution by M. A. Heaslet, San Jose State College, San Jose, Calif.

The Fibonacci sequence is determined by $v_{n+2}=v_{n+1}+v_n$ with $v_1=v_2=1$. We shall show that

$$\text{(1)} \qquad \text{arccot } v_2 - \text{arccot } v_3 - \text{arccot } v_4 - \cdots - \text{arccot } v_{2n+1} = \text{arccot } v_{2n+2},$$

where the principal value of arccot is meant. We have

$$\text{arccot } v_{2n} - \text{arccot } v_{2n+1} = \text{arccot}\left[\frac{v_{2n}v_{2n+1}+1}{v_{2n+1}-v_{2n}}\right]$$

$$= \text{arccot}\left[\frac{v_{2n}v_{2n+1}+1}{v_{2n-1}}\right].$$

From the easily deduced relation $v_{m+1}v_{m+2}-v_mv_{m+3}=(-1)^m$ (see *Théorie des Nombres*, N. Kraitchik, page 3) the above becomes on setting $m=2n-1$,

$$\text{(2)} \qquad \text{arccot } v_{2n} - \text{arccot } v_{2n+1} = \text{arccot } v_{2n+2}.$$

Summing the members of (2) for $n=1, 2, \cdots, n$, we obtain (1). Since the right member of (1) approaches zero as n becomes infinite, we have finally

$$\text{(3)} \qquad \text{arccot } v_2 = \sum_{n=1}^{\infty} \text{arccot } v_{2n+1}.$$

If we set $v_{2n+1}=u_{n+1}$, $v_2=v_1=u_1=1$, we may write this result as

$$\text{(4)} \qquad \text{arccot } u_1 = \sum_{n=1}^{\infty} \text{arccot } u_{n+1},$$

where $u_1=1$, $u_2=2$, $u_{n+2}=3u_{n+1}-u_n$, as is easily verified.

Solved also by L. S. Johnston, B. P. Hoover, and E. P. Starke.

Editorial Note. In Johnston's solution arccot x was denoted by $C(x)$ and the formula

$$C(v_{2m}) = C(v_{2m+1}) + C(v_{2m+3}) + \cdots + C(v_{2m+2k-1}) + C(v_{2m+2k})$$

was deduced. This can be written

$$C(v_{2m}) = \sum_{k=1}^{\infty} C(v_{2m+2k-1}).$$

Another result was obtained by denoting the inverse of coth x by $H(x)$ and showing that

$$H(v_{2n+1}) - H(v_{2n+2}) = H(v_{2n+3}), \qquad n \geqq 1,$$

and then that

$$H(v_{2m+1}) - H(v_{2m+2}) - H(v_{2m+4}) - \cdots - H(v_{2m+2k}) = H(v_{2m+2k+1}), \quad m \geqq 1.$$

This last result is proved by using the functional relation

$$H(v_{2n+1}) - H(v_{2n+2}) = H\left[\frac{v_{2n+1}v_{2n+2} - 1}{v_{2n+2} - v_{2n+1}}\right],$$

and $v_{2n+1}v_{2n+2}-v_{2n}v_{2n+3}=1$.

A general theorem was proved for which the above are special cases. Let the sequence $t_1, t_2, \cdots$ satisfy $t_{n+2}=t_{n+1}+t_n$, $t_3t_1-t_2=a^2$, where a^2 is not necessarily positive. Then any function $F(t)$ which satisfies

$$F(t_n) - F(t_{n+2}) = F\left[\frac{t_{n+2}t_n + (-1)^n a^2}{t_{n+2} - t_n}\right]$$

also satisfies $F(t_n)=F(t_{n+1})+F(t_{n+2})$.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

The Mathematical Association has for sale at 50 cents a copy a few copies of the following numbers of *Runkle's Mathematical Monthly*: Volume II: March, April, June, July, and September 1860 (with index), mostly lacking covers. Also one copy each of Volume III, October 1860, and March 1861, for 60 cents each. These may be had by sending to Secretary W. D. Cairns.

Several Benjamin Peirce Instructorships at Harvard University are open for the academic year 1939–40. These instructorships are ordinarily awarded to men who have recently received the Ph.D. degree or have had equivalent training.

Those interested in applying should write to the Chairman of the Division of Mathematics.

There will be a meeting of the Chapters of Pi Mu Epsilon at Richmond Virginia on Wednesday December 28 at 8:00 A.M. in the Hotel William Byrd.

Professor E. T. Bell of the California Institute of Technology has been awarded the gold medal of the Commonwealth Club of California for his book entitled *Men of Mathematics*. The medal is awarded for "the best scholarship and research work published during 1937."

The University of St. Andrews conferred an honorary degree upon Dean G. D. Birkhoff of Harvard University on July 5, at the tercentenary celebration of the birth of James Gregory.

A dinner was given in honor of Professor Emeritus Virgil Snyder and Mrs. Snyder on Friday evening, September 9, at the Hotel Woodstock, New York. The number attending was thirty-six. A silver bowl was presented on which is engraved: "Presented to Professor Virgil Snyder, Cornell University, by his Ph.D.'s as a token of appreciation," and also the names of the thirty-nine mathematicians who have written doctoral theses with Professor Snyder.

Assistant Professor Reinhold Baer of the University of North Carolina has been appointed an associate professor at the University of Illinois.

D. C. Baillie has been appointed a lecturer at the University of Toronto.

Dr. R. H. Bardell of the University of Wisconsin Extension Division at Milwaukee has been promoted to an assistant professorship.

Dr. R. P. Boas, Jr., National Research Fellow in mathematics, will spend the year 1938–39 at the University of Cambridge.

On an exchange arrangement, Dr. A. W. Boldyreff has been appointed an associate professor at the University of Illinois for the year 1938–39, while Dr. O. K. Bower is an assistant professor at the University of Arizona.

Assistant Professor H. E. Bray of Rice Institute has been promoted to an associate professorship.

Dr. Randolph Church of the United States Naval Academy has been promoted to an assistant professorship.

W. G. Cochran of Rothamsted Experimental Station in Harpenden, England, has been appointed visiting lecturer in mathematics and statistics at Iowa State College for the fall quarter of 1938.

Associate Professor L. W. Cohen of the University of Kentucky is on leave of absence for 1938–39 to study at Brown University.

Associate Professor C. H. Currier of Brown University has retired with the title emeritus.

Professor Arnaud Denjoy of the University of Paris is in residence as an exchange professor at Harvard University for the first half of 1938–39.

Dr. A. H. Diamond of the University of California at Los Angeles has been appointed professor and head of the department of mathematics at Oklahoma Agricultural and Mechanical College.

Dr. C. H. Dowker has been appointed lecturer at the University of Western Ontario.

Dr. Nelson Dunford of Yale University has been promoted to the rank of assistant professor.

Dr. W. H. Erskine of Bethany College, West Virginia, has been promoted to an assistant professorship.

Assistant Professor H. P. Evans of the University of Wisconsin has been promoted to an associate professorship.

Associate Professor D. A. Flanders of New York University has returned after a year's study at the University of Copenhagen.

Dr. J. S. Frame of Brown University has been promoted to an assistant professorship.

Assistant Professor H. J. Gay of Worcester Polytechnic Institute has been promoted to a professorship.

At the University of Florida, Assistant Professors H. H. Germond and Z. M. Pirenian have been promoted to associate professorships.

Associate Professor B. P. Gill of the College of the City of New York has been promoted to a professorship.

W. L. Graves of the University of Pennsylvania has been appointed an assistant professor at Drury College.

Dr. J. A. Greenwood of Duke University has been promoted to an assistant professorship.

Dr. D. C. Harkin of Alabama Polytechnic Institute has been appointed tutor at Brooklyn College for 1938–39.

Dr. M. L. Hartung of Ohio State University has been appointed an assistant professor in education at the University of Chicago.

Associate Professor T. W. Hatcher of Virginia Polytechnic Institute has been promoted to a professorship.

Dr. E. A. Hedburg of the University of Missouri has been appointed to a professorship at Baylor University.

Associate Professor G. A. Hedlund of Bryn Mawr College has a year's leave of absence to study at the Institute of Advanced Study.

Dr. W. L. Hutchings of Rollins College has been promoted to an assistant professorship.

Incident upon the transfer of the School of Engineering from Chapel Hill to Raleigh, Professor T. F. Hickerson and Associate Professor R. M. Trimble of the department of civil engineering have been added to the mathematics staff at the University of North Carolina.

Dr. Nathan Jacobson of the University of North Carolina has been promoted to an assistant professorship.

Professor C. G. Jaeger of Pomona College is on leave of absence for the first semester of 1938–39 for study at Brown University.

Associate Professor F. E. Johnston of George Washington University has been promoted to a professorship.

Assistant Professor B. W. Jones of Cornell University is on leave of absence and is studying at the Institute for Advanced Study.

Assistant Professor G. M. Jones of Howard University has been promoted to an associate professorship.

Dr. S. H. Kimball of the University of Maine has been promoted to an assistant professorship.

Dr. Morris Kline has returned to New York University after spending a two years' leave of absence at the Institute for Advanced Study.

Assistant Professor M. S. Knebelman of Princeton University has been appointed professor and chairman of the department of mathematics at the State College of Washington.

Assistant Professor A. H. Larsen of Illinois State Normal University is on leave of absence for study at the University of Wisconsin.

Dr. J. K. L. MacDonald has been appointed to an assistant professorship at Cooper Union.

C. E. Marshall of the University of Illinois has been appointed to an assistant professorship at Oklahoma Agricultural and Mechanical College.

Assistant Professor Deane Montgomery of Smith College has been promoted to an associate professorship.

Assistant Professor H. S. Pollard of Miami University has been promoted to an associate professorship.

Dr. Walter Prenowitz of Brooklyn College has been promoted to an assistant professorship.

After two years' study at the Institute for Advanced Study, Dr. J. F. Randolph has returned to Cornell University with the rank of assistant professor.

Professor E. L. Rees, who has been associated with the University of Kentucky since 1907, has resigned to look after business interests and to travel.

E. K. Ritter of the University of Richmond has been promoted to an assistant professorship.

Associate Professor H. P. Robertson of Princeton University has been promoted to a professorship.

Dr. E. H. Rothe of William Penn College has been promoted to a professorship.

N. N. Royall, Jr., has been appointed to an assistant professorship at the Citadel.

Dr. O. F. G. Schilling has been re-appointed Johnston Scholar for the current year at Johns Hopkins University.

T. C. Sermon of the Michigan College of Mining and Technology has been promoted to an assistant professorship.

Assistant Professor J. H. Service of the Michigan College of Mining and Technology has been promoted to an associate professorship.

Dr. D. T. Sigley of the University of Kansas City has been appointed an assistant professor at Kansas State College.

Professor T. M. Simpson has been appointed acting dean of the Graduate School at the University of Florida.

Assistant Professor F. C. Smith of the College of St. Francis has been promoted to a professorship.

Assistant Professor C. E. Springer of the University of Oklahoma has a year's leave of absence for study at the University of Chicago.

Assistant Professor C. G. Stipe of the Michigan College of Mining and Technology has been promoted to an associate professorship.

Dr. Alvin Sugar of the University of California has been appointed head of the department of mathematics and physics at St. Francis College, Loretto, Pa.

Dr. C. C. Torrance of Case School of Applied Science will be a member of the Institute for Advanced Study during 1938–39.

Professor M. O. Tripp of Wittenberg College has retired with the title of Professor Emeritus.

Professor H. S. Vandiver of the University of Texas is on leave of absence for the first semester of 1938–39.

Assistant Professor W. G. Warnock of Fort Hays Kansas State College has been appointed to an assistant professorship at the University of Alabama.

Dr. K. W. Wegner of the University of Minnesota has been appointed to a professorship at the College of St. Catherine.

Assistant Professor S. S. Wilks of Princeton University has been promoted to an associate professorship.

Dr. C. P. Winsor has been appointed an assistant professor of mathematics and statistics at Iowa State College.

The following appointments to instructorships have been announced:

University of Alabama: Dr. C. R. Cassity
Agricultural and Mechanical Junior College, Arkansas: C. V. Wetzig
Brooklyn College: Dr. J. J. DeCicco
Brown University: Dr. S. E. Warschawski
University of Cincinnati: Dr. W. C. Taylor
College of the City of New York: Dr. Max Shiffman
Columbia University: Dr. J. V. Wehausen
Cornell University: Dr. Fritz Herzog, Dr. S. Saslaw, Edward Stephany
Crosby-Ironton Junior College, Minnesota: J. D. Novak
University of Florida: T. C. George, E. C. Phillips
Hunter College: Dr. Madeline Levin, Dr. Annita Tuller
University of Illinois: Dr. P. R. Halmos, Dr. Olaf Helmer
Kansas State College: Robert Rawhouser
Keuka College: H. F. Archibald
University of Kansas: Edison Greer
University of Kentucky: J. M. Boswell, V. W. Pfeiffer
Michigan State College: Dr. G. B. Van Schaak
University of Michigan: Edith R. Schneckenburger
University of Minnesota: Dr. H. H. Campaigne
University of New Mexico: Dr. Douglas Derry
University of North Carolina: Whitfield Cobb, R. E. Smith
Ohio State University: Dr. G. E. Albert
University of Oklahoma: J. C. Van de Carr
Pennsylvania State College, Dubois Center: Dr. M. E. Shanks
University of Pittsburgh, Johnstown Center: John Breiland
Princeton University: Dr. C. B. Tompkins II
Queens College: Dr. N. A. Hall, Dr. Philip Hartman, Dr. Leo Zippin

University of Rochester: Dr. C. E. Clark
Rutgers University: R. E. Traber
San Antonio Junior College: L. H. Tulloch
Springfield Junior College, Illinois: N. W. Wells
Syracuse University: Dr. A. P. Cowgill
Texas Agricultural and Mechanical College: Dr. B. W. Brewer, Walter Jennings
Texas Technological College: Dr. C. P. Brady, Lida B. May
University of Texas: W. B. Coleman, R. E. Greenwood, Charles Moran
Vassar College: Dr. Audrey Wishard
West Virginia University: M. L. Vest
Wright Junior College, Chicago: Dr. Ruth G. Mason
Yale University: T. J. Benac, E. S. Grable.

Dr. T. W. Edmondson, professor emeritus of mathematics and former acting dean of the Graduate School of New York University, died November 4, 1938, at the age of sixty-nine. He retired from teaching in 1934, after thirty-eight years on the faculty at New York University. He was a charter member of the Mathematical Association.

THE SEMICENTENNIAL CELEBRATION OF THE AMERICAN MATHEMATICAL SOCIETY

Founded at Columbia University in 1888 as the New York Mathematical Society, the American Mathematical Society very appropriately celebrated its semicentennial anniversary by unusually interesting meetings at Columbia on September 6–9. The Society was very fortunate in having present its founder, Professor Thomas Scott Fiske, whom it honored suitably at a "Birthday Dinner." It also took this occasion to express its deep appreciation of the hospitality accorded it by Columbia University, in whose halls over half of its regular meetings have been held, and in whose buildings its central office and library have been housed for half a century.

Letters of felicitation were received from a large number of learned societies and from President Franklin D. Roosevelt and from Prime Minister W. L. Mackenzie King of the Dominion of Canada. Addressed to Dean R. G. D. Richardson, secretary of the Society, the letter from President Roosevelt reads as follows:

Please extend my greetings to the American Mathematical Society on the occasion of the celebration of its fiftieth anniversary. I trust that genuine satisfaction will come to its members as they contemplate the contribution which the Society has made during the half century of its existence.

It is sometimes difficult to comprehend the values accruing to society from mathematics. It is deeply rooted in social progress as a large part of the technological advance made in recent centuries would have been impossible had it not been for the constant refinement of the essential tool of technology—mathematics. Social sciences also owe a large debt to mathematics. Future advances in the social sciences will be largely dependent upon mathematical treatment of their data.

I congratulate the members of the American Mathematical Society on the important contributions which mathematics has already made. I trust that the years ahead will find mathematicians making significant contributions, as in the past.

Prime Minister King's letter reads:

I shall be pleased if you will extend to the American Mathematical Society, on the celebration of the fiftieth anniversary of its inception, my greetings and warm congratulations upon attaining this significant anniversary.

It is well that we should reflect, on occasions such as this, on the extent to which the learned societies of this continent have contributed to the strength and substance of our national and international life. The work of the American Mathematical Society fills a distinguished place among those academic fellowships which have done so much to keep alive, in our institutions of learning, that integrity of thought which is one of the proudest of our common traditions.

It is my hope that the deliberations of the Society's Jubilee Meeting will be attended by much good fellowship and a lively appreciation of the binding character of the academic fraternity which has so long and so happily prevailed between our two countries.

The first day of the celebration was devoted to seven sectional meetings at which over one hundred research papers were communicated. The scientific part of the remainder of the celebration consisted of ten addresses by invitation, which reviewed aspects of the development of mathematics in America during the past fifty years and indicated lines of progress for the future. The speakers and titles of their addresses were as follows:

R. C. Archibald, Brown University: History of the American Mathematical Society, 1888–1938.

G. D. Birkhoff, Harvard University: Fifty years of American mathematics.

E. T. Bell, California Institute of Technology: Fifty years of algebra in America, 1888–1938.

G. C. Evans, University of California: Dirichlet problems.

E. J. McShane, University of Virginia: Recent developments in the calculus of variations.

J. F. Ritt, Columbia University: Algebraic aspects of the theory of differential equations.

J. L. Synge, University of Toronto: Hydrodynamical stability.

T. Y. Thomas, University of California at Los Angeles: Recent trends in geometry.

Norbert Wiener, Massachusetts Institute of Technology: The historical background of harmonic analysis.

R. L. Wilder, University of Michigan: The sphere in topology.

The first of these addresses, together with a full account of the meetings, will appear in full in the January 1939 number of the *Bulletin* of the Society. In greatly amplified form Professor Archibald's paper appears as Volume I of the Society's *Semicentennial Publications*. The other nine addresses are given under the title of "Semicentennial Addresses" in Volume II. These two volumes were ready for distribution at the meetings.

The Association, without holding meetings of its own, joined in the celebration by the Society, and the following letter was formally presented:

The Mathematical Association of America, through its officers and trustees, felicitates the American Mathematical Society on fifty years of successful activity and guidance in the promotion of mathematical research in America. It has developed as a nucleus of an earnest and increasingly able group, respected and highly esteemed at home and abroad. It has exercised a strong influence not merely in pure mathematics but in the fields of secondary and collegiate mathematics.

The Mathematical Association of America is proud of the parent society and is grateful that the Society has uniformly given it a fine support and cooperation. May the next fifty years bring an even more notable advance in high and worthy scholarship.

THE SECOND ANNUAL MEETING OF THE SOUTHWESTERN SECTION

The second annual meeting of the Southwestern Section of the Mathematical Association of America was held at the University of New Mexico, Albuquerque, on Monday and Tuesday, April 25–26, 1938, in conjunction with the meeting of the Southwestern Division of the American Association for the Advancement of Science.

The attendance was eighty, including the following nineteen members of the Association: C. B. Barker, C. A. Barnhart, L. M. Bauer, J. W. Branson, Gordon Fuller, R. F. Graesser, Elizabeth M. Haskins, E. A. Hazlewood, E. R. Hedrick, C. A. Hutchinson, A. J. Kempner, H. D. Larsen, D. H. Leavens, Roy MacKay, L. E. Mehlenbacher, A. B. Mewborn, C. V. Newsom, P. K. Rees, R. S. Underwood.

On Monday evening there was a dinner for members of the Section and their guests, which was followed by an address by Professor A. J. Kempner on "The rôle of isomorphism in scientific systems." Following this address, the group attended the John Wesley Powell Memorial Lecture of the Southwestern Division of the American Association for the Advancement of Science, which was delivered by Provost E. R. Hedrick, of the University of California at Los Angeles, on "The relations of science to economics and to war." Tea was served on Tuesday afternoon by the New Mexico Alpha Chapter of Kappa Mu Epsilon.

At the business meeting the following officers were elected for next year: Chairman, R. S. Underwood, Texas Technological College; Vice-Chairman, L. E. Mehlenbacher, Arizona State Teachers College, Flagstaff; Secretary-Treasurer, H. D. Larsen, University of New Mexico.

Professor J. F. Branson presided as chairman of the "Symposium on teaching problems in mathematics"; Professor R. F. Graesser, chairman, presided at all other sessions.

The following papers were read, the first seven of which constituted the "Symposium."

1. "The function concept in elementary mathematical instruction and in advanced mathematics" by Provost E. R. Hedrick, University of California at Los Angeles.

2. "Some observations on the teaching of the mathematics of finance" by Professor H. D. Larsen, University of New Mexico.

3. "Report from the Joint Commission on the Place of Mathematics in Secondary Education" by Professor C. A. Hutchinson, University of Colorado.

4. "Teaching students to think on paper" by Professor H. B. Leonard, University of Arizona. (Read by Professor A. B. Mewborn.)

5. "Suggestions for reducing mortality in freshman mathematics" by Professor W. P. Heinzman, New Mexico State College. (Read by title.)

6. "Teaching aids for instructors in mathematics" by President H. C. Gossard, New Mexico Normal University, by invitation.

7. "A one-year college mathematics course" by Professor Bulah A. Liles, Texas College of Mines and Metallurgy, introduced by the Secretary.

8. "The efficiency of approximation formulas for determining the rate of interest in amortization schedules" by Wade Ellis, University of New Mexico, introduced by Professor H. D. Larsen.

9. "N-tic residues" by Professor R. S. Underwood, Texas Technological College.

10. "An arrangement of a given triangle and certain of its consecutive cevian triangles in a sequence which forms a convergent geometric progression" by Professor C. A. Barnhart, University of New Mexico.

11. "Invariant differential equations" by Dr. Gordon Fuller, New Mexico State College.

12. "On the behavior of certain entire functions in distant portions of the plane" by Professor C. V. Newsom, University of New Mexico.

13. "A null basis for simple closed curves" by Professor Roy MacKay, Eastern New Mexico Junior College.

14. "The perfect group of order 120, with illustrative model" by Professor J. B. Shaw, University of Illinois. (Read by title.)

15. "Some properties of the numbers 3, 5, 11, 17, and 41" by C. B. Barker, University of New Mexico.

16. "On the roots of algebraic equations with complex coefficients" by Professor A. J. Kempner, University of Colorado.

17. "Moments about the arithmetic mean of a hypergeometric frequency distribution" by Professor H. D. Larsen, University of New Mexico.

18. "A multiple correspondence in space" by Professor E. J. Purcell, University of Arizona. (Read by title.)

Abstracts of some of these papers follow, the numbers corresponding to the numbers in the list of titles:

1. This paper appeared in full in the August-September 1938 number of this MONTHLY.

8. The exact determination of an unknown rate of interest involved in an amortization plan requires the solution of an equation of degree $n+1$. This presents serious practical difficulties, and recourse is therefore made to various approximation formulas. Mr. Ellis discussed the relative efficiency of the more important of these formulas.

9. Professor Underwood generalized, under specified conditions, well known results concerning quadratic residues to apply to n-tic residues.

10. Professor Barnhart showed that, by keeping the corresponding cevian fixed, a sequence formed by a triangle and its consecutive cevian triangles, each in counter-clockwise order, when the cevian in question is either a median or an altitude, arranges itself into two alternate infinite sequences of homothetic triangles, and that the areas of all the triangles in the original sequence form a convergent geometric progression. The question whether this arrangement is possible for the other cevians merits further investigation.

11. The system of equations considered by Dr. Fuller is invariant, but their forms do not show their independence of coördinate systems. The author presented the equations in forms whose law of transformation is evident. Members of the new equations, obtained from the original system, were expressed as the components of a tensor.

12. Professor Newsom presented a revision of a previous theorem (C. V. Newsom, *Bulletin of the American Mathematical Society*, vol. 37, page 666, abstract 282). In the present paper, the author placed more liberal restrictions upon the general coefficient of the given series; namely, when the coefficient $g(n)$ is regarded as a function $g(w)$ of the complex variable $w=x+iy$, it must satisfy the following two conditions: (a) it must be single-valued and analytic throughout the finite w-plane, and (b) it must be such that for all values of x and y one may write $|g(w)| < Ke^{k*|y|}$, where K is a constant independent of x and y, and k is any given positive integer.

13. Professor MacKay showed that every simple closed curve of a peanian (continuous image of a line segment) is homologous to a linear combination of a subset of a particular sequence of simple closed curves of the continuum whose diameters form a null sequence. An application of this property is used to find a necessary and sufficient condition that certain classes of cyclic peanian continua have topological images in the plane.

14. Professor Shaw exhibited a model representing the group of order 120 based on Cayley's color representation. In this way the formation of products of the elements is simple. The order of every element is shown by the model.

15. In an expository paper, Mr. Barker discussed particular properties of the five prime numbers 3, 5, 11, 17, and 41 which apparently are possessed by no other numbers.

16. Professor Kempner extended the principle of symmetry of the roots of equations with real coefficients to equations with complex coefficients. As an incidental result, it was found that fifth degree equations with complex coefficients which are not solvable by radicals must either have their roots distributed symmetrically to the axis of reals, or they must have no real roots and no pairs of conjugate complex roots.

17. In a recent paper, Kirkman developed a method of continuation for obtaining the moments about the arithmetic mean of a binomial distribution. Professor Larsen extended the method to obtain the moments about the arithmetic mean of a hypergeometric distribution.

C. A. BARNHART, *Acting Secretary*

THE ANNUAL MEETING OF THE KENTUCKY SECTION

The twenty-first annual meeting of the Kentucky Section of the Mathematical Association of America was held at Morehead State Teachers College on Saturday, May 14, 1938, in conjunction with the annual meeting of the Kentucky Academy of Science. Professor D. E. South, chairman of the Section,

presided. In the absence of Professor Fehn, the chairman appointed Professor W. R. Hutcherson as temporary secretary.

There were forty-seven in attendance, including the following fifteen members of the Association: P. P. Boyd, M. C. Brown, H. H. Downing, L. A. Fair, Georgia M. Haswell, Charles Hatfield, W. R. Hutcherson, Fritz John, F. Elizabeth LeStourgeon, W. L. Moore, Sallie E. Pence, D. W. Pugsley, W. F. Smith, D. E. South, Guy Stevenson.

The following papers were presented:

1. "On representation of binary quadratic forms" by Professor N. B. Allison, Kentucky Wesleyan College, introduced by the Secretary.

2. "Maximum principle for elliptic differential equations" by Professor Fritz John, University of Kentucky.

3. "Calculus in biology; history in surveying—two illustrations" by Professor W. R. Hutcherson, Berea College.

4. "Configuration of double points of cubics of a pencil" by Dr. Sallie E. Pence, University of Kentucky.

5. "A rôle for mathematics in the sciences" by Professor W. L. Moore, University of Louisville.

6. "Summation of divergent series" by Mrs. A. S. Howard, University of Kentucky, introduced by Professor Cohen.

7. "The place of astronomy in the training of high school teachers" by Professor W. F. Smith, New River State College, West Virginia.

Abstracts of some of the papers follow, numbered in accordance with their place on the program:

2. Dr. John showed how the maximum principle is proved for potential functions and can be extended to the solutions of more general linear elliptic differential equations of the second order.

3. Professor Hutcherson stated in his first illustration that, in finding the consumption of sugar per protozoan in a medium where reproduction was obeying the law of logarithmic growth, the integral calculus was used. In his second illustration an old deed of a forty-acre farm, which has been a part of Berea College campus for nearly sixty years, was studied. Its corners were unknown, since the college owned the surrounding land. Mathematics relied upon the memory of oldest residents of the neighborhood concerning possible corners. In cross word puzzle fashion the plat was finally oriented.

4. Dr. Pence presented a paper on the configuration of the double points of the twelve nodal cubics of a pencil of cubics on nine associated points. After stating some properties for the general case, she studied the D_{12} configuration by means of the Geiser and Bertini transformations. She discussed certain special configurations. Among these it was shown that in the pencil of equianharmonic cubics there are only three distinct cubics having double points, each cubic being composed of three concurrent lines. It was also shown that the nodal cubics in the pencil of cubics invariant in the quadratic involution consist of six degenerate cubics, each consisting of an invariant line and a conic which cuts

the line in two double points. There are thus twelve double points which lie by sixes on four conics.

5. Professor Moore gave a brief account of a portion of the book *An Introduction to the Philosophy of Science* by A. C. Benjamin. This portion dealt with the use of symbols in scientific explanation. The speaker confined himself to the use of mathematical symbols as a medium for explanation.

6. The sum of an infinite series $\sum_0^\infty a_n$ can be defined as the limit of a sequence, $s_n = a_0 + \cdots + a_n$. To sum series for which the s_n oscillate, we seek another sequence σ_n which shall (a) have the same limit as s_n in those cases where $\lim s_n$ exists, (b) exist in cases where $\lim s_n$ does not exist. Mrs. Howard used the method of weighted means to determine σ_n and discussed its properties. From them she showed that at least the limit of the means of the partial sums of the Cauchy product $\sum_{n=0}^\infty (\sum_{i=0}^n a_i b_{n-i})$ of two convergent series exists and is equal to the product of the two series.

7. Professor Smith compared the teacher training requirements for certification of science teachers with the contents of ten different textbooks recently published. He found that not more than one-half of the content matter of these texts is covered by certificate requirements in Kentucky and West Virginia. In all the texts considered, with one exception, at least one whole chapter was devoted to the subject of astronomy with other chapters given over to allied subjects, but in neither state is astronomy included in any part of the teacher training program.

W. R. HUTCHERSON, *Secretary pro tem*

THE ANNUAL MEETING OF THE ROCKY MOUNTAIN SECTION

The twenty-second annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at the University of Colorado, Boulder, Colorado, April 15 and 16, 1938. There were three sessions. Professor C. A. Hutchinson presided at each. The Saturday morning session was a joint meeting with the mathematics section of the Eastern Division of the Colorado Educational Association.

The attendance was seventy-one, including the following twenty-four members of the Association: L. A. Aroian, C. F. Barr, J. R. Britton, I. M. DeLong, J. R. Everett, J. C. Fitterer, G. W. Gorrell, D. F. Gunder, I. L. Hebel, C. A. Hutchinson, L. Louise Johnson, A. J. Kempner, Claribel Kendall, A. J. Lewis, G. H. Light, S. L. Macdonald, A. E. Mallory, A. S. McMaster, W. K. Nelson, Greta Neubauer, O. H. Rechard, A. W. Recht, C. H. Sisam, H. C. Wiedeman.

At the business meeting the following officers were elected for next year: Chairman, C. F. Barr, University of Wyoming; Vice-Chairman, D. F. Gunder, Colorado State College.

Following a luncheon at the High School on Saturday, Professor A. J. Kempner, president of the Association, gave a very interesting address on "The situation in collegiate and secondary mathematics."

The following papers were presented:

1. "The type B Gram-Charlier series" by Professor L. A. Aroian, Colorado State College.

2. "Mathematics and science—after Keyser" by Professor S. L. Macdonald, Colorado State College.

3. "Teaching large sections in freshman mathematics" by Professor O. H. Rechard, University of Wyoming.

4. "Roots of algebraic equations with complex coefficients" by Professor A. J. Kempner, University of Colorado.

5. "Slide rules for the solution of two problems in spherical trigonometry" by Professor I. L. Hebel, Colorado School of Mines.

6. "Lagrangean multipliers" by Professor C. A. Hutchinson, University of Colorado.

7. "Cooperation between high school and college teachers of mathematics" by Professor A. E. Mallory, Colorado State College of Education, and F. A. St John, South High School, Denver.

8. "The contributions of high school geometry to the goals of education" by Professor A. J. Lewis, University of Denver.

9. "Preparation of teachers of mathematics with reference to the requirements of the North Central Association" by Professor A. E. Mallory, Colorado State College of Education.

10. "Approximate numbers" by Dr. J. R. Britton, University of Colorado.

11. "Report on the work of the Joint Commission" by Professor C. A. Hutchinson, University of Colorado.

12. "The use of placement examinations in pre-college mathematics" by Professor W. J. Hazard, University of Colorado, introduced by the Secretary.

Abstracts of some of the papers follow, the numbers corresponding to the numbers in the list of titles.

1. Professor Aroian showed how seven terms of the type B Gram-Charlier series may be used in fitting a frequency distribution. (The paper has appeared in the December 1937 issue of the *Annals of Mathematical Statistics*.)

2. Professor Macdonald discussed the boundary lines of science and mathematics—the distinction between the two and what each is and what each is not. The ideas were chiefly derived from two books by C. J. Keyser, namely: *Pastures of Wonder* and *Humanism and Science*.

3. Professor Rechard taught a college algebra class of 66 students and one in trigonometry containing 78 students in the fall and winter quarters respectively of the current year, employing a classroom procedure differing from the usual recitation method. Normal sections of each subject during each quarter, were taught by other members of the department. On the basis of the Ohio College Ability Test both types of sections were found to be comparable. Final examination grades showed some advantage for the large section in college algebra, but no significant differences either way were found for trigonometry.

5. Professor Hebel designed slide rules for the calculation of the time of sunrise (or sunset) at a given place on a given date, and the determination of

the distance between two points on the surface of the earth. The slide rule for the sunrise problem is a single-setting rule from which the time is read directly from the given latitude and the sun's declination, a table of declinations being given on the reverse side. The second slide rule is a two-setting rule based on the cosine law, from which the distance in statute miles is found from the latitudes of the two places and the difference in their longitudes; for greater accuracy distances have been limited to about 6,000 miles.

6. Professor Hutchinson gave a brief exposition of the subject of constrained maxima and minima, with a discussion of the use of multipliers, and applications in the field of adjustment of observations.

8. In this paper Professor Lewis attempted to show that high school geometry is particularly well fitted to contribute to some of the generally recognized goals of education.

9. The standards set forth by the North Central Association for mathematics teachers are minimum. Adequate preparation defined in terms of present conditions and curriculum trends of the high school imply a general and specialized preparation. The most serious result of these trends is the tendency to minimize the importance of long experience. Mathematics has been the victim of this criticism because immediate and final ends have been confused. Professor Mallory believes that the formal-training pattern of the mathematics teacher should include a broad general education, training in related subjects, professional training in general and special courses, and student teaching.

10. Dr. Britton gave a brief exposition of the rules of computation with approximate data.

11. A report was made by Professor Hutchinson of the progress of the work of the Joint Commission on the Place of Mathematics in Secondary Schools, and of the Commission's plans for the completion of their work.

12. Professor Hazard showed a curve of the scores made by 665 freshmen on a placement examination in pre-college mathematics. This curve was compared with the curve of standardized results from 8200 students and showed very much lower scores, varying from 50% to 80% of the standard. A list of suggested questions for such an examination was offered for criticism.

A. J. LEWIS, *Secretary*

THE SIXTH ANNUAL MEETING OF THE WISCONSIN SECTION

The sixth annual meeting of the Wisconsin Section of the Mathematical Association of America was held at the Columbus Community Club of St. Norbert College at Green Bay, Wisconsin, on May 14, 1938. The chairman of the Section, Professor Ethelwynn R. Beckwith of Milwaukee-Downer College, presided. The attendance was forty-five including the following twenty members of the Association: R. H. Bardell, Leon Battig, Ethelwynn R. Beckwith, May M. Beenken, W. W. Bigelow, L. A. V. DeCleene, Henry Ericson, R. C. Huffer, G. J. Kalcik, Elizabeth E. Knight, Morris Marden, Sister Mary Felice,

E. A. Nordhaus, R. E. Norris, G. A. Parkinson, H. P. Pettit, Irene Price, W. E. Roth, P. L. Trump, J. I. Vass.

Sessions were held in the morning and afternoon with luncheon in the Community Club at 12:30. At the close of the luncheon the Very Rev. A. M. Keefe of St. Norbert College graciously welcomed the visiting mathematicians and their guests. The chairman, Mrs. Beckwith, responded. There was also a short talk on the activities of the Intercollegiate Mathematical Association of Milwaukee by Miss Norma Fedders of Milwaukee-Downer College introduced by Mrs. Beckwith.

The business meeting was held at 1:30 p.m. at which the following officers were elected for the coming year: Chairman: R. C. Huffer, Beloit College; Secretary: G. A. Parkinson, University of Wisconsin Extension Division; Program Committee: W. E. Roth, University of Wisconsin Extension Division, and P. L. Trump, Wisconsin High School, Madison. An invitation to meet next year at Milwaukee State Teachers College was accepted by unanimous vote. Appreciation for the hospitality extended to the Section by St. Norbert College was expressed by a rising vote. After the afternoon session, tea was served. This was followed by a short visit to points of historic interest in and around Green Bay and West De Pere.

After the business meeting the afternoon session was devoted to a round-table discussion of the topic, "The survey course in college mathematics." The discussion was introduced by E. A. Nordhaus of the University Extension Division in Milwaukee. W. W. Bigelow of Beloit College, Sister Mary Felice of Mount Mary College, and Professor H. P. Pettit of Marquette University participated.

At the morning meeting the following papers were presented:

1. "On certain pythagorean numbers" by Leon Battig, University of Wisconsin Extension Division.

2. "The estimation of the total fish population of a lake" by Zoe E. Schnabel, University of Wisconsin, introduced by the Secretary.

3. "The number of circles covering a given point set" by R. B. Kershner, University of Wisconsin, introduced by the Secretary.

4. "Applications of mathematics to nuclear physics" by Professor Gregory Breit, Department of Physics, University of Wisconsin, by invitation of the Program Committee.

Abstracts of these papers follow, numbered in accordance with their listing above.

1. Series and recursion formulas for sets of pythagorean integers of which the smaller two differ by one can be found in the May 1914 issue of this MONTHLY in an article by George A. Osborne. In the present paper Mr. Battig presented a proof, using only algebraic means, that these series furnish all such integers, exhibited a related series in which the hypotenuse integer was followed by the perimeter of that triangle, and obtained a theorem on sums of squares of two successive terms of this series. Such sums are terms of the same series.

2. Miss Schnabel's paper was published in full in the MONTHLY for June–July 1938.

3. Mr. Kershner consider the following problem: Given a plane point set, how many circles of a given "small" radius ϵ are required to cover it? By geometrical methods, involving the use of the Euler relation, he obtained the following result: Let A denote the measure of the closure of the given set. Let $N(\epsilon)$ be the minimum number of circles of radius ϵ which will cover the set. Then $\epsilon^2 N(\epsilon) \to 2\sqrt{3}A/9$ as $\epsilon \to 0$.

4. Professor Breit discussed the mathematical methods which have proved of special value in the study of atomic nuclei. The non-Coulombian forces between two protons were compared with proton-neutron and neutron-neutron forces. Sufficient conditions for energy saturation were discussed; restrictions and conclusions were brought out. Comparison of calculations with the experiments indicates that the observed fine structure is greater than expected on the simplest form of theory. The possibility of still finer structure of nuclear levels was considered.

G. A. PARKINSON, *Secretary*

THE DIVISION OF ANGLES INTO EQUAL PARTS AND POLYGON CONSTRUCTION

W. B. GIVENS, Fresno State College

Angle trisection has long been of great interest due, primarily, to the restrictions placed on its solution. However, the division of angles into any number of equal parts is of greater importance because of the wider use which may be made of this knowledge. In an article published last year I developed a method for trisecting angles by the use of the strophoid as an auxiliary curve [1]. While investigating this problem, I discovered another method for solving it by the use of an entirely different curve; and the further fact that by its use, angles can also be divided into any number of equal parts.

In the time of the ancient Greeks, Hippias invented a device, known as the quadratrix, by means of which angles can be divided into equal parts. Gino Loria devotes a chapter in one of his books to a consideration of various sectrix curves which may be used for the division of angles into three or more equal parts [2].

This paper is given over, first, to a discussion of the general equation defining an auxiliary curve used for the division of angles; second, to the problem of constructing auxiliary triangles which are used in laying out the curve with compasses and ruler; and third, to the division of angles into equal parts and polygon construction.

In Figure 1, lay off a unit distance OC on the polar axis OX. Draw a line OA forming angle θ with OC. Then the problem is to locate a point D on line OA so that angle $ODC = \theta/n$, n being a positive integer. Let OD be the radius vector, r. Since angle $OCD = 180° - (n+1)\theta/n$ we see, by the law of sines, that the polar equation of the locus of D is

$$r = \sin\frac{n+1}{n}\theta / \sin\frac{\theta}{n}.$$

This locus is a closed curve; it has a period of $n\pi$; it is symmetrical with respect to the initial line; and it intersects the initial line once where $r=n+1$, $n-1$ times where $r=1$, and n times at the origin. Figure 1 gives the locus when $n=7$.

Points for the construction of the curve can be located with compasses and ruler. With C as a center and radius CO, draw a circle. At a point E on the circle, draw line EO, forming angle POC. With E as center and radius EO, draw a circle intersecting line CE at points F and F_1. With F as center and radius FO, draw a circle intersecting line CE at D and D_1. With F_1 as a center and radius F_1O, draw a circle intersecting line CE at points D_2 and D_3. Thus it is seen that

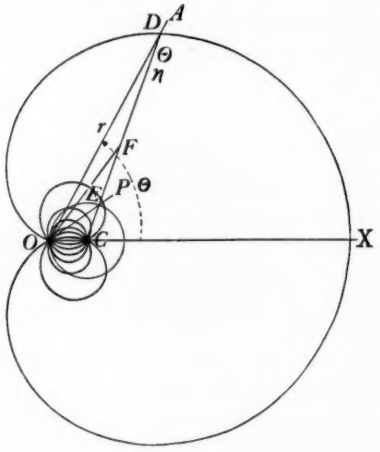

FIG. 1

FIG. 2

for each position of E on the circle, four points, D, D_1, D_2, and D_3 are located on the curve. In Figure 1, only point D is indicated by letter since it is the only one used in the constructions to follow. As E moves around the circle these four points will generate the curve.

Now draw line DO. If we assume angle $ODC=\theta/7$, then angle $DOF=\theta/7$, angle $FOE=$angle $OFE=2\theta/7$, and angle $EOC=$angle $OEC=4\theta/7$. Therefore angle $DOC=\theta/7+2\theta/7+4\theta/7=\theta$, and the curve for $n=7$ has been solved.

In this method of geometrical construction, the initial angle POC, drawn through a point E on the circle, is assumed and will be known as a *primary construction angle*. Hereafter, for ease of construction, this angle will be limited to 90° or less. The completed triangle ODC will be known as an *auxiliary triangle*.

If n is any positive integer, it may be expressed as powers of 2 in the form

$$n = a_0 2^k + a_1 2^{k-1} + a_2 2^{k-2} + \cdots + a_k 2^0,$$

where a_i is given the appropriate value of 1 or 0 [3]. By this method, auxiliary triangles can be constructed for any number of parts, even or odd; however, since the constructions involving the powers of 2 are more easily made by plane geometry, and since the construction for any other even number can be reduced to a construction for an odd number, hereafter, for simplicity, only odd integers will be considered. The primary construction angle always has a value of $2^k\theta/n$, the largest power of 2 contained in n, and the vertex angle ODC always has a value of θ/n. When the primary construction angle is 90°, then $\theta = 90°n/2^k$. Since a limitation of 90° will be placed on this angle, as stated for ease of construction, auxiliary triangles can be drawn for all values of θ lying between 0° and $90°n/2^k$.

Each auxiliary triangle is formed by the construction of successive isosceles triangles whose equal angles complete the series which can be expressed by the equation for n when $a_i = 1$; the equation for the series assumes the form

$$s_n = 2^k + 2^{k-1} + 2^{k-2} + \cdots + 1.$$

When $a_i = 1$ in the equation for n, this equation and the equation for s_n are identical; that is, for 3, 7, 15, 31, and 63 parts, all the numbers less than 100 which can be expressed by the equation, $n = 2^k - 1$. In constructing auxiliary triangles when n equals any of these numbers, all the isosceles triangles are drawn outside the primary construction angle, so that one of the equal angles of each may be added to it to complete angle θ. Figure 1 presents the general method for constructing auxiliary triangles when n has a value of any of the numbers just listed. All other odd numbers will be treated in the following paragraph.

In Figure 2, a problem is presented which illustrates the general procedure for the construction of auxiliary triangles for all odd numbers which cannot be expressed by the equation, $n = 2^k - 1$. An auxiliary triangle is drawn when $n = 11 = 2^3 + 2 + 1$ and the equation for $s_{11} = 2^3 + 2^2 + 2 + 1$. Since the number 2^2 does not appear in the equation for n, the isosceles triangle whose equal angles have a value of $2^2\theta/11$ in the construction will be drawn inside the primary construction angle. The primary construction angle has a value of $2^3\theta/11$ and is assumed as angle POC. With C as a center and radius CO, draw an arc of a circle intersecting PO at E. On PO lay off $EF = EC$. On FC produced, lay off $FG = FO$. Draw line GO. On CG produced, lay off $GA = GO$. Draw line AO. Then angle $OAC = 1/11$ angle AOB. Since the equal angles of the isosceles triangle FEC have a value of $2^2\theta/11$ in the construction, this triangle is drawn inside the primary construction angle POC as shown.

Now we come to the problem of dividing angles into equal parts and polygon construction. The number of regular polygons which can be constructed with compasses and ruler is few compared to the number which cannot be so constructed. As noted by Dickson [4], the number which can thus be constructed up to 100 is 24; up to 300 is 37 (all noted by Gauss); up to 1000 is 52; up to 1,000,000 is only 206. The method here described offers a means for the construction of regular polygons of any odd number of sides.

In Figure 3 a method is given for dividing an angle into seven equal parts; also the method for inscribing a regular polygon of seven sides. A right angle AOB is to be divided into seven equal parts. With a radius CO and center at C, draw an arc of a circle. Three or more primary construction angles are assumed and the essential constructions of the auxiliary triangles are completed as in Figure 1. A little practice will indicate the approximate positions for the assumed angles, so that a segment of the curve will intersect line AO at D. Draw line DC. Then angle $ODC = 1/7$ angle AOB. With D as a center and any radius, draw an arc of a circle cutting OD at L and CD at M. Locate a point on the arc such that arc LN is four times arc LM. Then arc LM is $1/7$ of the quadrant and arc LN is $1/7$ of the circle.

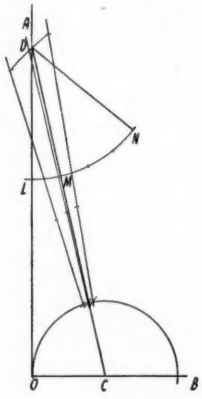

FIG. 3

FIG. 4

Figure 4 offers a method for dividing an angle into eleven equal parts, as well as one for inscribing a regular polygon of eleven sides. A right angle AOB is drawn. With C as a center and radius CO, draw an arc of a circle. Three or more primary construction angles are assumed and the auxiliary triangles drawn as in Figure 2, and so located that a segment of the curve will intersect line AO at D. Draw line DC. Then angle $ODC = 1/11$ angle AOB. With D as a center and any radius, draw an arc of a circle intersecting OD at L and CD at M. Locate a point on this arc such that arc LN is four times arc LM. Then arc LM is $1/11$ of the quadrant and arc LN is $1/11$ of the circle.

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ON ARCCOTANGENT RELATIONS FOR π

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In a recent number of the MONTHLY, J. W. Wrench* has brought up again the subject of arccotangent relations for π. This topic, despite the transcendence of π and of the arccotangent function, is actually a chapter of diophantine equations, and has always held a fascination for the devotees of that difficult but entertaining field.

The problem of expressing a rational multiple of π as a sum of arccotangents, however, can be solved in an infinite number of ways, most of them uninteresting. While I do not presume to set down any hard and fast rules for this indoor sport, still I should like to point out certain possibilities as well as certain inescapable facts which should not be overlooked by those seeking interesting new relations for π, and to give a rough scheme for comparing such relations with each other. I take this opportunity to subjoin a list of familiar and unfamiliar relations. This list is not guaranteed to contain all previously published relations, as there are most certainly a few others scattered through the literature which have escaped the writer's notice.

In such a relation as the famous Machin formula

$$\frac{\pi}{4} = 4 \operatorname{arccot} 5 - \operatorname{arccot} 239$$

it is intended that the arccotangents be evaluated by the Gregory series

$$\operatorname{arccot} x = \frac{1}{x} - \frac{1}{3x^3} + \frac{1}{5x^5} - \cdots .$$

Now it is clear that the coefficients, $1, 1/3, 1/5$, *etc.*, cannot be depended upon to help a great deal in making the terms tend to zero, and that for practical purposes these coefficients furnish only enough additional accuracy to take care of the few extra figures beyond those planned in advance to which one would prudently carry the calculation. In short, it is not far from the truth to say that the number of terms of the series which need to be taken to obtain a specified accuracy varies inversely as the common logarithm of x. Moreover, if one has a relation of n terms of the type

$$\frac{k\pi}{4} = \sum_{i=1}^{n} a_i \operatorname{arccot} m_i \tag{1}$$

then the amount of labor required by such a relation is proportional to

$$\sum_{i=1}^{n} \frac{1}{\log m_i}, \tag{2}$$

a quantity which we shall call the *measure* of the relation (1), and which we

* This MONTHLY, vol. 45, 1938, pp. 108–109.

shall adopt (with modifications, as mentioned later) as a guide to the discovery of more efficient arccotangent relations. For simplicity in calculation we use common logarithms in (2), although it is clear that any other logarithms would given proportional values to the measure.

Considerable typographical simplicity is achieved by writing

$$\operatorname{arccot} x = [x].$$

Thus Machin's formula becomes simply

$$[1] = 4[5] - [239],$$

and it has a measure of $1/\log 5+1/\log 239=1.8511$.

It is generally required that we deal only with angles whose cotangents are integers. This requirement is not imposed in the theorems that follow.

One of the methods commonly employed to derive one arccotangent relation from another is to replace one arccotangent by the sum of two arccotangents of larger numbers. Our first theorem shows that in general this only makes matters worse since this device produces a relation with a still higher measure. More precisely the theorem is:

THEOREM 1. *Let x, y, z be any positive real numbers such that*

$$[x] = [y] + [z], \tag{3}$$

then

$$(y - x)(z - x) = x^2 + 1, \tag{4}$$

and

$$\frac{1}{\log x} < \frac{1}{\log y} + \frac{1}{\log z}, \quad \text{if} \quad x > 2.88200803. \tag{5}$$

Proof. The relation (4) is well known* and is only another way of expressing the fact that if we take the cotangent of both members of (3) we obtain

$$x = (yz - 1)/(y + z).$$

To prove (5) we need the following lemma:

LEMMA. *If a, b, and t are positive, then*

$$\log\left(1 + \frac{t}{a}\right) \log\left(1 + \frac{b}{t}\right) < \frac{b}{a}, \tag{6}$$

where the logarithms are natural.

Proof. If the left member of (6) is denoted by ϕ we have on taking exponentials

* This formula has been attributed to Lewis Carroll.

(7) $$e^{\phi} = \left(1 + \frac{b}{t}\right)^{\log(1+t/a)}.$$

Noting that

$$e^{t/a} = 1 + \frac{t}{a} + \frac{t^2}{2a^2} + \cdots > 1 + \frac{t}{a},$$

so that

$$\log(1 + t/a) < t/a,$$

and replacing t by bu, (7) becomes

$$e^{\phi} < \left(1 + \frac{b}{t}\right)^{t/a} = \left\{\left(1 + \frac{1}{u}\right)^{u}\right\}^{b/a}.$$

Now it is well known that the quantity in the $\{\}$'s is less than e for all positive u. Hence $e^{\phi} < e^{b/a}$ or $\phi < b/a$, which proves the lemma.

Returning now to the proof of (5) we set $y - x = t$ so that $t > 0$, and we have by (4)

$$y = x + t, \qquad \log y = \log x + \log\left(1 + \frac{t}{x}\right),$$

$$z = x + (x^2 + 1)/t, \qquad \log z = \log x + \log\left(1 + \frac{x + x^{-1}}{t}\right).$$

Using the identity

(8) $$\frac{1}{p+q} + \frac{1}{p+r} - \frac{1}{p} = \frac{p^2 - qr}{p(p+q)(p+r)},$$

with

$$p = \log x, \qquad q = \log\left(1 + \frac{t}{x}\right), \qquad r = \log\left(1 + \frac{x + x^{-1}}{t}\right),$$

we find

$$\frac{1}{\log y} + \frac{1}{\log z} - \frac{1}{\log x} = \frac{\log^2 x - \log\left(1 + \frac{t}{x}\right)\log\left(1 + \frac{x + x^{-1}}{t}\right)}{\log x \log y \log z},$$

where the denoninator on the right is positive. Hence it suffices to show that the numerator is positive in case $x > 2.88200803$. Applying the lemma with

$$a = x, \qquad b = x + x^{-1},$$

we find at once that the numerator in question is greater than

$$\log^2 x - \frac{x + x^{-1}}{x} = \log^2 x - (1 + x^{-2}).$$

This function, which is obviously an increasing function of the positive variable x, has its real root at

$$x = 2.882008028.$$

This proves the theorem, at least for natural logarithms. But it is seen at once that if (5) holds for any system of logarithms, it holds for all others.

While condemning in general the substitution of the sum of two* arccotangents for a single arccotangent, we should, nevertheless, point out that in some special cases it is very desirable. In fact if one of the new arccotangents say $[y]$ appears elsewhere in the relation, then the measure is decreased by the positive amount

$$\frac{1}{\log x} - \frac{1}{\log z}$$

by this elimination of $[x]$. For example, in general, it would increase the measure of a relation containing [70] by .3796 if this arccotangent were replaced by [99]+[239]. However relation (33) contains both [70] and [99] and therefore when we eliminate [70] to obtain (34) there is a consequent decrease in measure. Another device commonly employed is to eliminate a term $[x]$ of an arccotangent relation by means of the identity

$$[x] = 2[2x] - [4x^3 + 3x]. \tag{9}$$

This scheme is of limited use only, as the following theorem shows:

THEOREM 2. *The measure of the left member of* (9) *is less than that of the right member if and only if* $x > 6.6760135$.

In fact

$$\frac{1}{\log 2x} + \frac{1}{\log (4x^3 + 3x)} - \frac{1}{\log x} = \frac{1}{\log x + \log 2} + \frac{1}{\log x + \log (4x^2 + 3)} - \frac{1}{\log x}.$$

In view of (8) this will be of the same sign as

$$\log^2 x - \log 2 \log (4x^2 + 3),$$

a function whose only positive zero is at $x = 6.6760135$.

Again, a relation may contain both $[x]$ and $[2x\ ]$, as for instance (28), then the use of (9) for $x > 6.676$ is justified, and we get (29) with a substantial reduction in measure. A generalization of (9) is

$$[x] = 2[(2x + h] - [(4x^3 + 4x^2h + 3x + h^2x + 2h)/(1 - 2xh - h^2)], \tag{10}$$

which is sometimes useful, as for instance in

$$[x] = 2[2x + 1/(2x)] + [16x^5 + 20x^3 + 5x], \tag{11}$$

* It is of course even worse to replace an arcotangent by the sum of three or more arccotangents.

but their use is again limited, depending on h. Further useful formulas are

$$[x] = 3[3x] - [(27x^4 + 18x^2 - 1)/8x], \tag{12}$$

$$[x] = 4[4x] - [(256x^5 + 160x^3 - 15x)/(80x^2 - 1)]. \tag{13}$$

These may be used to reduce the measure of a relation when x does not exceed 14.797 and 91.464 respectively. In general one may easily prove that as $n \to \infty$

$$n[nx] - [x] \sim [3x^2].$$

Hence we have the theorem which follows:

THEOREM 3. *Replacing* $[x]$ *by introducing the appropriate formula involving* $n[nx]$, *where* n *is large, will only serve to increase the measure in case* $x > n^{2+\epsilon}$, *where* $\epsilon = \sqrt{1 + \log 3/\log n} - 1 \to 0$.

Formulas (10)–(13) bring up the subject of arccotangents of rational numbers. At first thought their use would appear undesirable. However some rational numbers like 433.1 which appears in (31) are really no more difficult to handle than integers. Secondly, the arccotangent of a rational number may be expanded in many ways as a finite sum of arccotangents of integers. Moreover, it is possible by a certain algorithm* to obtain expansions whose measure is less than twice that of the original arccotangent. Thus in Euler's relation

$$[1] = 5[7] + 2[79/3]$$

the last term has a measure of .70407. But we may observe that

$$[79/3] = [26] - [2057],$$

where the measure of the right member is 1.0085. Finally, if the calculation of the terms of the Gregory series is done on a computing machine in the most efficient manner, there is hardly any difference between the calculation of the arccotangents of integers and of rationals. In the past, the terms

$$u_n(x) = \frac{1}{(2n + 1)x^{2n+1}}$$

of the Gregory series have been calculated by first computing a table of odd powers of $1/x$, formed by successive divisions by x^2, and then dividing by $2n+1$. This double calculation can be replaced by a single one by use of the recursion formula

$$u_{n+1}(x) = \frac{(2n + 1)u_n(x)}{(2n + 3)x^2}.$$

In case $x = p/q$, this becomes simply

$$u_{n+1}(p/q) = \frac{q^2(2n + 1)u_n(p/q)}{p^2(2n + 3)}.$$

* Duke Mathematical Journal, vol. 4, 1938, pp. 323–340.

In either case the value of u_{n+1} is obtained directly by the machine from the previous u_n.

In many of the relations given below there occurs [10]. This term, whose measure is 1 is, of course, much easier to calculate than another arccotangent of nearly the same size. For this reason we have modified the definition of measure so as to assign to [10] the measure 1/2. Furthermore, some relations include besides [10] the arccotangents of other powers of 10. In this case, since the additional arccotangents can be computed by merely recopying the significant figures in the terms of the series for [10], we have assigned to these arccotangents the measure* 0. Other modifications of the definition of measure might be given. For example Rutherford considered it easier to compute [70]–[99] rather than [239], in spite of the fact that their measures are 1.0431 and .4205 respectively, because of the ease with which the odd powers of 1/70 and 1/99 can be obtained by hand. However, these considerations must be ruled out if a machine is used.

In the following list the relations are arranged according to the size of the smallest cotangent. With each series is given its measure (modified if necessary).

(14) $[1]=[2]+[3]$, (5.4178) (Hutton, Euler)
(15) $[1]=[2]+[5]+[8]$, (5.8599) (Daze)
(16) $[1]=2[3]+[7]$, (3.2792) (Clausen)
(17) $[1]=3[4]+[19.8]$, (2.4322)
(18) $[1]=4[5]-[239]$, (1.8511) (Machin)
(19) $[1]=4[5]-[70]+[99]$, (2.4737) (Euler, Rutherford)
(20) $[1]=5[6]-[31.4375]-[117]$, (2.4364)
(21) $[1]=5[7]+2[79/3]$, (1.8873) (Hutton, Euler)
(22) $[1]=6[8]+[19.8]-3[268]$, (2.2904)
(23) $[1]=8[10]-[239]-4[515]$, (1.2892) (Klingenstierna)
(24) $[1]=8[10]+3[18]+2[100]+2[307]-3[515]+2[9901]$, (2.3177)
(25) $[1]=8[10]-2[452761/2543]-[1393]$, (1.2624)
(26) $[1]=8[10]-[100]-[515]-[371498882/3583]$, (1.0681)
(27) $[1]=8[10]-[100]-2[1000]+5[100000]-[719160]-\cdots$, $(<.8414)$
(28) $[1]=7[10]+2[50]+4[100]+[682]+4[1000]+3[1303]-4[90109]$, (1.9644) (Wrench)
(29) $[1]=7[10]+8[100]+[682]+4[1000]+3[1303]-4[90109]-2[500150]$, (1.5513) (Wrench)
(30) $[1]=8[10.1]-[239]+4[52525]$, (1.6280)
(31) $[1]=12[15]-[239]-4[433.1]$, (1.6500)
(32) $[1]=12[18]+8[57]-5[239]$, (1.7866) (Gauss)
(33) $[1]=12[18]+3[70]+5[99]+8[307]$, (2.2418) (Bennett)
(34) $[1]=12[18]+8[99]+3[239]+8[307]$, (2.1203)
(35) $[1]=16[20.05]-[239]-4[515]+8[1620050]$, (1.7182)
(36) $[1]=22[26]-2[2057]-5[3240647/38479]$, (1.5279)

* In the same way if a relation contained both $[x]$ and $[x\cdot 10^k]$ the measure of the latter should be taken as zero.

(37) $[1]=22[28]+2[443]-5[1393]-10[11018]$, (1.6343) (Escott)

(38) $[1]=12[38]+20[57]+7[239]+24[268]$, (2.0348) (Gauss)

(39) $[1]=44[57]+7[239]-12[682]+24[12943]$, (1.5860) (Wrench)

(40) $[1]=78[100]-2[682]+3[5396/3]+10[345737/243]-17[62575]$
$-34[500150]$, (1.6112)

(41) $[1]=160[200]-[239]-4[515]-8[4030]-16[50105]-16[62575]$
$-32[500150]-80[4000300]$, (2.2494) (Bennett)

(42) $[1]=3[239]+236[307]-12[19703]+240[93943]+8[10099/3]$
$-24[36101879/272]-80[2922754103/816]$, (1.8878)

(43) $[1]=2805[5257]-398[9466]+1950[12943]+1850[34208]+2021[44179]$
$+2097[85353]+1484[114669]+1389[330182]+808[485298]$,
(1.9568) (Gauss)

(44) $[1]=7854[10000]-[545261]-\cdots$, $(<.5986)$

A few remarks may be made about the above list. In the first place it is noted that the relations with many terms have large measures in spite of the fact that they involve arccotangents of large numbers. The most striking example of this is Gauss's remarkable (43), where a desperate effort has been made to eliminate arccotangents of small numbers. It would be rash to conclude from this that to discover relations of minimum measure, one should restrict the number of terms in the relation. In fact (44) is given merely to show that there exist relations whose measure are smaller than any preassigned positive number, but which contain a great many terms. It is clear that the sequence represented by the three dots of (44) is the arccotangent of a rational number, and it may be proved that this arccotangent has an expansion in a finite series of arccotangents of integers, whose total measure is less than that of [545261]. By replacing 10000 by a sufficiently larger number, the total measure of the relation thus obtained may be made as small as one pleases. As to the relation (27), if only 10 more terms were written down the resulting expression would differ from $\pi/4$ by less than 10^{-12000}. Nevertheless, (27) contains only a finite number of terms, in fact at most 108. Of course neither (27) nor (41) are of any practical use since they involve arccotangents of very large numbers indeed.

Wrench gives the following relation for $3[1]=3\pi/4$,

$$3[1] = 56[23] - [182] - 25[500] - 20[924] + 5[99557] + 10[298307], (2.2673),$$

and raises the question of the existence of primitive relations for $k\pi/4$ where $k>3$. It is easy to see that they may be obtained in unlimited numbers. In fact if the reader will select, for example, any 7 relations given above and add them together, he is almost certain to obtain a primitive relation for $7\pi/4$, that is one in which not all the coefficients of the arccotangents are divisible by 7.

Since the cotangent of $k\pi/n$ is an algebraic number, it is possible to give finite expressions for $k\pi/n$ in terms of arccotangents of algebraic integers. The following is an example of such a relation

$$\frac{\pi}{6} = 17[38] + 6[273] + 18[323.25] + [853] - 4[2072] - 2[14633] - [19703]$$
$$+ [12545\sqrt{3} + 21728], \quad (2.7732).$$

The successive terms in the Gregory series for this last arccotangent can be computed in the form $A_k + \sqrt{3}B_k$ in which the A's and the B's satisfy second order linear recurrence formulas with integer coefficients.

The practical-minded reader will naturally ask which of the 33 relations given above is the best to use for actually computing π to a very great number of decimals. This question is a hard one to answer; moreover, it is not exactly the question which interests the practical computer. The question should be: What pair of independent relations should we use to make two independent calculations of π to a very great number of places? This question is answered in the writer's opinion by (23) and (32). To be sure, both of these involve [239] but with different coefficients. That is to say, after one has computed the five arccotangents, $[x]$, for $x = 10, 18, 57, 239$, and 515, one has this equation of condition:

$$2[10] + [239] = 3[18] + 2[57] + [515]$$

as a final check on the whole work.

EXTENSIONS OF ALGEBRAIC SYSTEMS TO FORM FIELDS*

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The process of constructing the rational number system out of the natural numbers has been discussed by many authors. Some of these writers make the process seem unnecessarily complicated. The kernel of the construction consists of two applications of the process of extending a commutative semi-group to form a group. In order to see this clearly it is desirable to consider the process entirely abstractly, that is, on the basis of suitable postulates.

We shall let G denote a class of elements denoted by $a, b, \cdots$, which contains at least two elements. We consider also a binary operation $\circ$ "on GG to G," that is, $\circ$ makes correspond to each pair a, b of elements of G an element $a \circ b$ of G. Among the postulates to be considered are the following:

POSTULATE 1. The commutative law: $a \circ b = b \circ a$.

POSTULATE 2. The associative law: $(a \circ b) \circ c = a \circ (b \circ c)$.

POSTULATE 3. If $a \circ b = a \circ c$ then $b = c$.

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Elements of particular interest which may be present in a system $(G, \circ)$ are the "zero" and the "unit," defined as follows.

DEFINITION. An element z is called a *zero* of G in case $a \circ z = z$ for every a in G.

DEFINITION. An element u is called a *unit* of G in case $a \circ u = a$ for every a in G.

It is easily verified that a system $(G, \circ)$ satisfying the commutative law can have at most one zero z and at most one unit u, and that $z \neq u$ since G contains more than one element. Moreover, each of these special elements may be arbitrarily added to the system if it is missing.

A system $(G, \circ)$ satisfying Postulates 1, 2, 3, is called a commutative semi-group, and cannot have a zero. We wish to consider also *commutative semi-groups with a zero permitted*, for which Postulate 3 is replaced by the following:

POSTULATE 3Z. If b is not a zero and $b \circ c = b \circ d$, then $c = d$.

An example of such a system is constituted by the natural numbers and 0, with the operation $\circ$ defined to be multiplication. In a commutative semi-group with zero permitted, an element b satisfying $b \circ b = b$ is either a zero or a unit. For then $a \circ b \circ b = a \circ b$ for every element a, and if $b \neq z$, $a \circ b = a$, by Postulate 3Z, so that $b = u$. Another important property is that if $a \circ b = z$, then $a = z$ or $b = z$.

A *commutative group* has the following properties:

POSTULATE 4. There exists a unit u.

POSTULATE 5. For every a there is an element a^{-1} such that $a \circ a^{-1} = u$.

To define a *commutative group with a zero permitted* we replace Postulate 5 by the following:

POSTULATE 5Z. For every $a \neq z$ there is an element a^{-1} such that $a \circ a^{-1} = u$.

The first step in the extension of a commutative semi-group to form a group (with zero permitted) is the consideration of the class P of all pairs (a, a') of elements of G, with the second element a' restricted to be not zero. We define an operation $\circ$ in the class P by the formula $(a, a') \circ (b, b') = (a \circ b, a' \circ b')$. This operation is easily seen to obey the commutative and associative laws, but the properties of Postulates 3 and 3Z will usually fail to hold.

The next step is to divide the class P into sub-classes by means of an appropriate equivalence relation.

DEFINITION. $(a, a') \sim (b, b')$ in case $a \circ b' = b \circ a'$.

We see that the relation $\sim$ is symmetric, and also reflexive in the sense that every pair is equivalent to itself. To show that the relation is transitive, that is, that $(a, a') \sim (b, b')$ and $(b, b') \sim (c, c')$ imply $(a, a') \sim (c, c')$, requires the use

of the three Postulates 1, 2, and 3Z for the system $(G, \circ)$. Such an equivalence relation $\sim$ divides P into mutually exclusive sub-classes, each of which is determined by an arbitrary one of its members (a, a') and consists of all pairs $(b, b') \sim (a, a')$. We denote the sub-class determined by the pair (a, a') by $\{(a, a')\}$, and the class of all the sub-classes $\{(a, a')\}$ by R.

We note next that by means of the commutative and associative laws for the system $(G, \circ)$ we can show that if $(a, a') \sim (b, b')$ and $(c, c') \sim (d, d')$ then $(a, a') \circ (c, c') \sim (b, b') \circ (d, d')$. Consequently the following definition of the operation $\circ$ in R is well justified.

DEFINITION. $\{(a, a')\} \circ \{(b, b')\} = \{(a, a') \circ (b, b')\}$.

The operation $\circ$ in R also obeys the commutative and associative laws.

It is verified at once that if b is an arbitrary non-zero element of G, then $\{(b, b)\}$ is a unit in R, and $\{(z, b)\}$ is a zero in R provided G contains a zero z. Moreover, if a is also non-zero then $\{(b, a)\}$ is the inverse of $\{(a, b)\}$, so that $(R, \circ)$ is a group (with a zero if and only if a zero is present in G).

The element $\{(a \circ b, b)\}$ of R depends only on the element a of G, since $(a \circ b, b) \sim (a \circ c, c)$ for arbitrary non-zero elements b and c. The correspondence so set up between the subclass $R_G = [\text{all } \{(a \circ b, b)\}]$ of R and the class G is one-to-one, and $\{(a \circ b, b)\} \circ \{(c \circ d, d)\} = \{(a \circ c \circ b \circ d, b \circ d)\}$, that is, the correspondence establishes an *isomorphism* between the systems $(R_G, \circ)$ and $(G, \circ)$. In case $(G, \circ)$ satisfies Postulates 4 and 5Z, the class R_G is the whole of R, since then (a, b) may be written $(a \circ b^{-1} \circ b, b)$. We have also the following theorem:

THEOREM 1. *Let* $(G, \circ)$ *satisfy the Postulates* 1, 2, 3Z *and let* $(L, \circ)$ *satisfy* 1, 2, 4, 5Z. *Let* L' *be a subset of* L *such that* $(L', \circ)$ *is isomorphic to* $(G, \circ)$. *Then this isomorphism may be extended to an isomorphism between* $(R, \circ)$ *and* $(L'', \circ)$ *where* L'' *is a subset of* L *containing* L'.

In making the proof we shall use capital letters to denote the elements of L. We note that there is a zero Z in L' if and only if there is a zero z in G, and that they must correspond. Let A correspond to a, A' to a', B to b, B' to b', under the given isomorphism of G and L', and suppose that a' and b' are not zero. There is always a unique element C in L such that $A = A' \circ C$, and it is easily verified that $(a, a') \sim (b, b')$ if and only if also $B = B' \circ C$. Making C correspond to $\{(a, a')\}$, we see that there is thus established a one-to-one correspondence between R and a certain sub-class L'' of L. If $\{(b, b')\}$ corresponds to an element D, then $\{(a, a')\} \circ \{(b, b')\}$ corresponds to $C \circ D$. Moreover, if $\{(a \circ b, b)\}$ corresponds to E, then $E = A$, so that L' is a subset of L'', and the isomorphism established between L'' and R is in an appropriate sense an extension of the original isomorphism between L' and G.

We are now ready to consider an algebraic system consisting of a class G containing at least two elements, and two operations on GG to G which we now

denote by the usual symbols $+$ and $\times$ and call addition and multiplication. We suppose this system has the following properties:

POSTULATE I. $(G, +)$ satisfies Postulates 1, 2 and 3.

POSTULATE II. $(G, \times)$ satisfies Postulates 1, 2, and 3Z.

POSTULATE III. The distributive law: $(a+b)\times c=(a\times c)+(b\times c)$.

POSTULATE IV. If $a\neq b$ then there is an element c such that either $a=b+c$ or $b=a+c$.

In a system with these properties there is no zero for $+$, and an element b is a unit for $+$ if and only if it is a zero for $\times$. We shall use the customary notations 0 and 1, but note that neither of these elements is necessarily in the class G. In discussing such a system we have in mind the example of the natural numbers with ordinary addition and multiplication. But there are many other instances of importance and interest.

We first consider the extension of the system $(G, +)$ to form a group, and denote the system so obtained by $(R^+, +)$. It has the properties 1, 2, 4, and 5. Multiplication of elements of R^+ is defined as follows:

DEFINITION. $\{(a, a')\}\times\{(b, b')\}=\{(ab+a'b', ab'+a'b)\}$.*

We must show that if $(a, a')\sim(c, c')$ and $(b, b')\sim(d, d')$ then $(ab+a'b', ab'+a'b)\sim(cd+c'd', cd'+c'd)$. The given equivalences are by definition

$$a + c' = a' + c, \tag{1}$$

$$b + d' = b' + d. \tag{2}$$

These may also be written

$$a' + c = a + c', \tag{3}$$

$$b' + d = b + d'. \tag{4}$$

If we multiply (1) by b, (2) by c, (3) by b', and (4) by c' and add, we obtain $ab+bc'+bc+cd'+a'b'+b'c+b'c'+c'd=a'b+bc+b'c+cd+ab'+b'c'+bc'+c'd'$, from which the desired result follows by the uniqueness of subtraction.

It is easily verified that the multiplication so defined in R^+ obeys the commutative and associative laws, and is distributive with respect to addition. The unit for addition, which is the zero for multiplication, is $\{(a, a)\}$, so that to prove property 3Z for $(R^+, \times)$ we must show that if $a\neq a'$ and $\{(a, a')\}\times\{(b, b')\}=\{(a, a')\}\times\{(c, c')\}$ then $\{(b, b')\}=\{(c, c')\}$, that is, if $a\neq a'$ and

$$ab + a'b' + ac' + a'c = ab' + a'b + ac + a'c', \tag{5}$$

then

$$b + c' = b' + c. \tag{6}$$

* Here the sign $\times$ is omitted between elements of G.

Now by property IV there exists an element d such that either

$$a = a' + d \tag{7}$$

or

$$a' = a + d. \tag{8}$$

Plainly $d \neq 0$. If we substitute either (7) or (8) in (5) and apply Postulate 3 for $(G, +)$ and 3Z for $(G, \times)$ we obtain (6).

If G contains a unit 1 for multiplication, then $\{(a+1, a)\}$ is a unit for multiplication in R^+. Moreover, if also property 5Z holds for $(G, \times)$, the inverse of $\{(a'+d, a')\}$ is $\{(a'+d^{-1}, a')\}$ and the inverse of $\{(a, a+d)\}$ is $\{(a, a+d^{-1})\}$, so that $\{(a, a')\}$ has an inverse whenever $a \neq a'$. In this case the system $(R^+, +, \times)$ constitutes a field.

The theorems on isomorphism relating to semi-groups and their extensions have their analogs here. It is only necessary to show that the correspondences previously discussed are preserved under multiplication. We quote the analog of Theorem 1.

THEOREM 2. *Let* $(G, +, \times)$ *and* $(L, +, \times)$ *satisfy* I, II, III, IV, *and let* $(L, +)$ *satisfy also Postulates* 4 *and* 5. *Let* L' *be a subset of* L *such that* $(L', +, \times)$ *is isomorphic to* $(G, +, \times)$. *Then this isomorphism may be extended to an isomorphism between* $(R^+, +, \times)$ *and* $(L'', +, \times)$ *where* L'' *is a subset of* L *containing* L'.

We consider now the extension of the system $(G, \times)$ to form a group with a zero permitted, and denote the system so obtained by $(R^\times, \times)$. It has the properties 1, 2, 4, and 5Z. Addition of elements of $R^\times$ is defined as follows:

DEFINITION. $\{(a, a')\} + \{(b, b')\} = \{(ab'+a'b, a'b')\}$.

The verification that this definition is justified is simple and will be omitted. It is also easy to show that $(R^\times, +)$ satisfies Postulates 1, 2, and 3, and that the distributive law, Postulate III, holds. To establish property IV, let $\{(a, a')\} \neq \{(b, b')\}$, that is, $ab' \neq a'b$. Then there exists an element c such that $ab' = a'b + c$ or $ab' + c = a'b$. In the first case $\{(a, a')\} = \{(b, b')\} + \{(c, a'b')\}$ and in the second case $\{(a, a')\} + \{(c, a'b')\} = \{(b, b')\}$.

We recall that in case G contains 0, $\{(0, a)\}$ is a zero for $(R^\times, \times)$ and hence a unit for $(R^\times, +)$. Also if property 5 holds for $(G, +)$ it continues to hold for $(R^\times, +)$, and then the system $(R^\times, +, \times)$ is a *field*.

We have now shown that an algebraic system with the properties I, II, III, and IV can always be extended to form a field in at most two steps, and that (in the sense of isomorphism) no unnecessary elements have been introduced. When the extension is made with respect to addition first and multiplication afterwards we may denote the resulting system by $R^{+\times}$, and when the order is reversed, by $R^{\times+}$. In order to show that the two systems are isomorphic, we adopt standard forms to represent their elements. We first drop the $\{\ \}$, and

denote an element of R^+ by $(a, b)^+$, and an element of $R^{+\times}$ by $[(a, b)^+, (c, d)^+]^\times$, where $c \neq d$. We next note that $[(a, b)^+, (c, d)^+]^\times = [(b, a)^+, (d, c)^+]^\times$, so that by Postulate IV we may suppose $c = d + e$ where $e \neq 0$. Thus $[(a, b)^+, (c, d)^+]^\times = [(a, b)^+, (d+e, d)^+]^\times = [(a, b)^+, (b+e, b)^+]^\times$ and the last form we denote for short by $\alpha(e, a, b)$. It is easy to show that

(9) $\alpha(a, b, c) = \alpha(a', b', c')$ if and only if $a'b + ac' = ab' + a'c$;

(10) $\alpha(a, b, c) + \alpha(a', b', c') = \alpha(aa', ab' + a'b, ac' + a'c)$;

(11) $\alpha(a, b, c) \times \alpha(a', b', c') = \alpha(aa', bb' + cc', bc' + b'c)$.

Similarly an element of $R^{\times+}$ is denoted by $[(a, b)^\times, (c, d)^\times]^+$ where $b \neq 0 \neq d$. We see at once that $[(a, b)^\times, (c, d)^\times]^+ = [(ad, bd)^\times, (bc, bd)^\times]^+$. If we denote $[(b, a)^\times, (c, a)^\times]^+$ by $\mu(a, b, c)$, we find that μ has exactly the same properties (9), (10), and (11) as α has. This shows that $(R^{+\times}, +, \times)$ is isomorphic to $(R^{\times+}, +, \times)$.

THE MOTION OF A RELATIVISTIC PARTICLE IN A UNIFORM FIELD OF FORCE

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1. Introduction. One of the simplest and most familiar problems in classical dynamics is that of determining the motion of a particle subjected to a constant force. The conditions of the problem are approximately satisfied in the case of a projectile moving near the surface of the earth with not too great speed (so that the resistance of the air, the variation of the gravitational force from point to point on the actual trajectory, and other such complicating effects can be neglected), and the problem is commonly discussed in these terms. It will be recalled that the trajectory of the particle is found to be a parabola with its axis parallel to the lines of force of the field. In the subsequent sections of this article we shall occasionally refer to other details of the solution, in order to compare them with the corresponding details of the solution of the problem we are concerned with here; but it is unnecessary to give still another full discussion of this familiar classical problem.

In the following we shall be concerned, not with classical dynamics, but rather with the dynamics of the special theory of relativity. Briefly, the two systems of dynamics are distinguished by the fact that, whereas in classical dynamics the mass of a particle is a constant, in relativistic dynamics the mass is a non-constant function of the speed of the particle.

So far, not many problems in relativistic dynamics have been solved with any great completeness. In fact, a rather extensive search of the literature has failed to bring to light any full discussion of the relativistic analog of the simple classical problem described above.* For this reason, and because the relativistic

* The special case in which the particle moves parallel to the lines of force of the field is treated in L. Silberstein's, The Theory of Relativity, 1st and 2nd editions, pp. 189 *et seq.*

problem possesses some very interesting features, it has seemed worth while to give the following brief treatment of it.

2. The differential equations of motion and their solution. We consider a particle, of rest-mass m_0, moving in a fixed plane; and we let x and y denote the rectangular coördinates of the particle with respect to a fixed set of axes in the plane.*

In relativistic dynamics, as in classical dynamics, the force is equal to the time rate of change of the momentum of the particle. However, in relativistic dynamics the x- and y-components of momentum of the particle are not $m_0\dot{x}$ and $m_0\dot{y}$, as they are in classical dynamics, but rather

$$m_0\dot{x}\left[1-\frac{\dot{x}^2+\dot{y}^2}{c^2}\right]^{-1/2} \quad \text{and} \quad m_0\dot{y}\left[1-\frac{\dot{x}^2+\dot{y}^2}{c^2}\right]^{-1/2}.$$

Here $\dot{x}=dx/dt$, $\dot{y}=dy/dt$, and c is the speed of light. Consequently, the relativistic differential equations of motion of the particle are the following:

$$\frac{d}{dt}\left[m_0\dot{x}\left(1-\frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^{-1/2}\right]=X, \qquad \frac{d}{dt}\left[m_0\dot{y}\left(1-\frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^{-1/2}\right]=Y,$$

where X and Y are, respectively, the x- and y-components of the force acting on the particle. The positive value of $[1-(\dot{x}^2+\dot{y}^2)/c^2]^{-1/2}$ is the significant one; and wherever square roots appear in the following work it is understood that the positive values are intended.

We assume that the force is constant, in direction as well as in magnitude. Consequently, we can choose the coördinate axes so that X is identically zero, and then, since m_0 is a constant, the differential equations of motion can be written in the form

$$(1) \qquad \frac{d}{dt}\left[\dot{x}\left(1-\frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^{-1/2}\right]=0, \quad \frac{d}{dt}\left[\dot{y}\left(1-\frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^{-1/2}\right]=\frac{Y}{m_0}=-\gamma.$$

(We write $Y/m_0=-\gamma$ for the sake of convenience in comparing our results with the results for the classical case in their familiar form.)

Integrating equations (1), we obtain the relations

$$(2) \qquad \dot{x}\left(1-\frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^{-1/2}=\mu_x, \qquad \dot{y}\left(1-\frac{\dot{x}^2+\dot{y}^2}{c^2}\right)^{-1/2}=-\gamma t+\mu_y,$$

where μ_x and μ_y are constants of integration. The physical significances of these constants are determined by the fact that $m_0\mu_x$ and $m_0\mu_y$ are, respectively, the x- and y-components of the momentum of the particle at the instant $t=0$.

Equations (2) are easily solved for $\dot{x}$ and $\dot{y}$, the results being

$$(3) \qquad \begin{aligned} \dot{x} &= c\mu_x[(\gamma t-\mu_y)^2+(c^2+\mu_x^2)]^{-1/2}, \\ \dot{y} &= -c(\gamma t-\mu_y)[(\gamma t-\mu_y)^2+(c^2+\mu_x^2)]^{-1/2}. \end{aligned}$$

* We omit the very simple proof of the fact that a particle, moving in a uniform field of force, actually moves in a fixed plane determined by the initial conditions.

On integrating equations (3), we obtain the relations

$$(4)\qquad \begin{aligned} x &= x_0 + \frac{c\mu_x}{\gamma}\log\frac{(\gamma t-\mu_y)+[(\gamma t-\mu_y)^2+(c^2+\mu_x^2)]^{1/2}}{A-\mu_y},\\ y &= y_0 + \frac{Ac}{\gamma} - \frac{c}{\gamma}[(\gamma t-\mu_y)^2+(c^2+\mu_x^2)]^{1/2}, \end{aligned}$$

where x_0 and y_0 are constants of integration, and where

$$A = (c^2+\mu_x^2+\mu_y^2)^{1/2}.$$

Obviously, x_0 and y_0 are the coördinates of the particle at the instant $t=0$.

3. Discussion of the motion. Equations (4) represent the general solution of the system of differential equations (1). It remains to discuss the physical significance of this solution. The special case in which $\mu_x=0$ requires a separate discussion. Since this case is very simple, and since it has been treated by Silberstein, we shall not consider it explicitly here. Hence, in all of the following it is to be understood that $\mu_x\neq 0$.

It is an immediate consequence of the differential equations of motion that the (relativistic) momentum varies with time in our problem exactly as does the (classical) momentum in the analogous classical problem.

From the first of equations (4) we obtain the relations

$$(5)\qquad \begin{aligned} (\gamma t-\mu_y)+[(\gamma t-\mu_y)^2+(c^2+\mu_x^2)]^{1/2} &= (A-\mu_y)e^{\gamma(x-x_0)/c\mu_x},\\ -(\gamma t-\mu_y)+[(\gamma t-\mu_y)^2+(c^2+\mu_x^2)]^{1/2} &= (A+\mu_y)e^{-\gamma(x-x_0)/c\mu_x}. \end{aligned}$$

It follows from these equations and the second of equations (4) that we have the relation

$$y = y_0 + \frac{Ac}{\gamma} - \frac{c}{2\gamma}[(A-\mu_y)e^{\gamma(x-x_0)/c\mu_x}+(A+\mu_y)e^{-\gamma(x-x_0)/c\mu_x}],$$

which can obviously be written in the form

$$(6)\qquad y = y_0 + \frac{Ac}{\gamma} - \frac{c}{\gamma}(c^2+\mu_x^2)^{1/2}\cosh\left[\frac{\gamma}{c\mu_x}(x-x_0)-\frac{1}{2}\log\frac{A+\mu_y}{A-\mu_y}\right].$$

This is the equation of the trajectory of the particle. We see that *in the general case, in which* $\mu_x\neq 0$, *the trajectory is a catenary.*

In some of the following work we shall have to consider the coördinates x_v and y_v of the vertex of the catenary, which is the point at which $\dot{y}$ is zero. We note that these are given by the formulas:

$$(7)\qquad x_v = x_0 + \frac{c\mu_x}{2\gamma}\log\frac{A+\mu_y}{A-\mu_y},\qquad y_v = y_0 + \frac{Ac}{\gamma} - \frac{c}{\gamma}(c^2+\mu_x^2)^{1/2}.$$

The time at which the particle reaches a particular point (x, y) on the trajectory is given by the formula

$$t = \frac{\mu_y}{\gamma} + \frac{1}{\gamma}(c^2 + \mu_x^2)^{1/2} \sinh\left[\frac{\gamma}{c\mu_x}(x - x_0) - \frac{1}{2}\log\frac{A + \mu_y}{A - \mu_y}\right], \tag{8}$$

which is easily derived from equations (5).

It readily follows from equations (3), (8), and (6) that when the particle is at the point (x, y) the speed, $s = (\dot{x}^2 + \dot{y}^2)^{1/2}$, is given by either of the following equivalent formulas:

$$\begin{aligned} s^2 &= c^2 - \frac{c^4}{c^2 + \mu_x^2}\operatorname{sech}^2\left[\frac{\gamma}{c\mu_x}(x - x_0) - \frac{1}{2}\log\frac{A + \mu_y}{A - \mu_y}\right], \\ &= c^2 - \frac{c^6}{\gamma^2}\left[y - y_0 - \frac{Ac}{\gamma}\right]^{-2}. \end{aligned} \tag{9}$$

The equations (9) show that as t approaches $+\infty$ or $-\infty$, the speed approaches the constant c, instead of becoming infinite as it does in the classical case. It follows from this result regarding the speed and from the geometrical properties of the trajectory, or directly from equations (3), that as t approaches $+\infty$ or $-\infty$ the x-component of the velocity approaches zero, and the absolute magnitude of the y-component approaches c. We recall that in the corresponding classical problem $\dot{x}$ is a constant, and the absolute value of $\dot{y}$ approaches ∞ as t approaches $+\infty$ or $-\infty$. It is noteworthy that in the relativistic problem a force acting solely in the y-direction produces a non-zero component of acceleration in the x-direction. This is a manifestation of the fact that the x-component of momentum (which must be conserved) is, in the relativistic case, a function of $\dot{y}$ as well as of $\dot{x}$, instead of being a function of $\dot{x}$ only, as it is in the classical case.

4. Relativistic ballistics. Let us now assume that γ, μ_x, and μ_y are all positive. It follows from the first of equations (7) that the particle, when projected from the point (x_0, y_0) with the momentum $(m_0\mu_x, m_0\mu_y)$, reaches the line $y = y_0$ again at the point having the abscissa

$$x_1 = x_0 + \frac{c\mu_x}{\gamma}\log\frac{A + \mu_y}{A - \mu_y}.$$

We call $x_1 - x_0$ the range of the particle, and we denote it by the symbol R. Thus

$$R = \frac{c\mu_x}{\gamma}\log\frac{A + \mu_y}{A - \mu_y}. \tag{10}$$

Suppose, now, that the absolute value of the initial momentum, $m_0\mu = m_0(\mu_x^2 + \mu_y^2)^{1/2}$, is prescribed, but that the angle of fire is optional. What angle should be chosen in order to secure the maximum range? In the classical case this optimum angle of fire is known to be 45° for all values of μ. In the relativistic case the answer is not so simple.

We write $\mu_x = \mu\cos\theta$ and $\mu_y = \mu\sin\theta$ in (10), and we find that

$$\frac{dR}{d\theta} = \frac{c\mu}{\alpha\gamma(1-u^2)}\left[2(\alpha^2-u^2) - u(1-u^2)\log\frac{1+u}{1-u}\right],$$

where α and u are auxiliary variables defined by the equations

$$\alpha = \mu/(c^2+\mu^2)^{1/2}, \qquad u = \alpha\sin\theta. \tag{11}$$

Obviously, $dR/d\theta$ vanishes, and R is a maximum, if and only if the variables α and u satisfy the relation

$$\alpha^2 = u^2 + \tfrac{1}{2}u(1-u^2)\log\frac{1+u}{1-u}. \tag{12}$$

Since α is known when μ is given, equation (12) implicitly determines the value of u corresponding to the maximum range, and therefore also the optimum angle of fire. For the sake of distinctness, we shall henceforth denote the optimum angle of fire by the symbol θ_m.

Perhaps the simplest way to exhibit $\sin\theta_m$ explicitly as a function of α is to regard u in (12) and the second of equations (11) as a parameter, and to plot the α, $\sin\theta_m$-curve which these parametric equations represent. The curve obtained is shown in the accompanying figure.

In order to exhibit the functional relationship between $\sin\theta_m$ and μ explicitly, it is only necessary to add a non-uniform μ-scale defined by the first of equations (11); and this also has been done in the figure.

We observe that as μ approaches zero, θ_m approaches the classical value $\theta_m^{(0)} = 45°$, and that as μ increases indefinitely θ_m increases monotonically to a limiting value which is about $56°27'57''.01$.

The optimum angle of fire can be obtained from equations (11) and (12) in another manner, which is of some interest, and which we shall indicate briefly without going into any detail.

For the significant values of u, which all lie between zero and unity, equation (12) can be written in the form

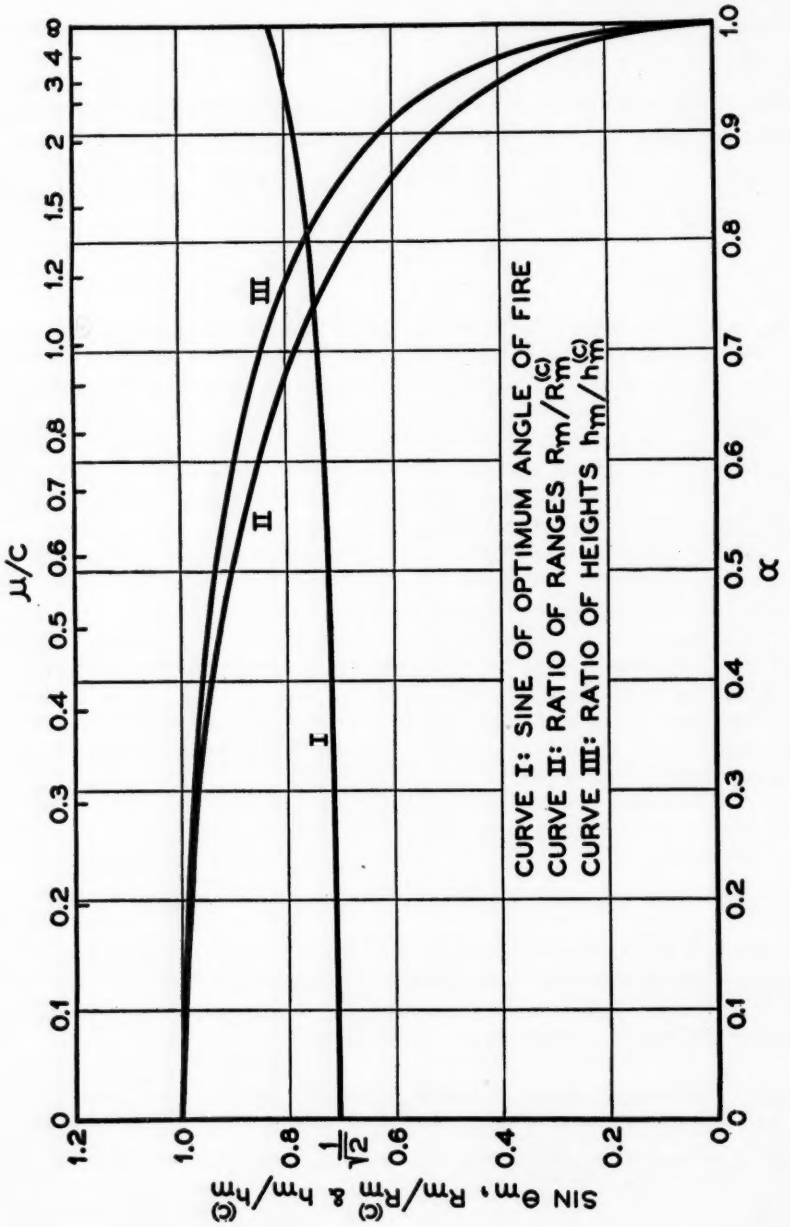
$$\alpha^2 = u^2 + u(1-u^2)(u + \tfrac{1}{3}u^3 + \tfrac{1}{5}u^5 + \cdots) = -2\sum_{n=1}^{\infty}\frac{u^{2n}}{(2n-3)(2n-1)}. \tag{12'}$$

It readily follows from the form of equation (12′) and the implicit function theorem that u^2 can be represented by a power series in the variable α^2. The coefficients of this series can be calculated successively in various obvious ways without any difficulty; and we obtain the result

$$u^2 = \frac{1}{2}\alpha^2 + \frac{1}{12}\alpha^4 + \frac{13}{360}\alpha^6 + \frac{307}{15120}\alpha^8 + \frac{5879}{453600}\alpha^{10} + \frac{12721}{1425600}\alpha^{12} + \cdots,$$

leading at once to the formula

$$\sin^2\theta_m = \frac{1}{2} + \frac{1}{12}\alpha^2 + \frac{13}{360}\alpha^4 + \frac{307}{15120}\alpha^6 + \frac{5879}{453600}\alpha^8 + \frac{12721}{1425600}\alpha^{10} + \cdots.$$


μ/c
α
SIN θm, Rm/Rm(c) & hm/hm(c)
CURVE I: SINE OF OPTIMUM ANGLE OF FIRE
CURVE II: RATIO OF RANGES Rm/Rm(c)
CURVE III: RATIO OF HEIGHTS hm/hm(c)
I
II
III
1/√2

The range obtained with the optimum angle of fire will be denoted by the symbol R_m. If the particle obeyed the laws of classical dynamics, and were projected with the initial momentum $m_0\mu$ at the classical optimum angle of fire, the range would be $R_m^{(c)} = \mu^2/\gamma$. From equations (10), (11), and (12) we easily obtain the relation

$$\frac{R_m}{R_m^{(c)}} = \frac{2\alpha(1-\alpha^2)^{1/2}(1-\alpha^{-2}u^2)^{3/2}}{u(1-u^2)}. \tag{13}$$

Equations (12) and (13) can be regarded as parametric equations of the curve representing $R_m/R_m^{(c)}$ as a function of α. The curve is shown in the figure. As μ approaches zero, R_m approaches the classical value $R_m^{(c)}$; and as μ increases indefinitely the ratio of R_m to the classical value approaches zero. (It has been remarked that Einstein, being a man of peace, does not allow us to shoot as far with given cartridges as did the more bellicose Newton!) Of course, as μ increases indefinitely, so also does the actual range R_m; but, for large values of μ, R_m is asymptotically proportional only to μ ($R_m \sim K c\mu/\gamma$; $K \doteq 1.3255$), instead of being proportional to μ^2 as is $R_m^{(c)}$.

For small values of α, $[R_m/R_m^{(c)}]^2$ can be represented by a power series in the variable α^2, as follows:

$$[R_m/R_m^{(c)}]^2 = 1 - \frac{2}{3}\alpha^2 - \frac{8}{45}\alpha^4 - \frac{22}{315}\alpha^6 - \cdots.$$

We shall not stop to give the rather commonplace derivation of this formula.

Although the height h_m to which the particle rises above the initial point, when it is projected with given initial momentum at the optimum angle of fire, is of secondary importance, it nevertheless possesses a certain interest.* It follows from the second of equations (7) and equations (11) that this height is

$$h_m = y_v - y_0 = \frac{c\mu}{\alpha\gamma}\left[1 - (1-u^2)^{1/2}\right],$$

where u is the function of α defined by equation (12).

In the classical case the particle, projected with initial momentum $m_0\mu$ at the classical optimum angle of fire, rises to a height $h_m^{(c)} = \mu^2/(4\gamma)$ above the initial point. It is easily shown that

$$\frac{h_m}{h_m^{(c)}} = \frac{4(1-\alpha^2)^{1/2}}{\alpha^2}\left[1 - (1-u^2)^{1/2}\right]. \tag{14}$$

Equations (12) and (14) can be regarded as parametric equations of the curve, shown in the figure, representing the ratio $h_m/h_m^{(c)}$ as a function of α. As μ approaches zero, the ratio approaches unity; as μ increases indefinitely the ratio

* Of course, if we desire to make the height to which the particle rises a maximum, with a prescribed value of μ, we must take $\theta = 90°$.

decreases to zero. Of course, as μ increases indefinitely the actual height h_m increases without limit; but, as is the case in regard to the ranges, the relativistic height h_m is asymptotically proportional to μ, while the classical height $h_m^{(c)}$ is proportional to μ^2.

We can show without much difficulty that $[h_m/h_m^{(c)}]^2$ can be represented, for small values of α, by a series as follows:

$$[h_m/h_m^{(c)}]^2 = 1 - \frac{5}{12}\alpha^2 - \frac{599}{2880}\alpha^4 - \frac{27973}{249120}\alpha^6 - \cdots .$$

By the time of flight we mean the time required for the particle, projected from the point (x_0, y_0), to reach the target on the line $y = y_0$.

It is clear that the time of flight is merely twice the time required for the y-component of momentum, which varies linearly with the time, to reach the value zero; hence the time of flight t_1 is given by the formula

$$t_1 = \frac{2\mu}{\gamma} \sin\theta, \tag{15}$$

where the symbol μ has the same significance as before, and θ is the angle of fire (which is not necessarily the optimum angle of fire). This formula can be derived directly from the second of equations (4).

Now we observe that (15) holds also in the classical case. Thus, for given values of the angle of fire and the absolute value of the initial momentum, the times of flight in the classical and relativistic cases, respectively, are equal.

Let t_{1m} denote the time of flight of the relativistic particle projected with the initial momentum $m_0\mu$ at the optimum angle of fire; and let $t_{1m}^{(c)}$ denote the time of flight of the classical particle projected with the same initial momentum at the classical optimum angle of fire. Then we have the relation

$$\frac{t_{1m}}{t_{1m}^{(c)}} = 2^{1/2} \sin\theta_m .$$

Thus $t_{1m}/t_{1m}^{(c)}$, except for a constant factor, is represented as a function of α by curve I of the figure.

QUESTIONS, DISCUSSIONS, AND NOTES

EDITED BY R. E. GILMAN, Brown University Providence, Rhode Island

The Department of Questions, Discussions, and Notes in the Monthly is open to all forms of activity in collegiate mathematics including the teaching of mathematics, except for specific problems, especially new problems, which are reserved for the department of Problems and Solutions.

AN ELEMENTARY PROOF OF THE PROBLEM OF PONCELET PENTAGONS

C. W. WILLIAMS, Washington and Lee University

Let us consider two conics I and II. From a point on I, draw a tangent to conic II, and from the second intersection with I, draw a second tangent to II. Let this operation be repeated. If we return after n such steps, to the original point of conic I, then it is possible to choose the original point on I arbitrarily, and return to it after exactly n such steps.

This theorem is due to Poncelet, and its proof, in the general case, depends on elliptic functions. When such an n-gon exists, inscribed to one conic and circumscribed to a second, we shall say that the two conics are in Poncelet Alignment.

For the case $n=5$, we shall offer a proof, making use only of elementary functions. This method is also applicable to cases, $n=3, 4$.

Let us choose the coördinate system so that conic I circumscribes the reference triangle; *i. e.*; then the conic has the equation,

$$\text{I:} \quad xy + yz + zx = 0, \tag{1}$$

or the parametric representation:

$$x = 1, \qquad y = -t, \qquad z = t/(1-t). \tag{2}$$

The coördinates of a line joining two points of I, with parameters t_1 and t_2, will be:

$$(t_1t_2): \quad t_1t_2, \qquad 1, \qquad [1-(t_1+t_2)+t_1t_2]. \tag{3}$$

Let us now choose five points on I with parameters: $0, \infty, 1, a, b$; the first three corresponding to the three vertices of the triangle of references: (1, 0, 0), (0, 1, 0), (0, 0, 1). We then form a pentagon with these five points as vertices in the given order. The equation of conic II (in line coördinates) inscribed in this pentagon will be:

$$\begin{aligned}\text{II:} \quad & ab(1-a)(1-b)v^2 - b(1-a)(1-b)uv \\ & - ab(1-a)vw - (a-b)wu = 0.\end{aligned} \tag{4}$$

Thus conic II is uniquely determined by the choice of the above five points, and the two conics are in Poncelet Alignment.

Let t be the parameter of any point on conic I. The coördinates of the line joining it to the point of parameter r, from (3), are

$$(tr): \quad tr, \qquad 1, \qquad [1-(t+r)+tr]. \tag{5}$$

In order that this line be tangent to II, it is necessary that these coördinates satisfy equation (4); *i.e.*,

$$(6)\qquad \begin{aligned}&[(a-b)(1-t)t]r^2+[t^2(a-b)+t(a^2b-ab^2+b^2-a)+ab(1-a)]r\\&\quad+[ab(1-a)(t-b)]=0.\end{aligned}$$

Denote the roots of (6) by r_1 and r_2. Then the lines (tr_1) and (tr_2) are tangents of conic II.

Similarly, denoting the roots of the equation,

$$(7)\qquad \begin{aligned}&[(a-b)(1-s)s]r'^2+[s^2(a-b)+s(a^2b-ab^2+b^2-a)+ab(1-a)]r'\\&\quad+[ab(1-a)(s-b)]=0,\end{aligned}$$

by r_1' and r_2', the lines (sr_1') and (sr_2') are tangents of conic II, drawn from the point of I, with parameter s.

Let us choose s so that one of the roots of (7) will be equal to one of the roots of (6). Algebraically, this condition becomes:

$$(8)\qquad \begin{aligned}&[(t-a)(t-b)]s^2-[t^2(a+b)+t(ab^2-a^2b-2ab-b^2-a)+ab(a+1)]s\\&\quad+[ab(t-1)(t-a)]=0.\end{aligned}$$

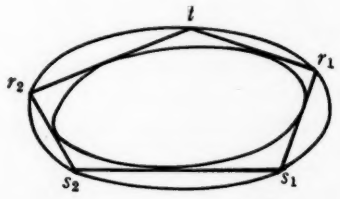
Denoting the roots of (8) as s_1 and s_2, our condition states geometrically that the two lines (r_1s_1) and (r_2s_2) are tangents of conic II. From (8) we have,

$$(9)\qquad \begin{aligned}s_1+s_2&=[t^2(a+b)+t(ab^2-a^2b-2ab-b^2-a)+ab(a+1)]/(t-a)(t-b),\\s_1\cdot s_2&=[ab(t-1)(t-a)]/(t-a)(t-b),\end{aligned}$$

so that the line (s_1s_2) has coördinates:

$$(10)\qquad (s_1s_2):\quad ab(t-1)(t-a),\quad (t-a)(t-b),\quad t(1-a)(1-b)(t-b).$$

But these coördinates satisfy the equation (4), so that (s_1s_2) is a tangent of II also.



Therefore, if we start at any point t, of I and draw a tangent to II, the second intersection with I is the point r_1 (or r_2). From r_1 we go on to s_1; and then to s_2; to r_2; and then finally back to t, after exactly five steps.

Since all pairs of conics in Poncelet Alignment for $n=5$, are projectively equivalent to the pair just discussed, if there exists one pentagon, circumscribing one conic and inscribing another, there exists an infinite number of such pentagons.

COMMENTS ON A PAPER BY DR. DOOLE

FRANCES E. BAKER, Mt. Holyoke College

The following note calls attention to certain inaccuracies found in a paper, *Integration of certain simple step functions*, by H. P. Doole, which appeared in this MONTHLY, vol. 44, 1937, pp. 222–227.

The author has based his development upon definitions of the Bernoulli polynomials and numbers as given by Bromwich, *Introduction to the Theory of Infinite Series*, 1926. (*Cf.* Doole, p. 223, after equation (4)). These definitions may be written as follows: (*Cf.* Bromwich, pp. 297–300.)

(a) The numbers B_n: $\quad B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \cdots$

from the expansion of

$$\frac{x}{e^x - 1} = 1 - \frac{x}{2} + B_1 \frac{x^2}{2!} - B_2 \frac{x^4}{4!} + B_3 \frac{x^6}{6!} - \cdots .$$

(b) The polynomials $B_n(x) = x^n - \frac{n}{2} x^{n-1} + \frac{n(n-1)}{2!} B_1 x^{n-2}$

$$- \frac{n(n-1)(n-2)(n-3)}{4!} B_2 x^{n-4} + \cdots .$$

Thus

$$B_1(x) = x;\ B_2(x) = x^2 - x;\ B_3(x) = x^3 - \tfrac{3}{2}x^2 + \tfrac{1}{2}x;\ B_4(x) = x^4 - 2x^3 + x^2; \cdots .$$

In the Doole paper, (a) and (b) have been employed to reach an erroneous result in equation (6).

Using $[x]$ to mena the greatest integer contained in x, and $\int_0^{[x]} [x]^p$ to signify the finite sum $\sum_0^{[x]-1} x[]^p$, the author obtains

$$(6)\quad \int_0^x [x]^p = \frac{B_{p+1}([x])}{p+1} + (-1)^{p+1/2} \left\{ \frac{1 + (-1)^{p+1}}{2} \right\} \frac{B_{p+1/2}}{p+1} + (x - [x])[x]^p,$$

(p a positive integer.)

It is readily verified (*e.g.* for $p=1$, 3) that the second term in the right hand member of (6) is superfluous. The correct result may be rapidly derived as follows, using the notation of finite differences (with p, x, positive integers):

(c)
$$\Delta B_{p+1}(x) = B_{p+1}(x+1) - B_{p+1}(x) = (p+1)x^p;$$

$$\sum \Delta B_{p+1}(x) = (p+1) \sum x^p = B_{p+1}(x) + C;$$

$$(p+1) \sum_0^{x-1} x^p = B_{p+1}(x) + C \Big]_0^x = B_{p+1}(x) - B_{p+1}(0).$$

In the Bromwich notation, $B_{p+1}(0) = 0$, for $p \geqq 0$; hence

$$\sum_{0}^{x-1} x^p = \frac{B_{p+1}(x)}{p+1}.$$

The last equation is a fundamental result in the theory of the Bernoulli polynomials. (*Cf.* Bromwich, §102, p. 304.)

For the step-functions, (6) should then read

$$(6)' \qquad \int_0^x [x]^p = \frac{B_{p+1}([x])}{p+1} + (x - [x])[x]^p.$$

A similar error appears in equations (7), (9), (12), (13), (14), and (20) of the Doole paper. Rewriting these equations to conform to the Bromwich definitions, one has:

$$(7)' \qquad \int_0^x (x - [x])[x]^p = \frac{1}{2}\left\{\frac{B_{p+1}([x])}{p+1} + (x - [x])^2 \cdot [x]^p\right\}.$$

$$(9)' \qquad \int_0^x (x - [x])^m \cdot [x]^p = \frac{1}{m+1}\left\{\frac{B_{p+1}[(x])}{p+1} + (x - [x])^{m+1} \cdot [x]^p\right\}.$$

$$(12)' \qquad \int_0^x ([x] - 1)^p = (-1)^p + \frac{B_{p+1}([x] - 1)}{p+1} + (x - [x])([x] - 1)^p.$$

$$(13)' \qquad \int_0^x \int_0^x \int_0^x [x] = \frac{B_4([x])}{4!} + \frac{(x - [x])^3[x]}{3!} + \frac{(x - [x])^2[x]([x] - 1)}{2.2!} + \frac{(x - [x])[x]([x] - 1)(2[x] - 1)}{2.3!}.$$

By an induction, (14)′ is reached:

$$(14)' \qquad \int_0^x \cdots k \cdots \int_0^x [x] = \frac{B_{k+1}([x])}{(k+1)!} + \sum_{s=0}^{k-1} \frac{(x - [x])^{k-s} \cdot B_{s+1}([x])}{(k-s)!(s+1)!} = \sum_{s=0}^{k} \frac{(x - [x])^{k-s} \cdot B_{s+1}([x])}{(k-s)!(s+1)!}, \quad \text{(where } 0! = 1\text{)}.$$

$$(20)' \qquad \int_0^x \left(x - \frac{[nx]}{n}\right)^m \left(\frac{[nx]}{n}\right)^p = \frac{1}{m+1}\,\frac{1}{n^{m+p+1}}\,\frac{1}{p+1}\left\{B_{p+1}([nx])\right\} + \left(\frac{[nx]}{n}\right)^p \frac{(x - [nx]/n)^{m+1}}{m+1}.$$

As part of an honors paper on finite integration presented for her bachelor's degree from Mount Holyoke College this June, Miss Mary P. Kendall studied the Doole paper. She observed that although (6) could not be checked for particular values of p, on the basis of the Bromwich definitions, it could be checked on the assumption that Doole had used the Steffansen-Nörlund definition of the Bernoulli polynomials, while retaining the older (Bromwich) definition of the Bernoulli numbers.

According to Steffansen (*Interpolation*, §13, pp. 119–121):

(d) The numbers B_n: $B_0 = 1$; $B_1 = -\frac{1}{2}$; $B_n = \sum_{v=0}^{n} \binom{n}{v} B_v$, $(n > 1)$; $B_{2k+1} = 0$, $(k > 0)$.

(e) The polynomials $B_n(x) = \sum_{v=0}^{n} \binom{n}{v} B_v x^{n-v}$.

Thus,

$$B_0(x) = 1; \quad B_1(x) = x - \tfrac{1}{2}; \quad B_2(x) = x^2 - x + \tfrac{1}{6};$$

$$B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x; \quad B_4(x) = x^4 - 2x^3 + x^2 - \frac{1}{30}; \cdots.$$

The polynomials defined either by (b) or by (e) satisfy relation (c).

Supposing Doole to have used such a mixture of definitions, his equations (6), (7), (9), (12), (13), (20) may be verified. However, on this basis, (14) will not reduce to (1) for $k=1$, nor to (4) for $k=2$, nor to (13) for $k=3$, as it of course should. Equation (14) is incorrect according to either the Bromwich or the Steffansen definition.

Except for a misprint in (24), where the second term of the right member should be positive, equations (24) and (25) are correct on the basis of the Bromwich definitions (a) and (b).

The errors noted above do not, of course, invalidate these and other valuable results obtained by Doole in his interesting paper.

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Steffansen, J. F., Interpolation, The Williams and Wilkins Company, Baltimore, 1927.

ON THE DEFINITION OF INFLECTION POINT

G. M. Ewing, University of Missouri

If we examine the definitions of inflection point given in textbooks on the calculus, certain inconsistencies appear. An inflection point may be defined as a point of a curve at which

(A) the direction of concavity changes [1];

(B) the tangent line crosses the curve [2];

(C) the slope has a maximum or minimum [3], or in other ways [4].

For our present purposes a curve is a set of points of the euclidian plane which can be represented by parametric equations $x=x(t)$, $y=y(t)$, $T_1 \leqq t \leqq T_2$, with $(x(t_1), y(t_1)) \neq (x(t_2), y(t_2))$ for $t_1 < t_2$, except possibly for $t_1 = T_1$ and $t_2 = T_2$. It is convenient to specify certain curves by equations of the form $y=f(x)$, $x=\phi(y)$, $F(x, y)=0$, *etc.*

A curve is said to be concave toward the x-axis (away from the x-axis) if given any three points $(x(t_i), y(t_i))$, $i=1, 2, 3$, such that $x(t_1)<x(t_2)<x(t_3)$, the determinant

$$D(t_1, t_2, t_3) \equiv \begin{vmatrix} y(t_1) & x(t_1) & 1 \\ y(t_2) & x(t_2) & 1 \\ y(t_3) & x(t_3) & 1 \end{vmatrix}$$

is greater than zero (less than zero).

The equations of certain curves, together with the indicated inflection points are set down for reference:

(1) $y = x^3$, $(0, 0)$.

(2) $y = x^{1/3}$, $(0, 0)$.

(3) $y = x^3$, $x < 0$;
$= x^2$, $x \geqq 0$; $(0, 0)$.

(4) $y = x^2$, $x < 0$;
$= x^{1/2}$, $x \geqq 0$; $(0, 0)$.

(5) $y = x^2$, $x < 0$; $(0, 1)$.
$= -x^2 + 1$, $x \geqq 0$.

(6) $x^2 + y^2 = 1$; $(1, 0)$.

(7) $x = t^2$; $(0, 0)$.
$y = t^4$, $t < 0$;
$= 2t^4$, $t \geqq 0$.

(8) $x = y^{2/3}$; $(0, 0)$.

(9) $y = \int_{-1}^{x} \phi(x)dx$; $(0, 0)$.

$\phi(x) = 2x^2 + x^2 \sin \frac{1}{x}$, $x \neq 0$;
$= 0$, $x = 0$.

(10) $y = 2x^3 + x^3 \sin \frac{1}{x}$, $x \neq 0$;
$= 0$, $x = 0$; $(0, 0)$.

In the following table, check marks indicate which of the properties (A), (B), and (C) are possessed by each of these examples.

Example	A	B	C	Remarks
1	√	√	√	[according to all writers.[6]]
3	√	√	√	$f''(0)$ does not exist. Hence not an inflection point.
2	√	√		Slope $\frac{dy}{dx}$ is not defined for $x=0$.
8	√	√		
4	√			There is no tangent line at $x=0$.
5	√			$f(x)$ is discontinuous at $x=0$.
6	√			
9		√	√	
10		√		
7			√	

In framing a definition it seems desirable to formulate it so that it will apply to as large a class of curves as possible; and hence properties depending upon

certain derivatives are to be avoided. It seems desirable that such points as those of examples (1), (2), and (3) be classed as inflection points whereas those of the remaining examples may well be excluded. The following definition is suggested.

Inflection Point. An inflection point is a point $(x(\bar{t}),\ y(\bar{t}))$ of a curve $C: x=x(t);\ y=y(t)$ if (and only if) the following conditions are satisfied:

(1) $x(t)$ and $y(t)$ are continuous for $t=\bar{t}$;

(2) there exist numbers $t_1<\bar{t}$ and $t_2>\bar{t}$ such that $D(\tau_1, \tau_2, \tau_3)>0$ (or <0) for $t_1\leqq\tau_1<\tau_2<\tau_3\leqq\bar{t}$, and $D(\tau_1, \tau_2, \tau_3)<0$ (or >0 respectively) for $\bar{t}\leqq\tau_1<\tau_2<\tau_3\leqq t_2$.

(3) for every linear form $ax+by+c$ which vanishes for $x=x(\bar{t}), y=y(\bar{t})$ there exist numbers $t_3<\bar{t}$ and $t_4>\bar{t}$ such that $ax(t)+by(t)+c>0$ (or <0) for $t_3\leqq t<\bar{t}$ and $ax(t)+by(t)+c<0$ (or >0 respectively) for $\bar{t}<t\leqq t_4$.

In descriptive language an inflection point is a point of a curve such that (1)′ the curve is continuous at the point, (2)′ the direction of concavity changes at the point, and (3)′ every straight line through the point has an arc of the curve on each side of it. Condition (2) is not the same as (2)′, as shown by example (7); but (2)′ is a consequence of (2) and (3). Some of the properties of an inflection point as just defined are as follows:

PROPERTY 1. Inflection points are unchanged by translations or rotations of axes.

PROPERTY 2. If the point $(x(\bar{t}),\ y(\bar{t}))$ is a point of inflection of a curve C, which is of class C'' (*i.e.*, $x''(t)$ and $y''(t)$ are continuous) for all values of t in an interval to which t is interior, and if $x'(t)$ and $y'(t)$ are not both zero, then $x'y''-y'x''$ vanishes for $t=\bar{t}$.

COROLLARY. If C can be represented in the form $y=f(x)$, and if $f''(x)$ is continuous, then $f''(x)=0$ at the point of inflection.

PROPERTY 3. If C has a tangent line at an inflection point, the tangent line crosses the curve.

The last is immediate from defining Property 3. The proofs of the other stated properties are simple.

Note by the Editor. Doubtless some readers will think that the author has sacrificed simplicity and clarity for generality. We may agree that the discussion is interesting whether the proposed definition of an inflection point should be used in a textbook or not.

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W. A. Granville, *et. al.*, Elements of the Differential and Integral Calculus, Ginn and Co., 1934, p. 79.

2. W. B. Fite, Advanced Calculus, MacMillan, 1938, p. 76.

R. Courant, Vorlesungen über Differential- und Integralrechnung, Springer, 1930, vol. 1, p. 127.

3. J. V. McKelvey, Calculus, MacMillan, 1937, p. 59.

4. G. H. Hardy, Pure Mathematics, Cambridge Press, 1925, p. 272.

RECENT PUBLICATIONS

EDITED BY TOMLINSON FORT, Lehigh University

All books for review should be sent directly to the editor of this department, at the Mathematical Association of America, 531 West 116th Street, New York, N. Y., and not to any of the other editors or officers of the Association.

NEW BOOKS RECEIVED

One Hundred Problems in Consumer Credit. By Charles H. Mergendahl and LeBaron R. Foster. (Pollak Pamphlet, No. 35.) Newton, Massachusetts, Pollak Foundation for Economic Research, 1938. 55 pages. $0.10.

Introduction à la mécanique non-linéaire. By Nicolas Kryloff and Nicolas Bogoliuboff. (Les Méthodes approchées et asymptotiques de la mécanique non-linéaire.) (Annales de la Chaire de la Physique Mathématique, vols. 1–2.) (Publié par l'Académie des Sciences de la RSS d'Ukraine.) Kiev, Academy of Sciences, 1937. 366 pages. (In Russian.)

Euclide. L'optique et la catoptrique. By Paul Ver Eecke. (Oeuvres traduites pour la première fois du grec en français avec une introduction et des notes. Ouvrage publié sous les auspices de la Fondation Universitaire de Belgique.) Paris et Bruges, Desclée de Brouwer et Cie, 1938. 47+126 pages. 75 fr.

General Mathematics. A One Year Course. By Harris Crandall and F. Eugene Seymour. New York, D. C. Heath and Company, 1937. 8+389 pages. $1.28.

Applications aux jeux de hasard. By Émile Borel. (Fascicule II de Tome IV: Applications diverses et conclusion; Traité du Calcul des Probabilités et de ses Applications; publié sous la direction de M. Émile Borel.) Paris, Gauthier-Villars, 1938. 12+122 pages. 60 fr.

The Physical Treatises of Pascal: The Equilibrium of Liquids and the Weight of the Mass of the Air. Translated by I. H. B. Spiers and A. G. H. Spiers. (With notes and introduction by Frederick Barry.) New York, Columbia University Press, 1937. 23+194 pages. $3.25.

The Impossibility of War Risk Insurance. By Sir William P. Elderton. (A paper read before the Insurance Institute of London on March 15, 1938.) Cambridge. The University Press, 1938. 16 pages. $0.25.

REVIEWS

Comptes-Rendus du Deuxième Congrès International de Récréation Mathématique. Brussels, Librairie du Sphinx, 1937, 103 pages.

This volume records the second meeting of the International Congress on Recreational Mathematics, including a full account of most of the twenty-two papers presented. The Congress was held in Paris and attended by eighty-eight delegates from ten different countries.

The first and longest paper was presented by M. A. Decerf on the relationships between mathematics and other branches of learning. The sciences furnish countless applications of mathematics in the way of functional relationships and examples of coördinate systems. History is replete with mathematical scientists

who have provided the stepping stones for the progress of civilization. The fine arts depend mathematically on rhythm, harmony, graceful patterns, and proportions. Even grammar cooperates with mathematics, fitting ordinary words to technical uses and conversely. In psychology, logic, philosophy and even the study of spiritual values, mathematics plays an important rôle. This paper is indeed inspiring to the teacher of mathematics.

The second paper was presented by M. Sainte-Laguë on Geometrical Forms in Nature. While enumerating the occurrence of Fibonacci series, spirals, and regular and semi-regular polyhedra among the flora, fauna, and inorganic manifestations of nature, the author emphasizes the rather surprising fact that pentagonal forms, such as appear in the starfish and many kinds of flowers, are entirely absent from all forms of crystal structure.

M. H. Barolet presented a brief paper on the calendar, in which he gave a simple rule for determining the day of the week corresponding to any given date in the modern era.

Dr. Bruno Berneis presented a little skit attributed to Lewis Carroll, in the form of an argument between Euclid and other ancient characters, in which the former defends geometry against the corrupting tendencies of non-euclidean assumptions.

General E. Cazalas discussed Dr. Erich Stern's treatment of magic squares, with particular reference to the linking of previous theories and the enumeration of various types.

T. R. Dawson presented solutions of the *RN*-Line Problem. In this problem one seeks to place counters on n of the squares of an $n \times n$ field, no two of which are in the same row, column, or slant line making arctan 2 with any side. Most of the solutions occur in overlapping sets.

F. Denk's Struwwelpeter Problem is unique and interesting. He determines analytically conditions of maximum chaos through a study of what he terms primitive permutations.

Jaques Devisme gave sidelights on the Postage Stamp Problem. In this problem, one seeks to determine the number of ways in which two given stamps in a sheet may be torn out with a connecting strip of stamps accompanying them. The number of turns in such a strip, and the number of such strips of different shapes which may be taken from a given sheet, give rise in their determination to problems of innocent appearance but highly complicated solutions.

A. Gloden furnished an interesting study of a system of diophantine equations, by means of which the sum of any two like even powers may be expressed as the sum of two squares in all possible ways.

M. Godron presented an exhaustive study of the famous old checker game of "Give-Away," showing how, under given initial conditions, twelve white pieces could force a single black piece to take them all.

E. Huber-Stockar described the "Devil's Checker-Board," a set of fifteen pieces which fit into an 8×8 checkered square in more than five thousand differ-

ent ways. He showed how Bayes's Theorem could be made to predict an upper limit to the number of possible solutions to this problem.

M. Kraitchik presented a list of large prime numbers between 10^{12} and $10^{12}+10^4$, together with the smallest prime factor of each odd composite number in this range.

M. Poulet, who is also interested in the prime factors of large numbers, presented a table of composite numbers, N, between 313 and 10^8, such that $2^{N-1}\equiv 1$ (modulo N), giving the largest prime factor of each such N. It is interesting to note that 52% of the values listed for N terminate in the digit 1. This table contains 251 values of N which satisfy Fermat's equation $B^{N-1}\equiv 1$ (modulo N) for all values of B prime to N. The table facilitates the identification of large prime values of N.

P. Sergesco gave a brief outline of the lives and works of several early medieval French mathematicians, several of whom seem to have forerun later and better known mathematicians in discoveries usually attributed to the latter.

Dr. Erich Stern discussed the possibilities of his method of building and enumerating magic squares by splitting them into pairs of auxiliary squares, which may be written down with very few restrictions and then combined linearly to produce the desired magic squares.

In the final paper appearing in this volume, M. V. Thébault explained how to determine squares which consist of sets of double digits, with their square roots also consisting of double digits. This was worked out first for the decimal system, and later for every possible basis of enumeration. M. Thébault explained how the congruence machine of M. A. Gérardin was used in solving some of these problems, such as the determination of the fact that, with 733 as base, $(43\ 86\ 172\ 345)^2 = 2\ 393\ 96\ 279\ 2\ 393\ 96\ 279$.

On the whole, this series of papers is presented in a manner sufficiently elementary to be within the comprehension of the majority of college seniors majoring in mathematics, and for those high school mathematics teachers who read French easily should furnish a most enjoyable way of whiling away long winter evenings.

W. F. CHENEY, JR.

Le Problème du Cavalier généralisé, Extrait des Comptes-Rendus. By E. Huber-Stockar, Brussels, Librairie du Sphinx, 1935. 16 pages. 2 belgas.

The author graciously dedicates to M. Maurice Kraitchik this generalization of the Chess Knight's Tour Problem. An ordinary Knight's moves may be symbolized by four vectors of equal magnitude, one in each quadrant, together with their opposites. Such a knight moves about on an 8×8 checkerboard, attempting to visit every square once but none twice, and end a normal jump from his starting-point. Euler made an exhaustive study of this problem in 1758, which has been the starting point of most of the subsequent investigations of the subject.

The generalized knight of this paper is also symbolized by four vectors, one in each quadrant, together with their opposites, but here the four vectors may be entirely unrelated in magnitude and direction. The field is also generalized to be of any size, provided only that it is a rectangle whose sides are integers.

The author derives functions of the rectangular components of the vector moves of his generalized knight, the relative primeness of which functions tells whether or not the given knight can tour a field of given dimensions. The more difficult task of determining an acceptable route for a knight which is known to be capable of traversing a given field, is solved in some of the simpler cases.

A few typographical errors were noted, as follows:

page 7, line 22, for "B et $-i$" read "B et -1";

page 9, line 2, for p_4+q_4i, read p_4-q_4i;

page 11, row III, column A of table, for a_2 read a_3.

There seems to be a needless multiplicity of symbolism in some parts of this paper, but, aside from that possible defect, the presentation is straightforward and interesting. It should be a valuable addition to the previously published material in this field.

W. F. CHENEY, JR.

Leerboek der Technische Theoretische Mechanica. Eerste Deel: Algemeene Grondslagen der Theoretische Mechanica. By Ir. J. F. Schuh. Groningen, P. Noordhoff, 1937. 14+312 pages. Paper fl. 7.25. Cloth fl. 8.50.

In this first volume of a work to be devoted to technical theoretical mechanics the general principles are derived. A second volume, to appear later, will deal with specific technical problems.

The first book is directed especially towards students at technical universities. It would seem, however, that (excluding the introductory chapter on vectors and a few very nice descriptive pages on Newton's axioms) its abstractness will appeal mainly to those who like theoretical mechanics for its own sake.

The average engineering student will be hampered by having conceptions such as Coriolis acceleration introduced by means of a deluge of symbols without further comments about what is involved. He will be disappointed also in not finding any examples or problems at the end of each chapter to really familiarize himself with the subject. The twenty-two examples given in the second half of the book average more than five pages and are really treatises in themselves. Eighty-one problems, chosen exclusively from those given at the University of Delft, conclude the first volume. Typical for this kind of "technical "problems is #19: On an immovable rough sphere (radius r, center M, highest point H) is placed a particle A (mass m) with a velocity zero in a location where $\sphericalangle HMA = 60°$. The particle is subject to gravity (acceleration g), dry friction (coefficient of friction $\frac{1}{2}\sqrt{3}$) and an air resistance which is proportional to the square of the velocity and equal to $m\sqrt{3}/2r$ at a velocity $1 \cdots$."

R. P. KROON

MATHEMATICS CLUBS

EDITED BY E. H. C. HILDEBRANDT, New Jersey State Teachers College

All reports of club activities, suggestions, topics with references, and other material of interest to clubs should be sent to E. H. C. Hildebrandt, New Jersey State Teachers College, Upper Montclair, N. J.

MATHEMATICS EXHIBITS

Mathematics clubs have been sending us reports which indicate a growing interest in construction and demonstration of various mathematical devices and models. Usually a project of this kind is begun for some temporary purpose, such as the necessity for a device for class room teaching or the need of an exhibit or display of models for visiting high school pupils. Once this work has been begun, it seems advisable to add to, rather than to duplicate existing material in succeeding years, until presently a display case is acquired through club funds or other sources and the mathematical exhibit becomes a reality.

Such collections or exhibits need not be devoted exclusively to geometric models. They may also include models to illustrate projection, a binomial and a trinomial cube, link-motion apparatus, instruments for drawing curves, curve tracers, pantographs, planimeters, various models of curved surfaces formed by intersecting layers of stiff paper or a series of stretched strings, wire devices for soap bubble surfaces, sun dials and clocks, calculating machines, nomographs, a variety of slide rules (straight, circular, cylindrical, spiral, and gridiron types), instruments and models for solving equations, maps (air survey, geodetic triangulation, *etc.*), map projection apparatus, surveying and astronomical instruments, working models of the normal probability curve, prime number sieve and devices showing other number relations, mathematical games and amusements, newspaper clippings concerning mathematics, and other things of mathematical origin and interest.

Descriptions of original models have appeared from time to time in the MONTHLY and other mathematics journals. A complete bibliography on mathematics exhibits is not available. If your club has developed a bibliography, this department would be glad to place such material at the disposal of other clubs. An article on this subject appeared in an earlier volume in this department under the heading "Club Topics."*

The following books contain material for the construction of devices and instruments:

BOOKS FOR CLUBS

General

40. *Solid Geometry*, by L. Lines. London, MacMillan, 1935. 287 pages. $2.20. Presents subject matter of the usual conventional course in solid geometry plus additional theorems based on trigonometry. Also chapters on centroids, rabatment, polyhedra, semi-regular and star polyhedra, space lattices, sphere packs, and patterns and crystals.

* Sell, A homemade mathematics exhibit. This MONTHLY, vol. 40, 1933, pp. 555–556.

41. *Anfertigung Mathematischer Modelle*, by K. Giebel. Leipzig, Teubner, 1915. 52 pages. Number 16, Mathematisch-Physikalische Bibliothek series. Discusses the construction of models used in algebra, geometry, and trigonometry. Also contains a chapter on the advantages and use of such materials as cardboard, glass, wood, lead, brass, and celluloid.
42. *Vielecke und Vielflache*, by Max Bruckner. Leipzig, Teubner, 1900. 227 pages, 12 double plates appended. Not a recent publication, but the pictures of several hundred polyhedra and star polyhedra give suggestions for additional models for the mathematics collection. They can be made of cardboard or stiff paper. The accompanying explanations and theoretical matter can form the basis for several club programs.
43. *Craftsmanship in the Teaching of Elementary Mathematics*, by F. W. Westaway. London, Blackie and Sons, 1931. 665 pages. 15 sh. As the title suggests, this book treats methods of teaching mathematics in schools of Great Britain. The last few chapters on wave motion, mechanics, geometrical optics, map projection, harmonic motion, polyhedra, mathematics in biology, proportion and symmetry in art, and numbers—their unexpected relations, may serve as introductions to these topics in club meetings.
44. *The Fifty-Nine Icosahedra*, by H. S. M. Coxeter, P. DuVal, H. T. Flather, and J. F. Petrie. Mathematical Series No. 6. Toronto, Univ. of Toronto Press, 1938. 26 pages+20 plates. $1.00. Describes and discusses polyhedra obtainable from the five Platonic solids by extending or producing the faces until they meet again, the rotational symmetry of the original solid being preserved. (An illustrated chapter on polyhedra written by H. S. M. Coxeter will appear in the forthcoming 11th edition of Mathematical Excursions and Essays, by W. W. R. Ball. See Books for Clubs, No. 39.)
45. *Mathematical Snapshots*, by H. Steinhaus. New York, Stechert, 1938. 135 pages. $2.50. An English translation of a Polish text relating to applications of mathematics to other fields, the construction of various polyhedra and surfaces, and a discussion of a number of mathematical instruments, devices, and amusements. It would undoubtedly prove to be an interesting addition to a mathematics club library.

CLUB REPORTS

1937–38

Mathematics Club, George Washington University

This group meets twice monthly and encourages the presentation of papers by its members. These included: Book Reviews, *Men of Mathematics* and *Mathematics for the Million;* Infinite radicals; Weierstrass's example of a continuous function which has no derivative; A numerical method of recording color; A problem in conformal mapping; A simple method of reducing quadratic forms to a sum of squares; The game of Nim; The history of probability; Some problems on ovals. Guest speakers and their topics included: Dr. E. W. Woolard, U. S. Weather Bureau, The geometry of geography; Dr. R. J. Seeger, The king of the sciences; Dr. Florence Mears, Infinite series.

Secretary, Rosedith Sitgreaves; Adviser, Dr. Florence Mears.

Mathematics Club, University of Rochester

Two of the papers presented before this club were by freshmen members, one of whom received the prize for the best paper presented during the year. Undergraduate members were encouraged to present their papers to some member of the department of mathematics before presentation at the meeting, resulting in a higher standard of excellence. These included: The factorization of integers; Babbage's calculating machines; Elliptic functions; Theory of finite groups; The classification of the cubic; Nomograms; Waves and particles in modern physical theory; Vector analysis. Dr. S. E. Warschawski and Dr. A. S. Galbraith of the faculty spoke on Divergent series, and The generalization of the derivative, respectively. The year's program closed with a picnic at Letchworth Park.

Mathematics Club, Wayne University

A series of teas at the monthly meetings gave the members of the club an opportunity to meet socially with the faculty. The following papers were presented: Transcendence of π; A theoretical lens; A new method for the solution of cubic equations; Divisibility rules by the remainder theorem; Rolle's method of cascades; Linear operators.

The Graduate Mathematical Round Table, University of Illinois

Papers presented by members were: The determination of all algebraic invariants by tensor algebra; Metric differential geometry of a general surface; Lorentz transformations; The theory of ideals; Functions satisfying the Cauchy-Riemann equations; Some elementary properties of monotone functions and Stieltjes's integrals with applications to the theory of probability; Ergodic theory; Generalization of two theorems of Eckardt.

President, James F. Heyda; Secretary, Kaj L. Nielsen.

William S. Hall Mathematics Club, Lafayette College

The outstanding papers presented were: The measurement of angles; Complex numbers; Geometry on the sphere; Ruler and compass constructions; Construction by special devices; Mathematical induction; Theory and applications of the polar planimeter. At one meeting Professor H. S. Grant of Rutgers University was the guest speaker on Farey's series. A picnic was held as the closing meeting of the year.

President, Edward Meister; Vice-President, Thomas Bagg; Secretary, Harold Day; Librarian, Peter Schenk; Adviser, Dr. Smith.

Mathematics Club, Oshkosh State Teachers College

Part of each monthly meeting of this group was devoted to mathematical recreations. The following topics were presented: Numerology; Problems in the teaching of mathematics; Mathematics used in various occupations; Mathematics of crystallography; Mathematics encountered in highway construction; The preparation of teachers of mathematics; Surveying and surveying instruments; History of geometry; and Readings from Stephen Leacock. The graduating members of the club were entertained at a mathematical party by the faculty advisors and the year closed with a picnic on the shores of Lake Winnebago.

Mathematics Club, Rutgers University

Two joint meetings were held with the Mathematics Club of the New Jersey College for Women, both open to the public. At the annual spring banquet, Professor Bennett of Brown University lectured on "The quadrangle." At the other meeting, Joseph Hilsenrath spoke on "Graphical representation of complex roots." Students and faculty presented papers at the monthly meetings on such topics as: An original system of algebra; Number problems; Mathematical recreations; Intuitive geometry; An introduction to the geometry of the complex plane.

President, Abraham Gelfond; Vice-President, Samuel Ginberg; Secretary-Treasurer, W. Elmer Lancaster; Faculty Adviser, Professor E. P. Starke.

PROBLEMS AND SOLUTIONS

EDITED BY OTTO DUNKEL, H. L. OLSON, AND W. F. CHENEY, JR.

ELEMENTARY PROBLEMS

Send all communications about Elementary Problems and Solutions to W. F. Cheney, Jr., Dept. Box 35, Connecticut State College, Storrs, Connecticut.

The Department of Elementary Problems welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 352. *Proposed by F. A. Alfieri, New York, N.Y.*

Show that the product of the six segments, cut by the interior angle-bisectors from the sides of any rational triangle, must be the square of a rational number.

E 353. *Proposed by J. E. Trevor, Cornell University.*

Mrs. Black and Mrs. Brown bought cloth, each paying as many cents per yard as she bought yards. Mrs. Green and Mrs. White bought groceries. Mrs. Green spent one cent more than the excess of twice Mrs. Black's payment over seven-ninths of Mrs. Brown's. Mrs. White spent as much as the sum of two-thirds of Mrs. Brown's payment and one-half of Mrs. Green's. Mrs. Black's expenditure was equal to five-sixths of Mrs. Green's payment, plus one-third of Mrs. White's. The total expenditure was more than one dollar and less than ten thousand dollars. How much money, in dollars and cents, did each woman spend?

E 354. *Proposed by Harry Polachek, Columbia University.*

A given point P lies inside a circle of given diameter AB. Find points X on AB such that $PX = XQ$, where XQ is perpendicular to AB, and Q is a point on the circle.

E 355. *Proposed by K. W. Miller, Utilities Research Commission, Chicago.*

If $a_1, a_2, a_3, \cdots, a_n$ are any set of n arbitrary numbers whatsoever, real or complex, rational or not, with or without repetitions or gaps, zero and unity excluded, which are assigned serial numbers i from 1 to n, respectively, in any arbitrarily designated order (which may bear no relation to natural sequence or to order of magnitude), prove the identity:

$$\frac{1}{a_1} + \sum_{i=1}^{i=n} \frac{1}{a_i} \prod_{j=1}^{j=i-1} \left(\frac{a_j - 1}{a_j} \right) = 1 - \prod_{i=1}^{i=n} \left(\frac{a_i - 1}{a_i} \right).$$

(This is a very fruitful identity, in that there are many infinite sets for which the product on the right will vanish. If the a's are all equal, this reduces to a geometric progression. The set of positive integers gives factorial series. If the a's are the primes, the resulting series involves Euler's "indicator.")

SOLUTIONS

E 288 [1937, 487]. *Proposed by W. B. Campbell, Drexel Institute.*

If $\cos(a+bi)=p+iq$, where a, b, p, and q are real, find explicit expressions for all permissible values of $\cos a$, $\sin a$, $\sinh b$, in terms of p and q, classifying where necessary, and showing that they all lead to real values of a and b.

Comment by the proposer.

The solution of this problem as published in March gives the correct numerical values of $\sin a$, $\cos a$, $\sinh b$, satisfying $\cos(a+bi)=p+iq$, but does not indicate the choice of signs, which must always be consistent with the determining equations,

$$\text{(A)} \qquad \cos a \cosh b = p, \qquad \sin a \sinh b = -q.$$

While the solutions were derived on the assumption that $pq \neq 0$, it can be shown directly from (A) that they are valid also in the special cases, if we remember that $(x^2)^{1/2}=|x|$, and choose all signs carefully. On the a, b diagram the results lie either all on the a-axis, or on two lines equidistant from it. If $c=a+bi$ is a solution, $b \neq 0$, the set of solutions consists of the points, $c+2n\pi$, and their reflections through the origin, $-c-2n\pi$, with n taking on all integral values. If $q=0$ and $p<-1$, then $a=(2n+1)\pi$, and $b=\pm\cosh^{-1}|p|$. If $q=0$ and $-1 \leqq p \leqq 1$, $a=2n\pi \pm \cos^{-1} p$, $b=0$. If $q=0$ and $1<p$, then $a=2n\pi$, $b=\pm\cosh^{-1}p$. If $p=0$, either $a=(2n-\frac{1}{2})\pi$, $b=\sinh^{-1}q$, or else $a=(2n+\frac{1}{2})\pi$, $b=-\sinh^{-1}q$.

E 315 [1938, 117]. *Proposed by F. A. Alfieri, New York, N.Y.*

In any plane triangle prove that the ratio of the sum of the squares of the sides to four times the sum of the cotangents of the angles is equal to the area.

Solution by E. S. Smith, Boulder, Colorado.

By the Law of Cosines, $a^2+b^2-c^2=2ab\cos C$, and similarly for the other two angles. Adding these three equations gives

$$a^2 + b^2 + c^2 = 2(ab\cos C + bc\cos A + ca\cos B).$$

The basic trigonometric formula for the area is $K=\frac{1}{2}bc\sin A$, from which $bc=2K/\sin A$. Multiplying both sides by $\cos A$ gives

$$bc\cos A = 2K\cot A.$$

Adding this to the two corresponding formulae derived from it by cyclic permutation of the letters gives

$$ab\cos C + bc\cos A + ca\cos B = 2K(\cot A + \cot B + \cot C).$$

By substituting this right-hand expression in the previous equation, one gets

$$a^2 + b^2 + c^2 = 4K(\cot A + \cot B + \cot C)$$

from which the desired relation obviously follows.

Note: This theorem is given as an exercise on page 145 of Dr. John Casey's *Treatise on Plane Trigonometry*, 1888, London and Dublin.

Also solved by Martha L. Bassett, W. E. Buker, J. O. Chellevold, W. B. Clarke, Fred Discepoli, Wm. Douglas, J. F. Heyda, I. Kaplansky, E. R. Keller, L. M. Kelly, C. A. Klug, L. W. Joos, Herman Levy, Yindell Luke, D. L. MacKay, Yetta V. Maizlish, E. A. Nordhaus, L. J. Paige, P. W. A. Raine, A. V. Richardson, E. P. Starke, Charles Stuckey, H. Tate, J. E. Thompson, H. B. Thornton, C. W. Trigg, R. H. Urbano, Paul Weisz, Maud Willey, G. A. Williams, Samuel Zuckerman, and the proposer.

E 316 [1938, 117]. *Proposed by J. E. Trevor, Cornell University.*

The profile of a certain straight road is the curve,

$$125x^2y + 5y - 1 = 0,$$

where x and y are measured in miles. In miles per hour and with reference to the road, the speed at which Mr. Smith walks is a linear function of the slope of the road, is four miles per hour on the level, and becomes zero when the slope reaches the value 0.8. Smith lives on the road at $x_0 = -0.8$ miles. He left home on foot at 9:35 A.M., saying to his son, "Telephone your mother that I will meet her at the crest of the hill at ten o'clock." Did he arrive in time for this appointment?

Solution by the proposer.

When Smith walks s miles along the road in t hours, the given data determine that the relation between his speed and the slope is

$$ds/dt = 4 - 5dy/dx. \tag{1}$$

We have also

$$ds/dx = \sqrt{1 + (dy/dx)^2}, \tag{2}$$

and from these equations we find

$$\frac{dt}{dx} = \frac{dt}{ds}\,\frac{ds}{dx} = \frac{\sqrt{1 + (dy/dx)^2}}{4 - 5dy/dx}. \tag{3}$$

Here, by the equation of the profile, $dy/dx = -10x/(25x^2+1)^2$, it appears that the crest of the hill is at $x=0$. The time required by Smith to walk from his home to the crest is obtained by integrating (3) from $x = -0.8$ to $x=0$. Dividing this interval into eight equal parts and evaluating the integral by Simpson's Rule, we find the elapsed time to be, very approximately, 0.429 hours, or 25.7 minutes. So Smith arrived about forty-two seconds late for his appointment.

E 317 [1938, 117]. *Proposed by Wm. F. Cheney, Jr., Connecticut State College.*

PRYP, *LARAP*, and *LLUMP* are three numbers in harmonic progression, where each letter is the code representative of some digit in the decimal system. Solve the code.

Solution by W. C. Rufus, University of Michigan.

The relationship may be written, $LARAP(PRYP+LLUMP)=2\cdot PRYP\cdot LLUMP$. As a first approximation, we substitute zero for each letter but L and P, and obtain roughly, $10{,}000{,}000\cdot L\cdot P+110{,}000{,}000\cdot L^2=22{,}000{,}000\cdot L\cdot P$. Since P is thus roughly $9\cdot L$, L must be 1 and P must be 9. Inserting these values, and solving for $LARAP$, since A is less than 2, it must be 0. We now have $9RY9$, $10R09$, and $11UM9$ in harmonic progression. Factors of the mean must be found in the extremes. When $R=5$, the mean is $3\cdot 31\cdot 113$, or 10509. The first number is $95Y9$, readily seen to be $3\cdot 31\cdot 103$, or 9579. The third is $103\cdot 113$, or 11639.

Also solved by M. L. Constable, Daniel Finkel, I. Kaplansky, E. P. Starke, Maud Willey, and the proposer.

E 318 [1938, 117]. *Proposed by N. A. Court, University of Oklahoma.*

A tetrahedron, $KLMN$, is inscribed in the tetrahedron $ABCD$, and O is a fixed point in space. If the parallels AP, BQ, CR, and DS to the lines OK, OL, OM, and ON meet the respective faces of $ABCD$ in P, Q, R, and S, show that $OK/AP+OL/BQ+OM/CR+ON/DS=1$.

Solution by L. M. Kelly, Northeastern University.

Draw AO intersecting the opposite face in E. Then EKP is a straight line; and $OK/AP=OE/AE$. Similarly, $OL/BQ=OF/BF$, $OM/CR=OG/CG$, and $ON/DS=OH/DH$. But $OE/AE+OF/BF+OG/CG+OH/DH=1$. Hence the theorem. (See N. A. Court, page 115.)

Also solved by I. Kaplansky, W. T. Short, and the proposer.

E 319 [1938, 117]. *Proposed by Joseph Rosenbaum, Bloomfield, Connecticut.*

An electric light bulb is connected to n switches in such a way that the light is lit only when each switch is closed. Each switch is controlled by a push button, successive depressions of which will alternately open and close that switch. The push buttons are not provided with the usual marks, "on" and "off."

It is required to find in what order the push buttons should be pressed so that the greatest possible number of pushes which may be required to turn on the light, will be as small as possible. Generalize the solution to cover the case in which each switch closes only at every p-th push.

Solution by the Proposer.

Denoting the push buttons by $A_1, A_2, \cdots, A_n$, let P_n (if such exists) represent a succession of pushes on the buttons in the order $A_i, A_j, \cdots, A_l$, such that it will always put on the light, on or before the termination of the operation (the subscripts $i, j, \cdots, l$ are not necessarily different). Now let an additional switch A_{n+1} be introduced into the set. It is obvious that if the junction at A_{n+1} is closed, then the succession P_n will produce a light.

Hence if P_n does not produce a light then A_{n+1} is not closed, and hence, a

push on A_{n+1} will render it closed, so that the application now of P_n will produce a light. This is expressed by

$$P_{n+1} = P_n A_{n+1} P_n. \tag{1}$$

This recursion formula together with the obvious

$$P_1 = A_1 \tag{2}$$

enables us to find P_n for any given n. It also proves the existence of P_n for every n.

As regards the greatest possible number of pushes that this solution may require, it is seen from (1) that if g_n represents this number then

$$g_{n+1} = 2g_n + 1. \tag{3}$$

From (2),

$$g_1 = 1 \tag{4}$$

Hence

$$g_n = 2^n - 1. \tag{5}$$

Now the number of possible states of the set of the switches (as regards the *on* and *off* positions of the individual switches) is 2^n. One of these is the state of all the switches being *on*, which gives a light. Hence the number of states corresponding to no light is $2^n - 1$. So that, by (5), this number is the same as g_n. It is not difficult to prove now that there exists no solution with a corresponding g_n less than the one given by (5). It is to be observed, however, that (1), (2) do not give all solutions. For when $n = 3$ for example, there is the solution

$$P_3 = 1232123.$$

which is not given by (1), (2). Similarly, for $n = 4$ there are at least the three following solutions which are not given by (1), (2):

$$P_4 = 123412324321432,$$
$$P_4 = 123432341432343,$$
$$P_4 = 123121431321314.$$

An extension of the above reasoning to the more general case gives the recursion formula

$$P_{n+1} = P_n A_{n+1} P_n \cdots A_{n+1} P_n \tag{1'}$$

where P_n and A_{n+1} appear p and $(p-1)$ times respectively. The equation corresponding to (2) is easily seen to be

$$P_1 = A_1 A_1 \cdots A_1, \tag{2'}$$

where A_1 is applied $(p-1)$ times. Also, by means of the equations corresponding to (3) and (4), there is obtained

$$g_n = p^n - 1. \tag{5'}$$

As before, it can be proved that there can be no solution with a smaller g_n.

Here, too, it can be shown that there are solutions other than those given by (1'), (2').

The problem could be generalized further by assigning a different period to each switch.

Also solved by I. Kaplansky.

ADVANCED PROBLEMS

Send all communications about Advanced Problems and Solutions to Otto Dunkel, Washington University, St. Louis, Mo. All manuscripts should be typewritten, with double spacing and with margins at least one inch wide.

Problems containing results believed to be new or extensions of old results are especially sought. The editorial work would be greatly facilitated, if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well known textbooks or results found in readily accessible sources, will not be proposed as problems for solution in this department. In so far as possible, however, the editors will be glad to assist members of the Association in the solution of such problems.

PROBLEMS FOR SOLUTION

3896. *Proposed by W. B. Clarke, San José, Calif.*

Let P and Q be isotomic conjugate points with respect to triangle ABC. Find the locus of P if PQ is parallel to a side of the given triangle.

3897. *Proposed by V. Thébault, Le Mans, France.*

Let $ABCD$ be a rectangle inscribed in a circle with the center O; let PA, PB, PC, PD cut the circle again in A', B', C', D', where P is any given point in the plane of the figure; let A'', B'', C'', D'' be the feet of the perpendiculars from P on the sides of the quadrilateral $A'B'C'D'$; and let Q be the intersection of the diagonals of this last quadrilateral. Show that: (1) The points O, P, Q are collinear; (2) The diagonals of the quadrilateral $A''B''C''D''$ are perpendicular, and the ratio of their lengths is the same as that of the two sides of $ABCD$; (3) The point P and the intersection of the diagonals of $A''B''C''D''$ are the foci of a conic inscribed in this quadrilateral; (4) The Newton lines for $A'B'C'D'$ and $A''B''C''D''$ are perpendicular.

3898. *Proposed by Otto Dunkel, Washington University.*

A point is chosen on a rectangular hyperbola. In how many ways, and under what conditions, may two other points on the curve be selected so that the centroid of the three points will lie also on the curve?

3899. *Proposed by V. E. Pound, The University of Buffalo.*

Using elementary vector methods show that the angular precession of an

ideal frictionless top, spinning so rapidly that the precession is steady, is given by the well known formula

$$\Omega = \frac{C\omega_3 - (C^2\omega_3^2 - 4mghA\cos\theta)^{1/2}}{2A\cos\theta}.$$

SOLUTIONS

3795 [1936, 436]. *Proposed by R. Goormaghtigh, Bruges, Belgium.*

Let P be a point on the circumcircle of a triangle ABC; A_0, B_0, C_0 the points where AP, BP, CP meet BC, CA, AB; A_1, B_1, C_1 the points dividing AA_0, BB_0, CC_0 in the same ratio so that

$$AA_1{:}A_1A_0 = BB_1{:}B_1B_0 = CC_1{:}C_1C_0 = k,$$

and finally A_2, B_2, C_2 the points where the altitudes meet the perpendiculars at P to AP, BP, CP. Prove that the ratio of the areas $A_1B_1C_1$ and ABC equals

$$\frac{1-k}{(1+k)^2} + \frac{k^2}{4(1+k)^2}\,\frac{AA_2\cdot BB_2\cdot CC_2}{R^3},$$

R being the circumradius.

Solution by C. E. Springer, University of Oklahoma.

Let the points $A(1, 0, 0)$, $B(0, 1, 0)$, $C(0, 0, 1)$ be the vertices of the triangle of reference for a system of areal coördinates, where x=area PBC/Δ, y=area PCA/Δ, z=area PAB/Δ, Δ=area ABC. Hence

$$x + y + z = 1. \tag{1}$$

Since $P(X, Y, Z)$ is on the circumcircle of ABC we have

$$a^2YZ + b^2ZX + c^2XY = 0, \tag{2}$$

where a, b, c are the lengths of the sides opposite A, B, C. The area of the triangle $A_1(X_1, Y_1, Z_1)$, $B_1(X_2, Y_2, Z_2)$, $C_1(X_3, Y_3, Z_3)$ is given by

$$\begin{vmatrix} X_1 & Y_1 & Z_1 \\ X_2 & Y_2 & Z_2 \\ X_3 & Y_3 & Z_3 \end{vmatrix} \cdot \Delta. \tag{3}$$

The coördinates of A_0, B_0, C_0 are given by

$$\left(\frac{0, Y, Z}{Y+Z}\right), \qquad \left(\frac{X, 0, Z}{Z+X}\right), \qquad \left(\frac{X, Y, 0}{X+Y}\right)$$

so that the coördinates of A_1, B_1, C_1 are given by

$$\left(\frac{Y+Z,\ kY,\ kZ}{(1+k)(Y+Z)}\right), \qquad \left(\frac{kX,\ Z+X,\ kZ}{(1+k)(Z+X)}\right), \qquad \left(\frac{kX,\ kY,\ X+Y}{(1+k)(X+Y)}\right).$$

On substituting these values into the determinant (3) and using the relation $X+Y+Z=1$, the required ratio (r) takes the form

$$r = \frac{1-k}{(1+k)^2} + \frac{2k^2}{(1+k)^2} \cdot \frac{XYZ}{(Y+Z)(Z+X)(X+Y)}. \tag{4}$$

Now we notice that $AP \cdot AA_0 = 2\Delta \cdot AA_2/a$, where $2\Delta/a$ is the length of the altitude through A. Using the relations (2) and (1) we find in turn

$$\overline{AP}^2 \cdot \overline{AA_0}^2 = \left(-a^2 \frac{YZ}{X}\right)\left(-a^2 \frac{YZ}{X(Y+Z)^2}\right) = \left[\frac{a^2 YZ}{X(Y+Z)}\right]^2,$$

$$\frac{YZ}{X(Y+Z)} = \frac{2\Delta}{a^3} \cdot AA_2, \quad \frac{ZX}{Y(Z+X)} = \frac{2\Delta}{b^3} \cdot BB_2, \quad \frac{XY}{Z(X+Y)} = \frac{2\Delta}{c^3} \cdot CC_2. \tag{5}$$

On multiplying these three equations we obtain

$$\frac{XYZ}{(Y+Z)(Z+X)(X+Y)} = \frac{8\Delta^3}{a^3b^3c^3} \cdot AA_2 \cdot BB_2 \cdot CC_2 = \frac{AA_2 \cdot BB_2 \cdot CC_2}{8R^3},$$

since $R = abc/4\Delta$. Hence,

$$r = \frac{1-k}{(1+k)^2} + \frac{k^2}{4(1+k)^2} \cdot \frac{AA_2 \cdot BB_2 \cdot CC_2}{R^3}.$$

Solved also by the proposer.

Editorial Note. For the computation of AP and AA_0 Springer used the formula $-\overline{P_1P_2}^2 = \sum a^2(Y_1-Y_2)(Z_1-Z_2)$ for the distance between (X_1, Y_1, Z_1) and (X_2, Y_2, Z_2). The problem does not give any rule for the signs of AA_2, BB_2, CC_2 or for their product; and it will be shown that they do not have always the same sign. Suppose that ABC is a triangle with angles $A<B<C<\pi/2$, and consider first that P is on and within the arc segment BC. Then $X<0$, $0<Y<1$, $0<Z<1$, $Z+X=1-Y>0$, $X+Y=1-Z>0$. Hence, for such positions of P, the signs of AA_2, BB_2, CC_2, as defined in (5) of the solution, are all negative. When P is on and within the arc segment $C\overline{C}$, where chord $C\overline{C}$ is parallel to AB, the signs in order are $-$, $-$, $+$; when P is within $\overline{C}A$ the signs are again all negative, and so on. Instead of considering the coördinates of P we may consider the case where k is infinite, and then r = area $A_0B_0C_0/\Delta$ and its sign is that of the second term for r in the general formula. For the first special case above $A_0B_0C_0$ has the sense opposite to that of ABC and hence this ratio is negative. For the second position of P, $A_0B_0C_0$ has the same sense as ABC and the ratio is positive, and so on.

The proposer obtained this problem as follows: In his paper in *Mathesis*, 1932, p. 5, it was shown that, if $B_1B_2B_3$ is a triangle inscribed in the triangle $A_1A_2A_3$, and if M_1, M_2, M_3 are points on A_1B_1, A_2B_2, A_3B_3, respectively, such that $A_1M_1 : M_1B_1 = A_2M_2 : M_2B_2 = A_3M_3 : M_3B_3 = k$, then

$$(1+k)^2\Delta_m = (1-k)\Delta_a + k^2\Delta_b, \tag{1}$$

where Δ_m is the area of triangle $M_1M_2M_3$, *etc.* If for B_1, B_2, B_3 we take the feet of the altitudes of triangle ABC, and denote the M triangle by $A'B'C'$, then the above gives

$$(2) \qquad (1+k)^2 \frac{\text{area } A'B'C'}{\text{area } ABC} = (1-k) + k^2 \frac{AH \cdot BH \cdot CH}{4R^3},$$

by using the fact that the area of the orthic triangle is $2 \cos A \cos B \cos C \cdot$ area ABC. The formula of the problem is then obtained from (2) by a transformation of complex affinity, the properties of which were developed by the proposer in papers several years ago. By complex affinity $x = x_0$, $y = iy_0$, ABC is transformed into $A_0B_0C_0$, and the orthocenter H of ABC into a point P on the circle $A_0B_0C_0$, A_0P and B_0C_0 being symmetric with respect to the direction ξ making an angle $\pi/4$ with the basis line of the affinity. Then AH/R is transformed into $A_0P\ (\sin 2\theta_a)^{1/2}/R_0\ (\sin 2\theta_a \sin 2\theta_b \sin 2\theta_c)^{1/2}$ and $AH \cdot BH \cdot CH/4R^3$; into $A_0P \cdot B_0P \cdot C_0P/4R_0^3 \sin 2\theta_a \sin 2\theta_b \sin 2\theta_c$, where $2\theta_a$ is the angle between the altitude from A_0 of $A_0B_0C_0$ and the perpendicular at P to A_0P; and this proves the statement.

The methods of Springer give the result (1) and then (2) easily follows. In (2) it is observed that the second term on the right is positive if no angle of ABC is as large as $\pi/2$, and it is negative if one angle is greater than $\pi/2$.

3805 [1936, 581]. *Proposed by R. E. Gaines, University of Richmond.*

Determine a point P on $b^2x^2 - a^2y^2 = a^2b^2$ such that the tangent and normal lines at that point shall be normal and tangent respectively to $b^2x^2 - a^2y^2 = -a^2b^2$, and hence, that if the hyperbola and its conjugate be together considered as a single curve, $(b^2x^2 - a^2y^2)^2 - a^4b^4 = 0$, a rectangle may be drawn which is both an inscribed and a circumscribed figure.

Solution by Evelyn Jaeckle, Junior Student, University of Buffalo.

The equation of the normal at $P(x_1, y_1)$ on the given hyperbola is

$$(1) \qquad y = y_1[(a^2 + b^2)x_1 - a^2x]/b^2x_1.$$

In order for (1) to be the equation of a tangent to the conjugate hyperbola, the quadratic obtained by substituting (1) in the equation of the conjugate must have equal roots. This gives the conditions

$$(2) \qquad \begin{aligned} a^6y_1^2 + (a^2 + b^2)^2x_1^2y_1^2 - b^6x_1^2 &= 0, \\ b^2x_1^2 - a^2y_1^2 - a^2b^2 &= 0; \end{aligned}$$

and the solution of these equations gives for the coördinates of P

$$(3) \qquad x_1 = \pm a\left[\frac{(a^4 + b^4)^{1/2} + b^2}{a^2 + b^2}\right]^{1/2}, \qquad y_1 = \pm b\left[\frac{(a^4 + b^4)^{1/2} - a^2}{a^2 + b^2}\right]^{1/2},$$

where the sign of the inner radical has been chosen as positive in order to obtain real points. The point of tangency $Q(x_{11}, y_{11})$ is found to be

(4) $$x_{11} = -a^4/x_1(a^2+b^2), \qquad y_{11} = b^4/y_1(a^2+b^2).$$

The normal at Q to the conjugate has the equation

(5) $$(y-y_{11})/(x-x_{11}) = b^2x_1/a^2y_1;$$

and this reduces by use of previous results to the equation of the tangent to the hyperbola at $P'(-x_1, -y_1)$.

Because of symmetry it will be clear that the normal at P' is tangent to the conjugate at $Q'(-x_{11}, -y_{11})$, and the normal at the latter point is tangent to the hyperbola at P. There are two rectangles of the type $PQP'Q'$. One of these is obtained by using the plus sign for the exterior radicals in (3) for x_1 and y_1, while for the other the sign of x_1 is plus and that of y_1 is minus.

Solved also by W. B. Campbell, J. F. Locke, Otto J. Ramler, C. E. Springer, and the proposer.

Editorial Note. After determining P and Q as above, we easily verify that these two points lie on the circle with its center at the origin and with the radius $(a^4+b^4)^{1/4}$; and this completes the proof. For the symmetric $P'Q'$ of PQ with respect to the origin has the same properties as PQ, and it then follows that $P'Q$ and PQ' are tangents to the hyperbola and normals to the conjugate. Ramler identified the equations of the normal to the hyperbola and the tangent to the conjugate written in terms of their equal slopes, and found the value of the slope. From the equations of orthogonal tangents to the two hyperbolas and this value of the slope it was found that the vertices of the required rectangle lie on the above mentioned circle. It was shown that the radius of this circle is the mean proportional between the diagonals of the parallelogram with the sides a and b and with one angle of 45°. This gives a geometrical construction for the radius. Campbell and Springer used parametric equations for the two hyperbolas.

3807 [1936, 643]. *Proposed by J. R. Musselman, Western Reserve University.*

Any four straight lines are touched by a parabola. It is well known that the three midpoints of the diagonals of the four lines are collinear, on l. Show that, if the points of tangency of the parabola with the four lines are concyclic, the line l is the axis of the parabola.

I. *Solution by H. Tate, McGill University, Montreal, Canada.*

Let the corners of the quadrilateral formed by the four tangents be A, B, C, D in order and suppose AD and BC produced meet at X and AB and DC produced meet at Y: then the diagonals are AC, BD, and XY. Let the equation of the parabola be $y^2=4ax$ referred to the axis and tangent at the vertex as axes. Suppose that the parameters of the points of contact of AB, CD, BC, AD are given by $(at^2, 2at$ where $t=t_1, t_2, t_3, t_4$ respectively. It is well known that the necessary and sufficient condition that these points of contact be concyclic is $t_1+t_2+t_3+t_4=0$. The coördinates of the points of intersection of pairs of tangents are given by at_1t_2, $a(t_1+t_2)$ *etc.* The extremities of the diagonals AC; BD;

XY are given by at_1t_4, $a(t_1+t_4)$ and at_2t_3, $a(t_2+t_3)$; *etc.* The ordinate of the midpoint of each diagonal is therefore $a(t_1+t_2+t_3+t_4)/2$ which is zero since $t_1+t_2+t_3+t_4=0$. These three midpoints therefore lie on the line $y=0$ which is the axis of the parabola $y^2=4ax$.

II. *Solution by C. M. Sparrow, University of Virginia.*

Let Σ be the pencil of conics on the four lines, one of these being the parabola π. Let k be the line at infinity and K its point of contact with π. The tangent pairs from K to the conics of Σ are in involution. Among these pairs are the lines to opposite vertices of the 4-line. The double rays of the involution are apolar to these; one double ray is k, the other therefore passes through the midpoints of the diagonals, and is the line l, which is thus also a diameter of the parabola. Let l meet the parabola again at B.

Now form the polar figure of all this with respect to the parabola. The involution of rays on K becomes an involution of points on k. One double point is K, the other, L, is the pole of l; BL is thus the tangent at B. The pencil Σ becomes a pencil σ of conics on the points of contact, and conics of σ meet k in point pairs of the involution, apolar to K and L.

But if the points of contact are concyclic, the circular points are one of these pairs. In this case the tangent BL is perpendicular to the diameter BK, or l. That is, l is the axis.

Solved also by J. W. Clawson, K. W. Crain, R. E. Gaines, J. W. Peters, O. J. Ramler, W. T. Short, C. E. Springer, and F. Underwood.

Editorial Note. The well known theorem in the first part of the problem is proved in the solutions of the problem, all of which are analytic except one which appears above. It is rather curious that the analytic proofs may be made synthetic by dropping the x's of the points since these are unnecessary. Denote the four distinct points on the parabola by y_1, y_2, y_3, y_4, their perpendicular distances from the axis of the parabola with proper signs; there is no ambiguity since the points are on the parabola. The tangents to the parabola at y_1 and y_2 meet in a point P_{12} which is the polar of the chord joining y_1 and y_2, with the midpoint M_{12}. Then $P_{12}M_{12}$ is parallel to the axis, *i.e.* it is a diameter with the distance $(y_1+y_2)/2$ from the axis. Similarly, y_3 and y_4 determine a diameter with the distance $(y_3+y_4)/2$ from the axis. The midpoint of $P_{12}P_{34}$ is at the distance $(y_1+y_2+y_3+y_4)/4$ from the axis, and this is the midpoint of a diagonal of the quadrilateral formed by the tangents at the four points. Since the other two diagonals give the same result, we conclude that the midpoints of the diagonals of a quadrilateral lie on a straight line l which is a diameter of the parabola tangent to its four sides.

Now suppose that the points y_1, y_2, y_3, y_4 lie also on a circle: then by a known theorem each pair of opposite common chords is such that the bisector of the angle between the chords of a pair is parallel to a principal axis of the conic. Hence if we reflect the lower half of the curve in the axis the chord y_3y_4 becomes

a chord parallel to chord y_1y_2, and hence their midpoints now lie on a line parallel to the axis, *i.e.*, the two diameters coincide. Hence $y_1+y_2+y_3+y_4=0$; but this says that the line joining the midpoints of the diagonals is the axis of the parabola; and this completes the proof of the theorem of the problem. This last argument may be reversed to show that, if the line of the midpoints of the diagonals is the axis, the four points of contact lie on a circle.

NEWS AND NOTICES

Readers are invited to contribute to the general interest of this department by sending news items to R. G. Sanger, Eckhart Hall, University of Chicago, Chicago, Illinois.

N. V. Swets and Zeitlinger, Keizersgracht 471, Amsterdam (C), Holland announce that they have purchased the stock of back volumes 1–80 of the Mathematische Annalen and are ready to reprint those numbers now out-of-print. The announced price is $480.00, including index and library binding. Reprinting is contingent on a sufficient number of orders being received. Institutions desiring information should correspond with Swets and Zeitlinger. Volumes 81 and following are in print and can be supplied. One bound set of volumes 1–80 is immediately available at $650.00.

The mathematics department of Northwestern University held an "open house" on November 30 to allow visitors to inspect its new quarters. The occasion was marked by a celebration of the centennial of two great American mathematicians. Biographical sketches were given as follows: "Josiah Willard Gibbs," by Professor R. E. Langer of the University of Wisconsin. "George William Hill," by Dr. F. R. Moulton, Permanent Secretary of the A.A.A.S., Washington, D.C.

Associate Professor Marion E. Stark of Wellesley College is on leave of absence and is spending the fall at the University of Chicago.

Dr. Mary B. Haberzetle is teaching at Mount Holyoke, replacing Dr. Frances Baker who is on leave of absence.

THE SECOND ANNUAL WILLIAM LOWELL PUTMAN MATHEMATICAL COMPETITION

The second annual William Lowell Putnam Mathematical Competition, under the sponsorship of the Mathematical Association of America, will be held on Saturday, March 4, 1939. This Competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund, left by Mrs. Putnam in memory of her husband, is open to undergraduates in the United States and Canada who have not received a degree.

The contest in April 1938 aroused great interest and manifestly exercised a strong influence on mathematical preparation over the country. One hundred and sixty-three individuals from sixty-seven institutions, including forty-two teams, took part in the contest.

The examination consists of two three-hour examinations. The questions will be taken from the fields of calculus, with applications to geometry and mechanics, higher algebra (determinants and theory of equations), elementary differential equations, and geometry (advanced plane and solid geometry). Any college or university wishing to enter a team or individual contestants may secure an application blank from the Secretary of the Association, W. D. Cairns, Oberlin, Ohio, by a postcard request. All applications must be filed with the Secretary not later than February 15, 1939. If three candidates are presented from a college or university, they are to constitute a team; if more than three are presented from any one college or university, the team of three must be named on the application.

The examination may be given at any place where a team, or at least three candidates, can be assembled. Exceptions to the rule may be made by the Secretary in cases of unusual necessity. Sealed copies of the examinations will be sent to the supervisor of the examination in ample time for the examination day and are not to be opened before the hour set. At the supervisor's first opportunity after the afternoon examination the books are to be sent by registered mail or by express to the Secretary of the Association, who will forward them to a qualified reader chosen by the Association. The results will be announced a month after the examination date.

The prizes to be awarded to the departments of mathematics of the institutions with the winning teams are $500, $300, and $200 in the order of their rank. In addition, there will be prizes of $50, $30, and $20 awarded to the members of these teams according to the rank of the team, and a prize of $50 to each of the five highest contestants. Each of the winners will receive a suitable medal. Honorable mention will be given to the three teams next in order after the three winning teams and to the five individuals next in order after the five individual winners. For further encouragement of the Competition, there will be awarded at Harvard University* an annual $1000 William Lowell Putnam Prize Scholarship to one of the first five contestants; this to be available either immediately or on the completion of the student's undergraduate work.

A more complete announcement of the general regulations under which the plan is operated will be found in the January 1938 issue of the MONTHLY and in the announcement which is being mailed to colleges and universities in the United States and Canada.

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* Or at Radcliffe College, in the case of a woman.

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